NONLINEAR RESONANCE INTERACTIONS AND INDEX CYCLES IN THE ATMOSPHERE

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ABSTRACT

A barotropic spectral model is used to study the planetary-scale motions of an atmosphere whose wave ensemble modes are externally driven. Pertubations are induced by a barotropic analogue of thermal driving and by Ekman friction, bottom topography, and the vanished internal dissipation. The use of complete spectral expansions without truncation leads to that the nonlinear coupling equations between the low-index mode and the high-index mode are obtained by means of the random phase approximation and the projection operator techniques. The nonlinear coupling equations are entirely equivalent to the *Volterra* systems in ecology.

In the phase-plane, the orbits of the nonlinear coupling equations are the family of closed curves, indicating a bound, and periodic motion. The qualitative behaviors of low-index and high-index modes as functions of time picture the motion of atmospheric flows, with exchanges of energy between the low-index mode and the high-index mode by nonlinear resonance interaction. It is suggested that the phenomenon of blocking be exponentially grown of the low-index mode, the atmospheric motion then evolved to the high-index mode due to relaxation process. The results therefore lead to a plausible hypothesis concerning index cycles in the atmosphere discussed by Lorenz's early works.

I. INTRODUCTION

A fundamental characteristic of atmospheric motions on all scales is that the exchange of energy between various wavelengths is accomplished via nonlinear interactions. For the purpose of demonstrating this nonlinear behavior conveniently, we use the quasi-geostrophic equation because it has the advantage of being easily represented in spectral form (e.g., Lorenz, 1960; Charney and DeVore, 1979).

In a recent paper (Li, 1986), we have considered some simple aspects of the nonlinear transfer of energy between topographically forced wave ensemble and zonal mean flow. We may write down the following differential equation governing topographically forced waves growth:

$$\frac{d}{dT}W = aW - bW^2. \quad (1)$$

Where W represents the mean power carried by topographically forced wave ensemble. And

$$a = \frac{16\pi G^2 \tilde{\tilde{H}}^2}{|A_{k_0} - A_{k_2}|^2} > 0, \quad b = 4\pi G^2 > 0,$$

where $\frac{\tilde{\tilde{H}}^2}{|A_{k_0} - A_{k_2}|^2}$ is the topographic function parameter, Astonishingly, equations entirely equivalent to the equation (1) occur in laser physics (Haken, 1983):

$$\frac{d}{dt}\mathbf{n} = -k\mathbf{n} - k_1\mathbf{n}^2 \tag{2}$$

for $k>0$ there is no laser light emission whereas for $k<0$ the laser emits laser photons. The same equations apply to certain problems of ecology and population dynamics (Gause, 1964).

Observe that as $T\to\infty$,

$$W(T)\to\frac{a}{b}=\frac{4\tilde{\tilde{H}}^2}{|A_{k_0}-A_{k_2}|^2}. \tag{3}$$

Thus, regardless of its initial value, the mean power carried by topographically forced wave ensemble always approaches the limiting value which is related with the topographic function.

When the parameter $\tilde{\tilde{H}}^2$ vanished, the equations will be a completely different way:

$$\frac{d}{dT}W=-bW^2. \tag{4}$$

A preliminary work (Li, 1985) has shown that the energy of external thermodynamically forced finite-amplitude disturbance wave ensemble can be completely transferred into the zonal mean flow by nonlinear resonance interactions when $\tilde{\tilde{H}}^2=0$. It is somewhat a chain of atmospheric wave motions:

......Thermodynamically Forced Waves ......Zonal Mean Flow...... Topographically Forced Waves......

We shall therefore consider the external terms representing the bottom topographic forcing, the Ekman friction, and the vorticity source driven by thermodynamic effects. It is hoped that the solutions may lead to a plausible hypothesis concerning index cycles in the atmosphere.

II. MODEL

The index change itself will be a barotropic effect, even though the cause will be baroclinic (Lorenz, 1960). Our starting point is the spectral form of the vorticity equation for two-dimensional, incompressible flow. Its derivation from the Navier-Stokes equations is outlined in earlier papers (Lorenz, 1960; Li, 1985). In the notation of the latter, it is

$$\frac{\partial}{\partial t}\nabla^2\psi+2\frac{\partial}{\partial\lambda}\psi+J(\psi,\nabla^2\psi+\eta)=\frac{1}{\kappa}\nabla^4\psi+\frac{v}{2}\nabla^2(\psi-\psi^{\circledast}), \tag{5}$$

where ψ is streamfunction; η the height of bottom topography: κ the Reynolds number; and the second term in the right hand side of Eq. (5) represents the frictionally induced vorticity sink given by Ekman pumping and the vorticity source driven by thermodynamic effects. Compared with the basic equation of Charney and DeVore (1979) or of Yorden (1985), Eq. (5) is a *Spherical Model* instead of the *Channel Models.*

