Nonlinear Stability of Plane Rotating Shear Flow under Three–Dimensional Nondivergence Disturbances①

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ABSTRACT

Nonlinear stability criterion for plane rotating shear flow under three–dimensional nondivergence disturbances was obtained by using both variational principle and convexity estimate introduced by Arnold (1965) and Holm et al. (1985). The results obtained in this paper show that the effect of Coriolis force plays an important role in the nonlinear stability criterion, and the nonlinear stability property of the basic flow depends on both the distribution of basic states and the way the external disturbance acts on the states. The upper bound of the gradient of the mass density displacement from the equilibrium $k^2 = |\nabla[\rho(\vec{x},t) - \rho_e(\vec{x})]|^2 / [\rho(\vec{x},t) - \rho_e(\vec{x})]^2$ is determined by the basic states and one example was given to show the exact upper value of k. The remarks on Blumen's paper were also given at Section 4 of this paper.

I. INTRODUCTION

Blumen (1970) used variational method to study the stability of plane shear flow under three–dimensional nondivergence disturbances. In his model the basic velocity is only limited in the x–y plane without depending on the vertical coordinate. This model corresponds to unreal fluid system, it cannot be studied by using laboratory experiments, but it is valuable in theoretical studying.

In this paper, the Blumen's model was extended to include the Coriolis force which plays an important role in the geophysical fluid mechanics. The stability criteria show that the stability property depends on both the distribution of basic states and the external disturbance of basic states. This case exists in almost all three–dimensional flow (Abarbanel et al., 1986) and also exists in the nonlinear stability of triad–wave interactions (Ren shuzhan, 1990). Therefore, the form of the external disturbances is important in studying the stability of three dimensional flows.

In a linearized fluid system, disturbances are composed of normal mode and continue spectra (Case, 1960; P. S. Lu et al., 1986). In recent years, more and more attentions concentrated on the continue spectra which, being proved, play an important role in the real atmosphere (Farrell, 1982; Zhang Minghua, 1986), although the general theory about the completeness of the spectra of the real atmosphere does not exist at present time. Generally speaking, the nonlinear stability criterion becomes a linear stability criterion when disturbances tend to be zero. In the example in Section 4 of this paper, we take basic density distribution $\rho_e = -rz$, then the exact upper bound of k^2 can be obtained. In the example of this section we take $\vec{V} = \vec{V}(y)$, and the disturbance of the mass density has the normal model

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form (This is just the case in Blumen's paper), then there always has the possibility of basic states being unstable, so we can conclude that stability parts are possible to compose of continue spectra. However, in the variational principle, the disturbances are finite small with any kind of form because we start from the original equations, so the general upper bound of k^2 can be derived by using the variational principle.

Strictly speaking, the criteria given by Blumen are only the linear stability ones in Liaponov meaning, not for nonlinear stability case. To study the nonlinear stability, the convexity estimate, which can be found in Abarbanel's paper, is needed.

We organized this paper in following order. In Section two the general equations were given and some basic relations between the basic states were derived, then variational principle and the formal stability criterion were introduced in Section three. Section four gave special form of the basic state of density, and the upper bound of K^2 was obtained. In Section five we got the nonlinear stability criterion for the basic states given in the example of Section four.

II. GOVERNING EQUATIONS

The fluid system is shown in Fig.1 and the governing equations are the Boussinesq equations

$$\partial_t \vec{v} + (\vec{v}\cdot\nabla)\vec{v} + f\vec{k}\times\vec{v} = -\nabla P - \rho g\vec{k}/\rho_s \tag{2.1}$$

$$\nabla\cdot\vec{v} = 0 \tag{2.2}$$

$$\partial_t \rho + \vec{v}\cdot\nabla\rho = 0 \tag{2.3}$$

Where $P=(\rho-\bar{\rho})/\rho_s$. $\bar{\rho}$ is the average value of the fluid mass density. ρ_s is the standard value of mass density. In the following seetion we suppose that $\rho_s=1$. f is the Coriolis factor and is a function of y. $\vec{k}$ is the unit vector in z direction.

