The Stability of Large–Scale Horizontal Air Motion in the Non–linear Basic Zephyr Flow under the Effect of Rossby Parameter

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ABSTRACT

The stability of large–scale horizontal motion in the atmosphere is discussed in this paper by using qualitative analysis theory of non–linear ordinary differential equations. Both the non–linear distribution of basic Zephyr flow and the variation of geostrophic vorticity along the latitude ($f=f_0+\beta\delta y$) are all included in this paper's mathematical model so as to analogue the background field of large–scale horizontal air motion more really in the rotating reference frame of the earth. Some significant results are drawn out from this paper and the conclusions of Li(1986)'s and Wan et al.(1990)'s are extended widely.

I. INTRODUCTION

With the rapid developing of non–linear differential equations' theory, scientists began to study the phenomena of bifurcation, catastrophe and chaos in the atmosphere. As to large–scale horizontal motion in the atmosphere, the previous result was demonstrated by using parcel method to discuss the inertial stability of air motion in the linear basic Zephyr flow while the geostrophic parameter f being taken as a constant. In 1986, Li Chongyin studyed the phenomena of bifurcation and catastrophe of large–scale horizontal air motion in cubic–form non–linear basic Zephyr flow by using qualitative analysis theory of non–linear differential equations while the geostrophic parameter being taken as a constant. Wan et al. (1990), considering the effect of Rossby parameter β, also discussed these phenomena in linear basic Zephyr flow. In this paper, basing on the real background field of large–scale horizontal air motion, the author, using the theory of bifurcation and catastrophe, puts the interaction of Rossby parameter β and quadric form non–linear basic Zephyr flow to the comprehensive study of the characteristics of large–scale horizontal air motion; meanwhile the distribution of cubic form non–linear basic Zephyr flow has also been analysed simply.

II. MATHEMATICAL MODEL

Casting aside the influence of viscous dissipation, the large–scale horizontal motion can be expressed as follows:

$$\frac{du}{dt}=fv-\frac{1}{\rho}\frac{\partial p}{\partial x} \quad , \tag{1}$$

$$\frac{dv}{dt}=-fu-\frac{1}{\rho}\frac{\partial p}{\partial y} \quad , \tag{2}$$

Supposing that the basic Zephyr flow accords with the geostrophic balance and that the distribution of environmental pressure field does not change with the horizontal motion of air package, we have

$$f\bar{v}=\frac{1}{\rho}\frac{\partial \bar{p}}{\partial x}=\frac{1}{\rho}\frac{\partial p}{\partial x}=0 \quad , \quad -f\bar{u}=\frac{1}{\rho}\frac{\partial \bar{p}}{\partial y}=\frac{1}{\rho}\frac{\partial p}{\partial y} \quad , \tag{3}$$

Substituting (3) into Eqs. (1)–(2) we obtain

$$\frac{du}{dt}=fv=f\frac{d(y-y_0)}{dt} \quad , \tag{4}$$

$$\frac{dv}{dt}=f(\bar{u}-u) \quad . \tag{5}$$

As far as the large–scale horizontal motion in the atmosphere is concerned, the effect of geostrophic parameter f' s changing along the latitude is very important, so we assume

$$f=f_0+\frac{\partial f}{\partial y}(y-y_0)=f_0+\beta(y-y_0) \quad , \tag{6}$$

where β is the Rossby parameter, $\beta=\left.\frac{\partial f}{\partial y}\right|_{y=y_0}=\frac{2\Omega\cos\varphi}{a}$, and it is considered as a constant.

Integrating Eq.(4), i.e., $du=fd(y-y_0)$ from y_0 to y , and noticing Eq.(6) we have

$$\int_{u_0}^{u} du=\int_{y_0}^{y}[f_0+\beta(y-y_0)]dy \quad .$$

Let $\eta=y-y_0$, we obtain the integral consequence

$$u=u_0+f_0\eta+\frac{\beta}{2}\eta^2 \quad .$$

Supposing the basic Zephyr flow distributes as parabolic type, i.e., $\bar{u}=\bar{u}(y_0)+\frac{\partial \bar{u}}{\partial y}\eta+\frac{1}{2}\frac{\partial^2 \bar{u}}{\partial y^2}\eta^2$, which may simulate the upper Zephyr flow more truthfully, then substituting u and $\bar{u}$ into Eq.(5) and arranging it in η order, one can eventually get the controlling equations of large–scale horizontal air motion as follows:

$$\frac{d\eta}{dt}=v \quad , \tag{7}$$

$$\frac{dv}{dt}=f_0\alpha+(\beta\alpha-f_0\xi_0)\eta-\left[\frac{1}{2}f_0\left(\beta-\frac{\partial^2\bar{u}}{\partial y^2}\right)+\beta\xi_0\right]\eta^2-\frac{\beta}{2}\left(\beta-\frac{\partial^2\bar{u}}{\partial y^2}\right)\eta^3 \quad , \tag{8}$$

where α represents the geostrophic deviation at the primary position, i.e., $\alpha=\bar{u}(y_0)-u(y_0)$, and ξ_0 represents the absolute vorticity of basic Zephyr flow at the initial position, i.e. $\xi_0=f_0-\frac{\partial \bar{u}}{\partial y}$. Eqs.(7)–(8) are a couple of non–linear autonomous dynamic system. From these two equations we can see that the factors affecting the air motion's stability are various.

III. THE HOMOGENEOUS BASIC ZEPHYR FLOW

First of all, we will discuss the constant basic Zephyr flow, i.e., $\bar{u}=\bar{u}(y_0)\equiv$ const, obviously, $\frac{\partial \bar{u}}{\partial y}=\frac{\partial^2 \bar{u}}{\partial y^2}=0$, $\xi_0=f_0$. Supposing the geostrophic deviation at the primary

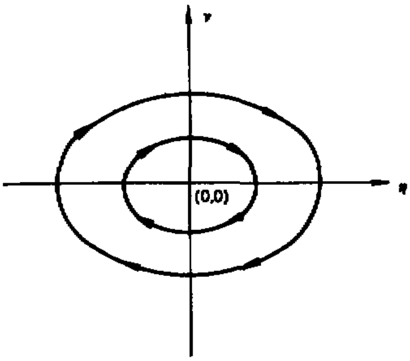

Fig.1. Periodic movement of stable center.

position is zero, from Eqs.(7)–(8) we get

$$\frac{d\eta}{dt} = v \quad , \tag{9}$$

$$\frac{dv}{dt} = -f_0^2\eta - \frac{3}{2}f_0\beta\eta^2 - \frac{\beta^2}{2}\eta^3 \quad . \tag{10}$$

1. *F–plane Approximation*

Let $f = f_0 = const$, thus $\beta = 0$, from Eqs.(9)–(10) we obtain

$$\frac{d\eta}{dt} = v \quad , \tag{11}$$

$$\frac{dv}{dt} = -f_0^2\eta \quad . \tag{12}$$

Distinctly, the very motion system delineates the linear large–scale horizontal air motion, and its equilibrium state is ($v = 0$; $\eta = 0$), and the characteristic roots of its characteristic equation are $\lambda^2 = -f_0^2$, that is to say, its characteristic roots are a couple of conjugate imaginary roots whose real parts are zero. Therefore, the equilibrium state (0,0) is a stable center, and the loci of air package's motion in the phase plane (η,v) are a group of elliptical orbits (Fig.1). Demonstratively, once the air package is disturbed, it will vibrate periodically along some fixed orbits around the equilibrium state (0,0) on the conditions of f–plane approximation and homogeneous basic Zephyr flow. The air motion is inertial stable.

2. *β–plane Approximation*

By taking in the effect of Rossby parameter β , i.e., $f = f_0 + \beta\eta$, the controlling equations can be presented as follows:

$$\frac{d\eta}{dt} = v \quad , \tag{9$'$}$$

$$\frac{dv}{dt} = -f_0^2 - \frac{3}{2}f_0\beta\eta^2 - \frac{\beta^2}{2}\eta^3 \quad . \tag{10$'$}$$

This motion system has three equilibrium states, ($v=0;\ \eta=0$), ($v=0;\ \eta=-f_0/\beta$) and ($v=0;\ \eta=\frac{2}{\beta}f_0$).

Successively, we will analyse the stability of every equilibrium state.