In order to write Eq. (5) in spectral form, both the streamfunction and the orographic function are expanded in spherical harmonics:

$$\psi(\vartheta,\lambda,t)=\sum_{l=0}^{\infty}\sum_{m=-l}^{+l}\psi_l^m(t)Y_l^m(\vartheta,\lambda) \tag{6a}$$

$$\eta(\vartheta,\lambda)=\sum_{l=0}^{\infty}\sum_{m=-l}^{+l}H_l^m Y_l^m(\vartheta,\lambda). \tag{6b}$$

The family of wavevectors with order l and m can be labelled as $k=(l_k, m_k)$ and Eqs. (6) become

$$\psi(t)=\sum_k \psi_k(t)Y_k \tag{6a}$$

$$\eta=\sum_k H_k Y_k. \tag{6b}$$

Note that Y_k is the eigenfunction of the spherical Laplacian

$$\nabla^2 Y_k=-\Lambda_k Y_k, \tag{7}$$

where $\Lambda_k=l_k(l_k+1)$. (8)

On substituting Eqs. (6), (7) into Eq. (5) and integrating $\int_0^{2\pi}d\lambda\int_0^{\pi}\sin\vartheta d\vartheta Y_k^*(\cdot)$, we obtain

$$\frac{d}{dt}\Psi_k(t)-\left[i\frac{2m_k}{\Lambda_k}-\frac{\Lambda_k}{\kappa}+\frac{v}{2}(1-\alpha_k)\right]\Psi_K(t)$$

$$=i\sum_{k'}\sum_{k''}[B_k(k',k'')\Psi_{k'}(t)\Psi_{k''}(t)+S_k(k',k'')\Psi_{k'}(t)H_{k''}], \tag{9}$$

where

$$\alpha_k=\frac{\psi_k^*}{\psi_k} \tag{10a}$$

$$B_k(k',k'')=\frac{1}{i}\frac{\Lambda_{k'}-\Lambda_{k''}}{2\Lambda_k}\int_0^{2\pi}d\lambda\int_0^{\pi}\sin\vartheta d\vartheta Y_k^* J(Y_{k'},Y_{k''}) \tag{10b}$$

$$S_k(k',k'')=\frac{1}{i}\frac{1}{\Lambda_k}\int_0^{2\pi}d\lambda\int_0^{\pi}\sin\vartheta d\vartheta Y_k J(Y_{k'},Y_{k''}). \tag{10c}$$

We may write out the constrained condition by dummy-variable

$$m_k=m_{k'}+m_{k''}. \tag{11}$$

The general solution of Eq. (9) can be written as

$$\Psi_k(t)=C_k(t)e^{-i\sigma_k t}. \tag{12}$$

By the multiple-time-scale properties of atmospheric motion, the amplitude can be considered to vary slowly with time. Let

$$t=t,\quad T=\varepsilon t;\quad \frac{d}{dt}=\frac{\partial}{\partial t}+\varepsilon\frac{\partial}{\partial T} \tag{13}$$

$$\Psi_k(t)=\varepsilon C_k(T)e^{-i\sigma_k t}. \tag{14}$$

On substituting Eqs. (13), (14) into Eq. (9), using the first-order equation of ε, we obtain the dispersion relation

$$\sigma_k=-\frac{2m_k}{l_k(l_k+1)}+i\left[-\frac{l_k(l_k+1)}{\kappa}+\frac{v}{2}(1-\alpha_k)\right]. \tag{15}$$

Obviously, the frequency now has a complex form. Let

$$\sigma_k=\sigma_{rk}+i\sigma_{ik}. \tag{16}$$

Similar to the discussion for the β-plane by Pedlosky (1979), we assume that the spin-down time is of the same order as the slower time scale, i.e. the nonlinear equation (9) admits approximate solutions which vary slowly over space and more slowly with time due to weakly nonlinearity and dissipative effects. We then have

$$\Psi_k(t) = \varepsilon C_k(T) e^{-i(\sigma_{rk} + i\sigma_{ik})t}. \tag{17}$$

The nonlinear coupling equation for amplitude can be obtained by using the second-order equation of ε:

$$\frac{\partial}{\partial t} C_k(T) + \sigma_{ik} C_k(T)$$

$$= i \sum_{m_k = m_{k'} + m_{k''}} [B_k(k', k'') C_{k'}(T) C_{k''}(T) + S_k(k', k'') C_{k'}(T) H_{k''}] e^{i\Delta\sigma_k t}. \tag{18}$$

It is considered that since the frequency spectrum of wavevector ensemble has finite bandwidth, the resonance interactions among wavevectors will not be coherent. Therefore, we must allow a frequency mismatch, i.e.

$$\Delta\sigma_k \equiv \sigma_k - (\sigma_{k'} + \sigma_{k''}), \quad |\Delta\sigma_k| \ll |\sigma_k|. \tag{19a}$$