The following process is just the same as that in Adarbanel et al.'s paper. Transform (2.1) into the following form

$$\begin{cases} \partial_t \vec{v} = \vec{v}\times\vec{\Omega} - \nabla(\frac{1}{2}|\vec{v}|^2 + P + \rho g z) + g\nabla\rho \\ \partial\rho/\partial t + \vec{v}\cdot\nabla\rho = 0 \\ \nabla\cdot\vec{v} = 0 \end{cases} \tag{2.1}$$

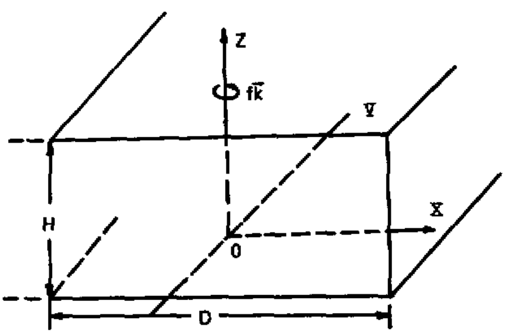

Fig.1. The model of three dimensional rotating fluid.

where $\vec{\Omega}=f\vec{k}+\nabla\times\vec{v}$.

It is easy to get the conservation equation of potential vorticity q from (2.1)′

$$\partial_t q+\vec{v}\cdot\nabla q=0 \tag{2.4}$$

where $q=\nabla\rho\cdot[\nabla\times\vec{v}+f\vec{k}]$.

At equilibrium (2.1)′ becomes

$$\vec{v}_e\times\vec{\Omega}_e-\nabla(\frac{1}{2}|\vec{v}_e|^2+P_e+\rho_e gz)+g\nabla\rho_e=0 \tag{2.5}$$

where subscript "e" denotes the variables at equilibrium and

$$\vec{v}_e\cdot\nabla\rho_e=0 \tag{2.6}$$

$$\vec{v}_e\cdot\nabla q_e=0. \tag{2.7}$$

From (2.5) we get

$$\vec{v}_e\cdot\nabla[\frac{1}{2}|\vec{v}_e|^2+P+\rho gz]=0. \tag{2.8}$$

Eq.s (2.6)–(2.8) implay taht the term $\frac{1}{2}|\vec{v}_e|^2+P_e+\rho_e gz$ is functionaly related to the variable q, then we can suppose that

$$B(q,\rho)=\frac{1}{2}|\vec{v}|^2+P+\rho gz \tag{2.9}$$

where B is the Bernoulli function.

Multiplying (2.5) by $\nabla\rho$ gives

$$\vec{v}_e=q_e^{-1}B_q(q_e,\rho_e)\nabla\rho_e\times q_e \tag{2.10}$$

Eq.(2.10) is important in following sections.

III. VARIATIONAL PRINCIPLE AND FORMAL STABILITY

The main idea about the variational principle is to construct the general Hamilton function H_c by using conserved quantities. From (2.1) and (2.2) we see taht the total energy E, the potential vorticity q and the mass density ρ are all conserved quantities. Note that E, ρ and q are different kinds of conserved quantities, in which q and ρ are conserved point by point, but E is not. After H_c was obtained we make H_c have small fluctuation about the equilibrium. If the $\delta^2 H_c$ has the definite sign (both positive and negative), we say that the equilibrium is formally stable, otherwise the equilibrium is possibly unstable (although we can not prove it).

The general Hamilton H_c was constructed by q, ρ and E $(E=\frac{1}{2}\int d^3x[\frac{1}{2}|v_e|^2+\rho gz])$

$$H_c=\frac{1}{2}\int d^3x[\frac{1}{2}|\vec{v}_e|^2+\rho gz]+\int d^3x\psi(\rho,q)+\oint_{\partial D}d^2x\lambda q$$

where $\psi(q,\rho)$ is an arbitrary function of q and ρ, and λ is a constant, ∂D is the boundary of the integral area.

a) The First Variation of H_c.

Taking $\vec{v}$ and ρ has a small variation about the equilibrium (basic states), then the variation of H_c in first order is

$$\delta H_c(\delta\vec{v},\delta\rho)=\int d^3x[\psi_\rho+gz-\vec{\Omega}\cdot\nabla\psi_q]\delta\rho+[\vec{v}-\psi_{qq}\nabla\rho\times\nabla q]\delta\vec{v}$$

$$+\oint_{\partial D}d^2x(\lambda+\psi_q)(\Omega\delta\rho-\nabla\rho\times\nabla\vec{v})\cdot\vec{n}$$