(1) As to equilibrium state ($v=0;\ \eta=0$), its characteristic equation is

$$\left|\begin{matrix} 0-\lambda & 1 \\ -f_0^2-3f_0\beta\eta-\frac{3}{2}\beta^2\eta^2 & 0-\lambda \end{matrix}\right|_{\substack{v=0\\ \eta=0}}=0 \quad ,$$

i.e., $\lambda^2=-f_0^2$, that is to say, the characteristic roots are a couple of conjugate imaginary ones whose real parts are zero ($Re\ \lambda=0$), and the hyperbolical property of the equilibrium state (0,0) is destroyed. So the linearization method cannot be used to analyse the stability of this very equilibrium state (0,0), the sequential–function categorizing method will be adopted to distinguish between centers and focus in equilibrium states of high–order non–linear motion system (Zhang, 1981).

Let $M=\eta,\ N=-\frac{1}{f_0}v,\ \tau=f_0t$; substituting them into Eqs. (9)′–(10)′, we have

$$\frac{dM}{d\tau}=-N \quad , \tag{13}$$

$$\frac{dN}{d\tau}=M+\frac{3\beta}{2f_0}N^2+\frac{3\beta^2}{2f_0^2}N^3 \quad . \tag{14}$$

We introduce polar coordinate (r,θ) as follows:

$$M=r\cos\theta \quad , \quad N=r\sin\theta \quad ,$$

and let

$$a=\frac{3}{2}\frac{\beta}{f_0} \quad , \quad b=\frac{\beta^2}{2f_0^2} \quad ,$$

substituting them into Eqs.(13)–(14) and arranging the two equations in power order of r , we finally get

$$\frac{dr}{d\tau}=a\sin\theta\cos^2\theta r^2+b\sin\theta\cos^3\theta r^3 \quad ,$$

$$\frac{d\theta}{d\tau}=1+a\cos^3\theta r+b\cos^4\theta r^2 \quad .$$

Combining the above two equations, simultaneously disposing the variate τ , we obtain

$$\frac{dr}{d\theta}=(\,a\sin\theta\cos\theta r^2+b\sin\theta\cos^3\theta r^3\,)\,/\,(\,1+(\,a\cos^3\theta r+b\cos^4\theta r^2\,)\,) \quad . \tag{15}$$

If r is tiny enough, Eq.(15) can be expanded into a power series of r , i.e.,

$$\begin{aligned}\frac{dr}{d\theta}&=(a\sin\theta\cos^2\theta r^2+b\sin\theta\cos^3\theta r^3)\cdot[1-(a\cos^3\theta r+b\cos^4\theta r^2)\\&\quad+(a\cos^3\theta r+b\cos^4\theta r^2)^2-(a\cos^3\theta r+b\cos^4\theta r^2)^3+......]\\&=a\sin\theta\cos^2\theta r^2+(b\cos^3\theta-a^2\cos^5\theta)\sin\theta r^3+(a^3\cos^8\theta-2ab\cos^6\theta)\sin\theta r^4\\&\quad+(2a^2b\cos^9\theta+ab\cos^9\theta-b^2\cos^7\theta-a^4\cos^{11}\theta)\sin\theta r^5+...... \quad .\end{aligned} \tag{16}$$

Suppose the solution of differential Eq.(15) as follows

$$r(\theta,c) = c + r_2(\theta)c^2 + r_3(\theta)c^3 + \ldots\ldots \quad , \tag{17}$$

where

$$r_2(0) = r_3(0) = \ldots\ldots = 0 \quad . \tag{18}$$

Substituting (17) into (16) and comparing the bilateral coefficients we have

$$\frac{dr_2}{d\theta} = a\sin\theta\cos^2\theta \quad , \tag{19}$$

$$\frac{dr_3}{d\theta} = 2a\sin\theta\cos^2\theta r_2 + (b\cos^3\theta - a^2\cos^5\theta)\sin\theta \quad , \tag{20}$$

$$\frac{dr_4}{d\theta} = a\sin\theta\cos^2\theta r_2^2 + 3(b\cos^3\theta - a^2\cos^5\theta)\sin\theta r_2$$

$$+ 2a\sin\theta\cos^2\theta r_3 + (a^3\cos^8\theta - 2ab\cos^6\theta)\sin\theta \quad , \tag{21}$$

$$\ldots\ldots \quad .$$

Integrating Eq.(19), and noticing the primitive condition (18), we get

$$r_2 = -\frac{a}{3}(\cos^3\theta - 1) \quad ,$$

apparently, r_2 is a periodic function with the period of 2π .