Because of such a finite bandwidth of the spectrum of each of the wavevector ensemble, we cannot select only three discrete wavevectors but we must sum up over many wavevectors that satisfy the constraint

$$m_k = m_{k'} + m_{k''}. \tag{19b}$$

To each of the wavevector triads teamed by dummy-variables k, k' and k'', we use the normalized complex amplitude defined by

$$A_k(T) = C_k(T) |B_k(k', k'')|^{-1/2} \tag{20a}$$

$$\tilde{\tilde{H}}_k = H_k |B_k(k', k'')|^{-1/2} \tag{20b}$$

we have

$$\frac{\partial}{\partial t} A_k + \sigma_{ik} A_k$$

$$= i \sum_{m_k = m_{k'} + m_{k''}} [G_k(k', k'') A_{k'} A_{k''} + F_k(k', k'') A_k \tilde{\tilde{H}}_{k''}] e^{i\Delta\sigma_k t} \tag{21}$$

where

$$G_k(k', k'') = |B_k(k', k'') B_{k'}(k'', k) B_{k''}(k, k')|^{1/2} \tag{22a}$$

$$F_k(k', k'') = \left[\frac{S_k(k', k'')}{B_k(k', k'')}\right] G_k(k', k''), \tag{22b}$$

Note that the k is the dummy-variable. Mathematically, in that case, it is an infinite set of first-order ordinary differential equation. Physically, in large scale atmospheric motion, it can be considered as the case that there are some planetary wave modes which effectively govern the behaviour as a wave ensemble and interfere with each other through the nonlinear resonance interactions. By means of cycle replacement of k_0, k_1 and k_2 to k, we obtain the amplitude coupling equations:

$$\frac{\partial}{\partial T} A_{k_0} + \sigma_{ik_0}$$

$$= i \sum_{m_{k_0} = m_{k'} + m_{k''}} [G_{k_0}(k', k'') A_{k'} A_{k''} + F_{k_0}(k', k'') A_{k'} \tilde{\tilde{H}}_{k''}] e^{i\Delta\sigma_{k_0} t} \tag{23a}$$

$$\frac{\partial}{\partial T}A_{k_1}+\sigma_{ik_1}$$

$$=i\sum_{m_{k_1}=m_k-m_{k''}}[G_{k_1}(k''^{*},k)A^{*}_{k''}A_k+F_{k_1}(k''^{*},k)A_{k''}\tilde{\tilde{H}}_k]e^{-i\Delta\sigma_{k_1}t} \tag{23b}$$

$$\frac{\partial}{\partial T}A_{k_2}+\sigma_{ik_2}$$

$$=i\sum_{m_{k_2}=m_k-m_{k'}}[G_{k_2}(k,k'^{*})A_kA^{*}_{k'}+F_{k_2}(k,k'^{*})A_k\tilde{\tilde{H}}^{*}_{k'}]e^{-i\Delta\sigma_{k_2}t}, \tag{23c}$$

where $k^*=(l_k,-m_k)$. And obviously, we have

$$A_{k^*}=A^*_k. \tag{24}$$

$\Delta\sigma_k$ is the frequency mismatch given by

$$\Delta\sigma_{k_0}=\sigma_{k_0}-(\sigma_{k'}+\sigma_{k''}) \tag{25a}$$

$$-\Delta\sigma_{k_1}=\sigma_{k_1}-(\sigma_k-\sigma_{k''}) \tag{25b}$$

$$-\Delta\sigma_{k_2}=\sigma_{k_2}-(\sigma_k-\sigma_{k'}). \tag{25c}$$

Without loss of generality, we may assume that

$$m_{k_0}>m_{k_1}>m_{k_2}>0. \tag{26}$$

III. SIMPLIFICATION OF THE NONLINEAR COUPLING EQUATIONS

Recall the second term in the right hand side of Eq. (5), the ratio α_k defined by Eq. (10a) is related to the budget between the frictionally induced vorticity sink given by Ekman pumping and the vorticity source driven by thermodynamic effects. In the atmosphere, the effect of heat flux from the Earth's surface has especially strong influence on the low-index mode due to the zonally distributed of continents and oceans, and the effect of Ekman friction, in the other hand, has more influence on the high-index mode due to the zonally averaged west wind being relatively large (see Palman and Newton, 1969). Therefore, the budget will be with a net recept to the relatively low-index motion mode k_0, i.e.

$$\alpha_0\equiv\frac{\psi^{\oplus}_{k_0}}{\psi_{k_0}}>1. \tag{27a}$$

To the relatively high-index motion mode k_2, the budget will be with a net expenditure, i.e.

$$\alpha_2\equiv\frac{\psi^{\oplus}_{k_2}}{\psi_{k_2}}<1. \tag{27b}$$

There must be a balance mode k_1 between mode k_0 and mode k_2 that the budget will be zero, i.e.

$$\alpha_1\equiv\frac{\psi^{\oplus}_{k_1}}{\psi_{k_1}}=1. \tag{27c}$$

In comparing with the Ekman friction, we may assume the internal dissipation to be vanished. And the bottom topography was also assumed, for simplicity, to be much smaller than the fluid depth. Hence, mode k_1, particularly, will represent the small topographically forced wave ensemble.