where $\vec{n}$ is the outward normal vector on the boundary. Obviously

$$\delta H_c(\delta q,\delta\rho)\big|_e=0$$

at equilibrium, it gives

$$gz=\vec{\Omega}_e\cdot\nabla\psi_q^{(e)}+\psi_\rho^{(e)} \tag{3.1}$$

$$\vec{v}_e=\psi_{qq}^{(e)}\nabla\rho_e\times\nabla q_e \tag{3.2}$$

$$\lambda=-\psi_q \tag{3.3}$$

Comparing (3.2) with (2.11) we have

$$\psi_{qq}^{(e)}=q_e^{-1}B_{qe}(q_e,\rho_e) \tag{3.4}$$

or $$\psi^{(e)}=q^e\left(\int^{q_e}B(s,\rho_e)/s^2ds+F(\rho_e)\right) \tag{3.5}$$

where $F(\rho_e)$ is an arbitrary function of ρ_e and will be taken as zero in the following section. Note that Eq.(3.5) is an important equation!

(b) The Second Variation of H_c.

Based on the δH_c we can get the second variation of H_c.

$$\begin{aligned}\delta^2H_c&=D^2H_c(\rho_e,\vec{v}_e)(\delta\rho,\delta\vec{v})^2\\&=\int d^3x\Big\{\left|\delta\vec{v}+\nabla\psi_q^{(e)}\times\nabla\delta\rho\right|^2+\left|\nabla\psi_q^{(e)}\cdot\nabla\delta\rho\right|^2-\left|\nabla\psi_q^{(e)}\right|\left|\nabla\delta\rho\right|^2\\&\quad+\psi_{\rho\rho}^{(e)}(\delta q)^2+\psi_{\rho\rho}^{(e)}(\delta q)^2+2\psi_{q\rho}\delta\rho\delta q\Big\}\end{aligned}$$

We can also write δ^2H_c in quadratic form by introducing $|\nabla\delta\rho|^2=k^2|\delta\rho|^2$

$$\delta^2H_c=\int d^3x\Big\{\left|\delta\vec{v}_e+\nabla\psi_q^{(e)}\times\nabla\delta\rho\right|^2+\left|\nabla\psi_q^{(e)}\cdot\nabla\delta\rho\right|^2+2\psi_{q\rho}\delta\rho\delta q$$
$$+\begin{bmatrix}\delta\rho\\\delta q\end{bmatrix}^T\begin{bmatrix}\psi_{\rho\rho}^{(e)}-|k|^2|\nabla\psi_q^{(e)}|^2 & \psi_{q\rho}\\ \psi_{q\rho} & \psi_{qq}\end{bmatrix}\begin{bmatrix}\delta\rho\\\delta q\end{bmatrix}\Big\} \tag{3.6}$$

To make $\delta^2H>0$ needs

$$\psi_{qq}^{(e)}>0 \tag{3.7}$$

$$\psi_{qq}^{(e)}(\psi_{\rho\rho}^{(e)}-|k|^2|\nabla\psi_q^{(e)}|^2)-\psi_{q\rho}^{(e)2}>0 \tag{3.8}$$

Eq.(3.8) gives the upper bound of K^2

$$K^2 < K_+^2 = \left[\frac{\psi_{\rho\rho} - \psi_{q\rho}^2 / \psi_{qq}}{|\nabla \psi_q|^2}\right]_e > 0 \tag{3.7$'$}$$

From (3.2) and (3.7) we have

$$\psi_{qq}^{(e)} = \vec{v}_e \cdot (\nabla\rho \times \nabla q) / |\nabla\rho \times \nabla q|^2$$

Eq.(3.8) also means that

$$\psi_{\rho\rho}^{(e)} > 0 \tag{3.9}$$

Eqs.(3.7)–(3.9) are the general stability criteria in three dimensional Boussinesq flows. In the following section we turn to study the stability for a special equilibrium.

IV. FORMAL STABILITY OF SPECIAL EQUILIBRIUM

In this section a special equilibrium was given. Suppose that the special equilibrium linearly depends on the vertical coordinate and the velocity was limited in the x–y plane, i.e.,

$$\vec{v}_e = \vec{v}_e(x,y), \quad \rho_e = \rho_e(z) = -rz, \quad f = f_0 + \beta y \quad (\beta \text{ is a constant})$$