Suppose Eq.(20)'s solution is

$$r_3 = g_3\theta + f_3(\theta) \quad ,$$

where $f_3(\theta)$ is a periodic function of 2π , meanwhile

$$g_3 = \frac{1}{2\pi}\int_0^{2\pi}[2a\sin\theta\cos^2\theta r_2 + (b\cos^3\theta - a^2\cos^5\theta)\sin\theta]d\theta$$

$$= \frac{1}{2\pi}\int_0^{2\pi}(2a\sin\theta\cos^2\theta)\cdot[-\frac{a}{3}(\cos^3\theta - 1)]d\theta$$

$$= 0 \quad ,$$

so r_3 is also a periodic function of 2π .

Using mathematical induction and basing on the orthogonality of triangular functions, i.e., $\int_0^{2\pi}\sin\theta\cos^n\theta d\theta = 0$, we can prove that $r_2, r_3, r_4, \ldots\ldots r_k \ldots\ldots$ are all periodic functions of 2π . Therefore, the equilibrium state ($v = 0$; $\eta = 0$) is still a stable center, its stability is not affected by Rossby parameter β at all.

(2) As regards equilibrium state ($v = 0$; $\eta = -f_0 / \beta$), its characteristic equation is

$$\begin{vmatrix} 0-\lambda & 1 \\ -f_0^2 - 3f_0\beta\eta - \frac{3}{2}\beta^2\eta^2 & 0-\lambda \end{vmatrix}_{\substack{v=0 \\ \eta=-\frac{f_0}{\beta}}} = 0 \quad ,$$

i.e., $\lambda^2 = \frac{1}{2}f_0^2$, the characteristic roots are $\lambda_1 > 0$ and $\lambda_2 < 0$, so the equilibrium state (0 , $-f_0 / \beta$) is hyperbolical; that means, the phase chart of this non–linear system is topologically equal to that of its relative linear system. Consequently, the equilibrium state $(0, -f_0 / \beta)$ is an unstable saddle, and the air package will part from the equilibrium position

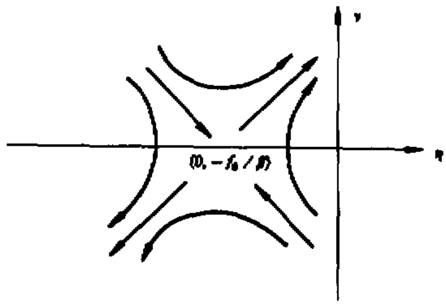

Fig.2. Unstable movement of saddle.

acceleratively once it is disturbed (Fig.2).

The aforementioned result shows that, even in the homogeneous basic Zephyr flow, there still has unstable air motion by which the importance of Rossby parameter β for large–scale horizontal air motion is just verified.

(3) *For the equilibrium state* ($\nu = 0$; $\eta = -2f_0 / \beta$), its characteristic equation is

$$\begin{vmatrix} 0-\lambda & 1 \\ -f_0^2 - 3f_0\beta\eta - \frac{3}{2}\beta^2\eta^2 & 0-\lambda \end{vmatrix}_{\substack{\nu=0 \\ \eta=-2f_0/\beta}} = 0 \quad ,$$

i.e., $\lambda^2 = -f_0^2$, the characteristic roots are a couple of conjugate imaginary roots whose real parts are zero, that means, the hyperbolical property of the equilibrium state $(0, -2f_0 / \beta)$ is destroyed. By means of sequential–function categorizing method the equilibrium state (0 , $-2f_0 / \beta$) can be easily testified to be a stable center.

3. *The Influence of Geostrophic Deviation*

In the preceding discussions, the geostrophic deviation at the initial position is not involved. Now we will study the effect of geostrophic deviation with the β–plane approximation, fom Eqs.(7)–(8) we have

$$\frac{d\eta}{dt} = \nu \quad , \tag{22}$$

$$\frac{d\nu}{dt} = f_0\alpha + (\beta\alpha - f_0^2)\eta - \frac{3}{2}f_0\beta\eta^2 - \frac{\beta^2}{2}\eta^3 \quad . \tag{23}$$

As regards this non–linear system, we just let $f_0 = \beta = 1$ for simplifying the discussion and making the physical meaning clear. From Eqs.(22)–(23) we have

$$\frac{d\eta}{dt} = \nu \quad , \tag{24}$$

$$\frac{d\nu}{dt} = \alpha + (\alpha - 1)\eta - \frac{3}{2}\eta^2 - \frac{1}{2}\eta^3 \quad . \tag{25}$$