We emphasize that because of a finite bandwidth of the spectrum of the modes, as mentioned previously, we must sum up over many wavevectors that satisfy the constraint conditions. And therefore, in such a case, the use of the complex Fourier amplitude of a single wavevector does not make sense and only the phase independent quantity such as the energy spectral density of wave ensemble is useful.

Consider the phase of the three wave modes changes rapidly at random times during the process of the nonlinear interaction. In such a case, the phases of wavevectors are distributed randomly on all the components of a wave ensemble. By the *Ergodic Hypothesis* (or, more accurately, the *Quasi-Ergodic Hypothesis*), time averages and phase averages must coincide. As a consequence, the wave ensemble statistical average can be superseded in favour of the time-average. Note that the time-averaged amplitude $\langle A_k\rangle$ vanishes, when averaged over a time τ which is much larger than the average period of the phase change but smaller than the nonlinear interaction time scale T. This occurs because

$$\langle A_k\rangle \equiv \frac{1}{\tau}\int_0^\tau |A_k(T)|e^{i\varphi_k(t)}dt, \tag{28}$$

where $\varphi_k(t)$ represents the phase of A_k which is rapidly varying compared with the time scale of T, hence

$$\langle A_k\rangle = |A_k(T)|\langle \cos\varphi_k(t) + i\sin\varphi_k(t)\rangle = 0. \tag{29}$$

And the average of the products of the two amplitudes which are statistically independent, such as $\langle A_{k'}A_{k''}^*\rangle$, has a non-vanishing value only when $k'=k''$. Hence

$$\langle A_{k'}A_{k''}^*\rangle = \langle |A_{k'}|^2\rangle \delta_{k',k''}. \tag{30}$$

The use of this relation is known as *Random Phase Approximation*.

We now construct the time differential equation for the $\langle |A_k|^2\rangle$' s. Note that $|A_k|^2$ is in the direct ratio of the wave energy and $\langle |A_k|^2\rangle$ is related to the energy spectral density, consequently $\langle |A_k|^2\rangle$ may represent the mean power which is carried by certain mode of wave ensemble. If we multiply Eq. (19a) by Ak_0^* and add to it the product of Ak_0 and the complex conjugate of Eq. (19a), then operate the wave ensemble average, we have

$$\frac{\partial}{\partial T}\langle |A_{k_0}|^2\rangle + v(1-\alpha_0)\langle |A_{k_0}|^2\rangle$$

$$= i\sum_{m_{k_0}=m_{k_1}+m_{k_2}} [G_{k_0}(k_1,k_2)\langle A_{k_0}^* A_{k_1}A_{k_2}e^{i\Delta\sigma_{k_0}t}\rangle$$

$$+ F_{k_0}(k_1,k_2)\tilde{\tilde{H}}_{k_2}\langle A_{k_0}^* A_{k_1}e^{i\Delta\sigma_{k_0}t}\rangle] + C.C. \tag{31}$$

In this expression, we have switched the dummy-variables k', k'' to k_1 and k_2 since the wave ensemble average is used. Furthermore, note that there is a quite difference between the two time scale t and T, the time variation of the amplitude A_k due to the nonlinear interaction is slow, compared to that associated with the phase which fluctuates rapidly. It can be regarded that the slowly relaxing variables as a "slow subspace" and the random phase as an element of "fast subspace" which is arthogonal of this" slow subspace". By means of the *Projection Operator Techniques*, we obtain (see Appendix A):

$$\frac{\partial}{\partial T}\langle |A_{k_0}|^2\rangle + v(1-\alpha_0)\langle |A_{k_0}|^2\rangle$$

$$= 2\pi G^2(\langle |A_{k_1}|^2\rangle\langle |A_{k_2}|^2\rangle - \langle |A_{k_2}|^2\rangle\langle |A_{k_0}|^2\rangle - \langle |A_{k_0}|^2\rangle\langle |A_{k_1}|^2\rangle$$

$$+ 2\pi F^2\tilde{\tilde{H}}^2\langle |A_{k_1}|^2\rangle. \tag{32a}$$

Similarly we can obtain equations for $\frac{\partial}{\partial T}\langle|A_{k_1}|^2\rangle$ and $\frac{\partial}{\partial T}\langle|A_{k_2}|^2\rangle$:

$$\frac{\partial}{\partial T}\langle|A_{k_1}|^2\rangle$$
$$+v(1-\alpha_1)\langle|A_{k_1}|^2\rangle=-2\pi G^2(\langle|A_{k_1}|^2\rangle\langle|A_{k_2}|^2\rangle-\langle|A_{k_2}|^2\rangle\langle|A_{k_0}|^2\rangle$$
$$-\langle|A_{k_0}|^2\rangle\langle|A_{k_1}|^2\rangle+2\pi F^2\tilde{H}^2\langle|A_{k_2}|^2\rangle \tag{32b}$$

$$\frac{\partial}{\partial T}\langle|A_{k_2}|^2\rangle+v(1-\alpha_2)\langle|A_{k_2}|^2\rangle$$
$$=-2\pi G^2(\langle|A_{k_1}|^2\rangle\langle|A_{k_2}|^2\rangle-\langle|A_{k_2}|^2\rangle\langle|A_{k_0}|^2\rangle-\langle|A_{k_0}|^2\rangle\langle|A_{k_1}|^2\rangle$$
$$+2\pi F^2\tilde{H}^2\langle|A_{k_0}|^2\rangle \tag{32c}$$

Recall the constraint (27), note that mode k_1 represents the small bottom topographically forced wave ensemble, its amplitude can never grow to an appreciable level due to the result of Eq. (3). i.e.

$\tilde{H}^2\ll 1$; $\langle|A_{k_1}|^2\rangle\ll\langle|A_{k_0}|^2\rangle,\langle|A_{k_2}|^2\rangle$.

The coupled Eqs. (32) then reduce to coupled equations between the low-index mode and the high-index mode given by

$$\frac{\partial}{\partial T}W_0=\gamma_0 W_0-2\pi G^2 W_0 W_2+\eta_0 \tag{33a}$$

$$\frac{\partial}{\partial T}W_2=-\gamma_2 W_2+2\pi G^2 W_0 W_2+\eta_2. \tag{33b}$$

Where $W_k\equiv\langle|A_k|^2\rangle$;

and $\gamma_0\equiv-v(1-\alpha_0))0$,

$\gamma_2\equiv v(1-\alpha_2))0$;

η_0,η_2 are small terms representing the influences related to the topographically forced wave ensemble.

IV. DISCUSSION AND CONCLUSION

In the absence of the small terms, we are astonished again to see that the nonlinear coupling equations (33) entirely equivalent to the *Volterra* systems in ecology (Volterra, 1931). We will analyze this system and derive several properties of its solutions. By the geometric theory of differential equations, we may study the orbit, or trajectory, in the W_0-W_2 phase-plane.

Observe first that Eqs. (33) have two equilibrium solutions:

$$(W_0(T)=0,\ W_2(T)=0) \tag{34a}$$

and

$$\left(W_0(T)=\frac{\gamma_2}{2\pi G^2},\ W_2(T)=\frac{\gamma_0}{2\pi G^2}\right) \tag{34b}$$

The first equilibrium solution is of no interest to us. This system also has the family of solutions:

$$\begin{cases}W_0(T)=W_0(0)e^{\gamma_0 T}\\ W_2(T)=0\end{cases} \tag{35a}$$ and $$\begin{cases}W_0(T)=0\\ W_2(T)=W_2(0)e^{-\gamma_2 T}.\end{cases} \tag{35b}$$

Thus, both the W_0 and W_2 axes are orbits of Eqs. (33). This implies that every solution $W_0(T)$, $W_2(T)$ of Eqs. (33) which starts in the first quadrant $W_0>0$, $W_2>0$, due to the definition $W_k=\langle|A_k|^2\rangle$ representing the mean power which is carried by the mode k, at time $T=T_0$ will remain there for all future time $T\geqslant T_0$.

The orbits of Eqs (33), for W_0, $W_2 \neq 0$ are the family of closed curves, indicating a bound and periodic motion (see Appendix B, also Fig. 1). The qualitative behaviors of W_0 and W_2 as functions of time are plotted in Fig. 2.

Physically, γ_0 and γ_2 can be regarded as the growth and the damping rates of W_0 and W_2 respectively. At an early stage of the linear instability, low-index mode W_0 forced by thermodynamical source will grow exponentially. When W_0 becomes larger than $\gamma_2/2\pi G^2$, the linear damping rate of high-index mode W_2 related to the Ekman pumping vorticity sink will shoot up suddenly at a rate faster than exponential and will stabilize the linear instablity of W_0 through the nonlinear coupling term W_0W_2 in the first equation of Eqs (33). This type of nonlinear resonance relaxation process will continue until the interactions with other modes diffuse the spectrum.

Goel *et al.* (1971), however, have pointed out the importance of the emission small terms, in our case are η_0, η_2 related to the bottom topography, in the dynamics of W_0 and W_2. They have shown that even if these quantities are small, their presence destroys the recurrence nature of W_0 and W_2 and the temporal oscillation damps out. To place the hypothesis on a firmer basis,it may be necessary to include some features about the general circulation to modify the Volterra equations. Nevertheless, this approach still demonstrates unmistakably a plausible hypothesis concerning index cycles in the atmosphere.