The pressure formula in the equilibrium,

$$P(z) = -g\int_0^z \rho(z)dz = \frac{1}{2}grz^2$$

From (2.4) we know that $q = q(x,y)$, therefore

$$\psi = q_e^{-1}\int^{q_e} B(s,\rho_e)/s^2 ds = q_e^{-1}\int^{q_e} \tilde{B}(s)/s^2 ds + \frac{rg}{2}z^2 \tag{4.1}$$

then $\psi_{q\rho} = 0$, and $\psi_{\rho\rho} = g/r > 0$. i.e., $r > 0$.
From (3.7)$'$ we have

$$\psi_{qq} = \vec{v} \cdot [\nabla\rho \times \nabla q] / |\nabla\rho \times \nabla q|^2 > 0 \tag{4.2}$$

From (3.8)$'$ we have

$$K^2 = \left(\frac{\nabla\delta\rho}{\delta\rho}\right)^2 < K_+^2 = \psi_{\rho\rho}^{(e)} / |\nabla\psi_q^{(e)}|^2$$

Since $\nabla\rho \perp \nabla q$, so it is easy to see that

$$\psi_{qq} = \frac{\vec{v} \cdot \vec{l}_0}{|\nabla q| \cdot |\nabla\rho|} > 0$$

then

$$K_+^2 = rg / [\vec{v} \cdot \vec{l}_0]^2 \tag{4.3}$$

where $\vec{l}_0$ is an unit vector:

$$\vec{l}_0 = \nabla\rho \times \nabla q / |\nabla\rho \times \nabla q|$$

If we take $\vec{v} \,/\!/\, \vec{l}_0$ then we have

$$K^2 = \max\left|\frac{\nabla\delta\rho}{\delta\rho}\right|^2 < K_+^2 = \frac{rg}{|\vec{v}_e|^2_{\max}}$$

For example, if we take $\vec{v} = \alpha y^2 / 2\vec{i}$, i.e., $\vec{I}_0 \,//\, \vec{v}$, immediately we get that

$$\psi_{qq} = \frac{|\vec{v}|}{(\alpha - \beta) r^2} > 0 \qquad \text{i.e., } \alpha > \beta$$

$$\psi_{\rho\rho} = g / r \quad > 0 \qquad \text{i.e., } r > 0$$

$$K^2 < K_+^2 = \frac{2rg}{\alpha H^2}$$

Remarks on Blumen's work:

(1) In Blumen's work, if $\vec{v} = \frac{1}{2}\alpha y^2 \vec{i}$, then ψ_{qq} is always positive and the equilibrium is always stable, but in our work, the equilibrium is possibly unstable if $\alpha < \beta$.

(2) Following Blumen, if we take the normal mode form as

$$\delta\rho = |\delta\rho| \exp[i(mx + ly - \sigma t)] \sin\frac{n\pi}{H} z$$

then from (4.2) it must be

$$|\vec{v}|^2 < (rg / (\tilde{m}^2 + n^2\pi^2 / H^2))^{1/2} \qquad (n = 1, 2 \cdots)$$

where $\tilde{m}^2 = m^2 + l^2$. Therefore, there always exist some values of n for fixed $\tilde{m}$ to violate the condition mentioned above, that means that almost all the normal modes are possibly unstable and the stable part of the disturbance is composed of continue spectra (or nonmodal part).

V. NONLINEAR STABILITY

As we said in the introduction that the formal stability is different from the nonlinear stability (see Holm et al., 1985). To study the nonlinear stability the convexity estimate was needed.

Suppose that there are some finite small variations in the equilibrium, and that the basic state of mass density $\rho_e = -rz$, $\vec{v} = \vec{v}(x,y)$ as in Section 4, then the variation of H_c is

$$\hat{H}_c(\Delta\vec{v}, \Delta\rho) = H_c(\vec{v}_e + \Delta\vec{v},\ \rho_e + \Delta\rho) - H_c(\vec{v}_e, \rho_e) - DH_c(\vec{v}_e, \rho_e)(\Delta\vec{v}, \Delta\rho) \tag{5.1}$$

where both $\Delta\vec{v}$ and $\Delta\rho$ are finite small. From (5.1) the following equation can be obtained

$$\hat{H}_c(\Delta\vec{v}, \Delta\rho) = \frac{1}{2}\int d^3x \Big\{ \Big[|\Delta\vec{v}| + \nabla\psi_q \times \nabla\Delta\rho \Big]^2 + |\nabla\psi_q \cdot \nabla\Delta\rho|^2 - |\nabla\psi_q|^2 |\nabla\Delta\rho|^2 + 2\hat{\psi}(\Delta\vec{v},\ \Delta\rho) \Big\} \tag{5.2}$$

where

$$\hat{\psi}(\Delta\vec{v},\ \Delta\rho) = \psi(q_e + \Delta q,\ \rho_e + \Delta\rho) - \psi(q_e,\ \rho_e) - DH(\rho_e,\ q_e)(\Delta\rho,\ \Delta q)$$