The motion system has three equilibrium states ($\nu = 0$; $\eta = -1$),($\nu = 0$; $\eta = -1 + \sqrt{1+2\alpha}$) and ($\nu = 0$; $\eta = -1 - \sqrt{1+2\alpha}$) , its characteristic equation is $\lambda^2 = (\alpha - 1) - 3\eta - \frac{3}{2}\eta^2$. By demonstrating, the stability's variation of every equilibrium state with α is

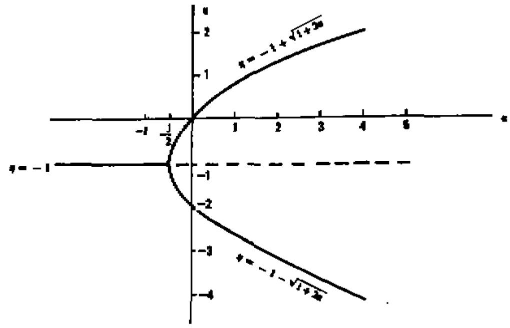

Fig.3. The effect of geostrophic deviation α on the stability of large–scale horizontal air motion ($f_0 = \beta = 1$). The bold line represents stable states, and the dashed line represents unstable states.

shown in Fig.3.

From Fig.3. we can see that if $\alpha < -1/2$, then there is only one kind of movement state, the stable equilibrium state ($v = 0$; $\eta = -1$) , and that if $\alpha > -1/2$, the air package's movement produces supercritical Hopf bifurcation, that means that the air package's movement bifurcates out two stable equilibrium states ($0, -1 + \sqrt{1 + 2\alpha}$) and ($0, -1 - \sqrt{1 + 2\alpha}$) while the equilibrium state (0, 0) becomes unstable with the increasing of α .

From the preceding discussions we can conclude that in the constant basic Zephyr flow, the motion system has only one stable equilibrium state ($v = 0$; $\eta = 0$) if the geostrophic parameter f' s variation along the latitude is not taken in, which means that the air motion is inertial stable, and that on the contrary, if the Rossby parameter β' s effect is involved in, the motion system will generate more than one equilibrium states (including an unstable one). If the effect of geostrophic deviation at initial position is further considered, the air movement will generate Hopf bifurcation with the variation of α .

IV. PARABOLIC NON–LINEAR BASIC ZEPHYR FLOW

Wan et al. (1990) had ever minutely researched the stability of large–scale horizontal air motion in linear basic Zephyr flow with f–plane approximation and β–plane approximation properly. In this section , the author will extensively analyse the air motion's stability in quadric form non–linear basic Zephyr flow.

Let $\bar{u} = \bar{u}(y_0) + \frac{\partial \bar{u}}{\partial y}\eta + \frac{1}{2}\frac{\partial^2 \bar{u}}{\partial y^2}\eta^2$, notice that if $\beta = \frac{\partial f}{\partial y}$, then $\beta - \frac{\partial \bar{u}}{\partial y^2} = \frac{\partial}{\partial y}\left(f - \frac{\partial \bar{u}}{\partial y}\right) = \frac{\partial \xi}{\partial y}$, where ξ represents the absolute vorticity of basic Zephyr flow at any position, from Eqs.(7)–(8) we obtain

$$\frac{d\eta}{dt} = v \quad , \tag{26}$$

$$\frac{dv}{dt} = f_0\alpha + (\beta\alpha - f_0\xi_0)\eta - \left(\frac{1}{2}f_0\frac{\partial \xi}{\partial y} + \beta\xi_0\right)\eta^2 - \frac{\beta}{2}\frac{\partial \xi}{\partial y}\eta^3 \quad . \tag{27}$$

1. *F–plane Approximation*

By taking the geostrophic deviation at initial position as a nil and noticing $f \equiv f_0$, the controlling equations can be expressed as follows:

$$\frac{d\eta}{dt} = v \quad , \tag{28}$$

$$\frac{dv}{dt} = -f_0 \xi_0 \eta + \frac{f_0}{2} \frac{\partial^2 \bar{u}}{\partial y^2} \eta^2 \quad . \tag{29}$$

This system has equilibrium states ($v = 0$; $\eta = 0$) and $\left(v = 0;\ \eta = 2\xi_0 \Big/ \frac{\partial^2 \bar{u}}{\partial y^2} \right)$.