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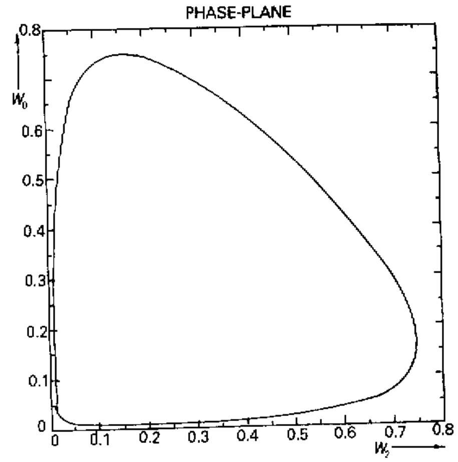

Fig. 1. A typical trajectories in the W_0-W_2 phase plane of the Volterra model for fixed parameter.

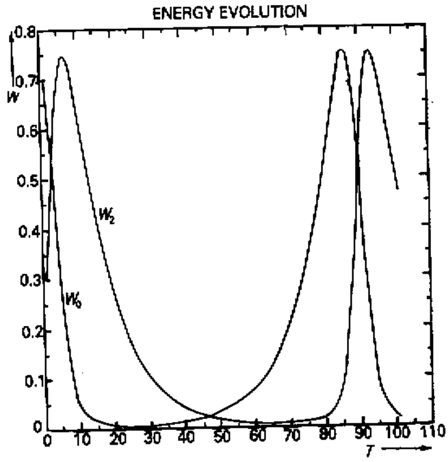

Fig. 2. Time variation of the energies of two modes W_0, W_2 corresponding to a trajectory of Fig. 1.

Appendix A

DERIVATION OF NONLINEAR COUPLING EQUATIONS

It is considered a situation very early in time when only the term of vanished internal dissipation is taken place (i.e. the dissipation-vanish limit). Thus A_k in the lowest order can be obtained by integrating the coupled Eqs (23) from $t=-\infty$ to t by means of the *Projection Operator Techniques* (see Berne, 1977). Hence,

$$A_{k_0}\approx\sum_{m_{k_0}=m_{k'}+m_{k''}}\frac{1}{\Delta\sigma_{rk_0}-i0^+}[G_{k_0}(k',k'')A^{(0)}_{k'}A^{(0)}_{k''}+$$

$$+F_{k_0}(k',k'')A^{(0)}_{k'}\tilde{H}_{k''}]e^{+i(\Delta\sigma_{rk_0}-i0^+)t} \quad (A.1a)$$

$$A_{k_1}\approx-\sum_{m_{k_1}=m_k-m_{k''}}\frac{1}{\Delta\sigma_{rk_1}+i0^+}[G_{k_1}(k''^*,k)A^{(0)*}_{k''}A^{(0)}_{k}+$$

$$+F_{k_1}(k''^*,k)A^{(0)*}_{k''}\tilde{H}_{k}]e^{-i(\Delta\sigma_{rk_1}+i0^+)t} \quad (A.1b)$$

$$A_{k_2}\approx-\sum_{m_{k_2}=m_k-m_{k'}}\frac{1}{\Delta\sigma_{rk_2}+i0^+}[G_{k_2}(k,k'^*)A^{(0)}_{k}A^{(0)*}_{k'}+$$

$$+F_{k_2}(k,k'^*)A^{(0)}_{k}\tilde{H}^*_{k'}]e^{-i(\Delta\sigma_{rk_2}+i0^+)t}, \quad (A.1c)$$

where the items $\pm i0^+$ make the integral convergent.

This causality is very important to show the properties of dissipation even though the dissipation-vanish limit is set (see Prigogine, 1975).

Let us first consider the time differential equation of $\langle|A_{k_0}|^2\rangle$. If we now make use of the expression

$$\langle A_kA_{k'}A_{k''}\rangle=\langle A_kA^{(0)}_{k'}A^{(0)}_{k''}\rangle+\langle A^{(0)}_{k}A_{k'}A^{(0)}_{k''}\rangle+\langle A^{(0)}_{k}A^{(0)}_{k'}A_{k''}\rangle \quad (A.2)$$

for the average of the products $A^*_{k_0}A_{k_1}A_{k_2}$ and $A_{k_0}A^*_{k_1}A^*_{k_2}$ in the Eq. (31), there will appear ten terms inside the bracket. To evaluate the average of each term, we take one conjugate pair out of the five pairs and designate it by the quantity a_1:

$$a_1\equiv G_{k_0}(k_1,k_2)\langle A^*_{k_0}A^{(0)}_{k_1}A^{(0)}_{k_2}e^{+i(\Delta\sigma_{k_0}-i0^+)t}\rangle+C.C. \quad (A.3)$$