Define that

$$|\nabla\Delta\rho|^2 = |K|^2 (\Delta\rho)^2 < K_+^2 (\Delta\rho)^2$$

$$\varphi(\rho, q) = \psi(\rho, q) - \frac{1}{2} |\nabla\psi_q(\rho_e,\ q_e)|^2 |k|^2 \rho^2$$

$$\varphi_{\rho\rho}(\rho,q)=\psi_{\rho\rho}-|\nabla\psi_q|^2|k|^2\geqslant\psi_{\rho\rho}-|\nabla\psi_q|^2|k_+|^2$$
$$\varphi_{qq}(\rho,q)=\psi_{qq}$$

and as in Section 4, that $\varphi_{q\rho}=0$.
To make $\delta^2 H_c>0$ needs

$$0\leqslant\alpha\leqslant\varphi_{qq}(\rho_e,q_e)<\infty \tag{5.3}$$

$$0\leqslant\begin{bmatrix}\Delta\rho\\ \Delta q\end{bmatrix}^T\begin{bmatrix}r & 0\\ 0 & \alpha\end{bmatrix}\begin{bmatrix}\Delta\rho\\ \Delta q\end{bmatrix}\leqslant\begin{bmatrix}\Delta\rho\\ \Delta q\end{bmatrix}^T\begin{bmatrix}\varphi_{\rho\rho} & 0\\ 0 & \varphi_{qq}\end{bmatrix}\begin{bmatrix}\Delta\rho\\ \Delta q\end{bmatrix} \tag{5.4}$$

where α and r are constants.
From the results above we can get the upper bound of K^2

$$K^2<K_+^2=(\varphi_{qq}\varphi_{\rho\rho}-r\alpha)/|\nabla\varphi_q|^2 \tag{5.5}$$

Eqs.(5.1) and (5.2) give the lower bound of $\delta^2 H_c$

$$\hat{H}_c(\Delta\vec{v},\Delta\rho)\geqslant\int d^3x\left\{\frac{1}{2}|\Delta\vec{v}+\nabla\varphi_q(\rho_e q_e)\times\nabla\Delta\rho|^2\right.$$

$$\left.+\frac{1}{2}\begin{bmatrix}\Delta\rho\\ \Delta q\end{bmatrix}^T\begin{bmatrix}r & 0\\ 0 & \alpha\end{bmatrix}\begin{bmatrix}\Delta\rho\\ \Delta q\end{bmatrix}\right\}>0 \tag{5.6}$$

where α and r are constants, and the upper bound of $\hat{H}_c$ is given by

$$\hat{H}_c(\Delta\vec{v},\ \Delta\rho)\leqslant\frac{1}{2}\int d^3\times\left\{\left[\Delta\vec{v}+\nabla\varphi_q(\rho_e,\ q_e)\times\nabla\Delta\rho\right]^2\right.$$

$$\left.+\begin{bmatrix}\Delta\rho\\ \Delta q\end{bmatrix}^T\begin{bmatrix}\bar{r} & 0\\ 0 & \bar{\alpha}\end{bmatrix}\begin{bmatrix}\delta\rho\\ \delta q\end{bmatrix}\right\} \tag{5.7}$$

where $\bar{\alpha}$ and $\bar{r}$ are constans.

The right hand of (5.6) and (5.7) define two kinds of modes with which (5.6) and (5.7) can be rewritten in the following form

$$\|\Delta\vec{v},\Delta\rho,\Delta q\|^2\leqslant\hat{H}_c(\Delta\vec{v},\Delta\overline{\rho})=\hat{H}_c(\Delta\vec{v}_0,\Delta\rho_0)\leqslant R\|(\Delta\vec{v}_0,\Delta\rho_0,\Delta q_0)\|^2$$

where $\Delta\vec{v}_0$ and $\Delta\rho_0$ are initial disturbances, and R is a constant.

Making the equilibrium be the form as in Section 4, we can easily get the nonlinear stability criterion.

If the equilibrium satisfies (5.3)–(5.4) and the K, which was defined in section 3 and bounded by (5.5), then the equilibrium is nonlinearly stable.

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