On substituting Eq. (A. 1a) into Eq. (31), the exponential term in the wave ensemble average will be eliminated, we then have

$$a_1=\sum_{m_{k_0}=m_{k'}+m_{k''}}\Big[\frac{|G_{k_0}(k',k'')|^2}{\Delta\sigma_{rk_0}+i0^+}\langle A^{(0)*}_{k'}A^{(0)*}_{k''}A^{(0)}_{k_1}A^{(0)}_{k_2}\rangle+$$

$$+\frac{G_{k_0}(k',k'')F^*_{k_0}(k',k'')\tilde{H}^*_{k''}}{\Delta\sigma_{rk_0}+i0^+}\langle A^{(0)*}_{k'}A^{(0)}_{k_1}A^{(0)}_{k_2}\rangle\Big]+C.C. \quad (A.4)$$

By the *Random Phase Approximation*, the term $\langle A^{(0)*}_{k'}A^{(0)}_{k_1}A^{(0)}_{k_2}\rangle$ becomes zero and the term $\langle A^{(0)*}_{k'}A^{(0)*}_{k''}A^{(0)}_{k_1}A^{(0)}_{k_2}\rangle$ becomes

$$\langle A^{(0)*}_{k'}A^{(0)*}_{k''}A^{(0)}_{k_1}A^{(0)}_{k_2}\rangle=\langle|A^{(0)}_{k''}|^2\rangle\delta_{k',k''*}\langle|A^{(0)}_{k_1}|^2\rangle\delta_{k_1,k_2^*}$$

$$+\langle|A^{(0)}_{k_1}|^2\rangle\delta_{k_1,k'}\langle|A^{(0)}_{k_2}|^2\rangle\delta_{k_2,k''}$$

$$+\langle|A^{(0)}_{k_2}|^2\rangle\delta_{k_2,k'}\langle|A^{(0)}_{k_1}|^2\rangle\delta_{k_1,k''}. \quad (A.5)$$

The first and the third terms on the right hand side of this expression do not contribute. We can derive a similar expression for the c.c. term. Substituting these expressions into a_1, we have

$$a_1=\sum_{m_{k_0}=m_{k_1}+m_{k_2}}\Big[\frac{|G_{k_0}(k_1,k_2)|^2}{\Delta\sigma_{rk_0}+i0^+}-\frac{|G_{k_0}(k_1,k_2)|^2}{\Delta\sigma_{rk_0}-i0^+}\Big]\langle|A^{(0)}_{k_1}|^2\rangle\langle|A^{(0)}_{k_2}|\rangle. \quad (A.6)$$

Using the relations

$$|G_{k_0}(k_1,k_2)|^2=|G_{k_1}(k_2^{\mp},k_0)|^2=|G_{k}(k_0,k_1^{\mp})|^2\equiv G^2 \quad (A.7a)$$

$$\Delta\sigma_{rk_0}=\Delta\sigma_{rk_1}=\Delta\sigma_{rk_2}\equiv\sigma_{rk_0}-(\sigma_{rk_1}+\sigma_{rk_2})\equiv\Delta\sigma_r \tag{A.7b}$$

the second and the third conjugate pairs inside the bracket in Eq. (31) can be also evaluated to give

$$a_2=-\sum_{n_{k_1}=n_{k_0}-n_{k_2}}\left[\frac{G^2}{\Delta\sigma_r+i0^+}-\frac{G^2}{\Delta\sigma_r-i0^+}\right]\langle|A^{(0)}_{k_2}|^2\rangle\ \langle|A^{(0)}_{k_0}|^2\rangle \tag{A.8}$$

$$a_3=-\sum_{n_{k_2}=n_{k_0}-n_{k_1}}\left[\frac{G^2}{\Delta\sigma_r+i0^+}-\frac{G^2}{\Delta\sigma_r-i0^+}\right]\langle|A^{(0)}_{k_0}|^2\rangle\ \langle|A^{(0)}_{k_1}|^2\rangle \tag{A.9}$$

In the fourth conjugate pair

$$\begin{aligned}a_4&\equiv F_{k_0}(k',k'')\tilde{H}_{k_1}\langle A^{\mp}_{k_0}A^{(0)}_{k_1}e^{i|\Delta\sigma_{rk_0}-i0^+|t}\rangle+C.C.\\&=\sum_{n_{k_0}=n_{k'}+n_{k''}}\left[\frac{F\tilde{H}G}{\Delta\sigma_r+i0^+}\langle A^{(0)\mp}_{k'}A^{(0)\mp}_{k''}A^{(0)}_{k_1}\rangle\right.\\&\left.+\frac{F^2\tilde{H}^2}{\Delta\sigma_r+i0^+}\langle A^{(0)\mp}_{k'}A^{(0)}_{k_1}\rangle\right]+C.C.\\&=\sum_{n_{k_0}=n_{k'}+n_{k''}}\left[\frac{F^2\tilde{H}^2}{\Delta\sigma_r+i0^+}-\frac{F^2\tilde{H}^2}{\Delta\sigma_r-i0^+}\right]\langle|A_{k_1}|^2\rangle\end{aligned} \tag{A.10}$$

The last conjugate pair, however,

$$\begin{aligned}a_5\equiv&-\sum_{n_{k_1}=n_k-n_{k'}}\left[\frac{F\tilde{H}G}{\Delta\sigma_r+i0^+}\langle A^{(0)\mp}_{k_0}A^{(0)\mp}_{k''}A^{(0)}_{k}\rangle\right.\\&\left.+\frac{F^2\tilde{H}^2}{\Delta\sigma_r+i0^+}\langle A^{(0)\mp}_{k_0}A^{(0)\mp}_{k''}\rangle\right]+C.C.=0.\end{aligned} \tag{A.11}$$

If we substitute these results into Eq. (31) and remove the superscript (0) which is no longer needed to consider the time development in the scale. Furthermore, we assume a continuous zonal wavenumber spectrum, the summation sign becomes integral. By means of the formulation of *Principal-Valued Integration*

$$\frac{1}{\Delta\sigma_r\mp i0^+}=P.V.\frac{1}{\Delta\sigma_r}\pm i\pi\delta(\Delta\sigma_r), \tag{A.12}$$

we finally obtain

$$\begin{aligned}&\frac{\partial}{\partial T}\langle|A_{k_0}|^2\rangle+v(1-\alpha_0)\langle|A_{k_0}|^2\rangle\\&=\iint dm_k d\sigma_r\delta[m_{k_0}-(m_{k_1}+m_{k_2})]\delta[\sigma_{rk_0}-(\sigma_{rk_1}+\sigma_{rk_2})]\\&\times[2\pi G^2(\langle|A_{k_1}|^2\rangle\langle|A_{k_2}|^2\rangle-\langle|A_{k_2}|^2\rangle\langle|A_{k_0}|^2\rangle\\&-\langle|A_{k_0}|^2\rangle\langle|A_{k_1}|^2\rangle)+2\pi F^2\tilde{H}^2\langle|A_{k_1}|^2\rangle].\end{aligned} \tag{A.13}$$

Because of the δ-function in the right side of Eq. (A.13), the non-trivial solution exists only when

$$m_{k_0}=m_{k_1}+m_{k_2} \tag{A.14a}$$

$$\sigma_{rk_0}=\sigma_{rk_1}+\sigma_{rk_2}. \tag{A.14b}$$

Eqs (A.14) are just the resonance constraints. By means of the resonance constraints and the characteristics of δ-function, we can obtain the integration in the right side of Eq. (A.13):

$$\frac{\partial}{\partial T}\langle|A_{k_0}|^2\rangle+v(1-\alpha_0)\langle|A_{k_0}|^2\rangle$$
$$=2\pi G^2(\langle|A_{k_1}|^2\rangle\langle|A_{k_2}|^2\rangle-\langle|A_{k_2}|^2\rangle\langle|A_{k_0}|^2\rangle$$
$$-\langle|A_{k_0}|^2\rangle\langle|A_{k_1}|^2\rangle)+2\pi F^2\tilde{H}^2\langle|A_{k_1}|^2\rangle. \quad (3.6a)$$

APPENDIX B

THE SOLUTION OF EQS. (33)

This solution can be found in most standard texbooks. It is included here for completeness.

The orbits of Eqs. (33), for W_0, $W_2\neq 0$ are the solution curves of the first-order equation

$$\frac{dW_2}{dW_0}=\frac{-\gamma_2 W_2+2\pi G^2 W_0 W_2}{\gamma_0 W_0-2\pi G^2 W_0 W_2}=\frac{W_2(-\gamma_2+2\pi G^2 W_0)}{W_0(\gamma_0-2\pi G^2 W_2)}. \quad (B.1)$$

This equation is separable, since we can write it in the form

$$\frac{\gamma_0-2\pi G^2 W_2}{W_2}\frac{dW_2}{dW_0}=\frac{-\gamma_2+2\pi G^2 W_0}{W_0}. \quad (B.2)$$

Consequently, $\gamma_0\ln W_2-2\pi G^2 W_2+\gamma_2\ln W_0-2\pi G^2 W_0=C_1$,
for some constant C_1. Taking exponentials of both sides of this equation gives

$$\frac{W_2^{\gamma_0}}{e^{2\pi G^2 W_2}}\frac{W_0^{\gamma_2}}{e^{2\pi G^2 W_0}}=C \quad (B.3)$$

for some constant C. Thus, the orbits of Eqs. (33) are the family of curves defined by Eq. (B.3), and these curves are closed as we now show in Fig. 1.

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