(1) As regards equilibrium state ($v = 0$; $\eta = 0$) , the characteristic roots of its characteristic equation are $\lambda^2 = -f_0 \xi_0$. Therefore, if $\xi_0 < 0$, the characteristic roots are $\lambda_1 > 0$ and $\lambda_2 < 0$, the equilibrium state (0, 0) is an unstable saddle; whereas, if $\xi_0 > 0$, the characteristic roots λ_1 and λ_2 are a couple of conjugate imaginary ones whose real parts are zero, using sequential–function catagorizing method we can prove that the equilibrium state (0,0) is a stable center.

(2) As to equilibrium state $\left(v = 0;\ \eta = 2\xi_0 \Big/ \frac{\partial^2 \bar{u}}{\partial y^2} \right)$, the characteristic roots are $\lambda^2 = f_0 \xi_0$. Therefore, if $\xi_0 < 0$, λ_1 and λ_2 are a couple of conjugate imaginary roots whose real parts are zero, so the equilibrium state $\left(0,\ 2\xi_0 \Big/ \frac{\partial^2 \bar{u}}{\partial y^2} \right)$ is a stable center; whereas, if $\xi_0 > 0$, then $\lambda_1 > 0$ and $\lambda_2 < 0$, so the equilibrium state (0, $2\xi_0 / (\partial^2 \bar{u} / \partial y^2)$) is an unstable saddle.

From above–mentioned discussions we can see that with changing the absolute vorticity of basic Zephyr flow at primary position from negative to positive, the stabilities of these two equilibrium states are all transformed at $\xi_0 = 0$, producing transcritical bifurcation (Fig.4). The stability's changing of equilibrium state (0, 0) accords with the common inertial stability criterion, meanwhile, the equilibrium state (0, $2\xi_0 / (\partial^2 \bar{u} / \partial y^2)$) behaves contradictorily. This phenomenon is caused by the heterogeneous distribution of basic Zephyr flow, i.e., $\frac{\partial^2 \bar{u}}{\partial y^2} \neq 0$.

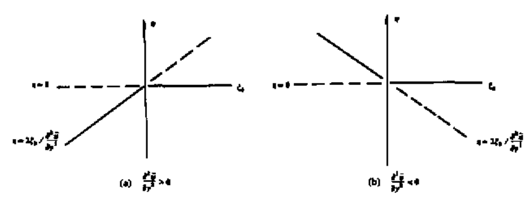

Fig.4. The stability of air motion in non–linear basic Zephyr flow,(f = const, $\alpha = 0$).

2. *β–plane Approximation*

Still assuming $\alpha = 0$ at $y = y_0$, noticing $f = f_0 + \beta\eta$, from Eqs.(26)–(27). we obtain the controlling equations as follows:

$$\frac{d\eta}{dt} = v \quad , \tag{30}$$

$$\frac{dv}{dt} = -f_0\xi_0\eta - \left(\frac{1}{2}f_0\frac{\partial\xi}{\partial y} + \beta\xi_0\right)\eta^2 - \frac{\beta}{2}\frac{\partial\xi}{\partial y}\eta^3 \quad . \tag{31}$$

This system has equilibrium states $(v = 0;\ \eta = 0)$,$(v = 0;\ \eta = -f_0 / \beta)$ and $\left(v = 0;\ \eta = -2\xi_0 / \frac{\partial\xi}{\partial y}\right)$, and its derivative matrix presents as

$$\begin{bmatrix} 0 & 1 \\ -f_0\xi_0 - \left(f_0\frac{\partial\xi}{\partial y} + 2\beta\xi_0\right)\eta - \frac{3}{2}\frac{\partial\xi}{\partial y}\eta^2 & 0 \end{bmatrix}_{(\eta, v)}$$

(1). *First, let's fix* $\frac{\partial\xi}{\partial y}$ *and take* ξ_0 *as the controlling parameter*

a). As regards to equilibrium state $(v = 0;\ \eta = 0)$) , its characteristic equation is $\lambda^2 = -f_0\xi_0$. So, if $\xi_0 < 0$, then $\lambda_1 > 0$ and $\lambda_2 < 0$, the equilibrium state (0, 0) is hyperbolical, i.e., the system's phase chart around the equilibrium state (0, 0) is topologically equal to that of its relative linear system, and (0 ,0) is an unstable saddle. Whereas, if $\xi_0 > 0$, then the characteristic roots are a couple of conjugate imaginary ones whose real parts are zero, the hyperbolical property of the equilibrium state (0, 0) is destroyed; demonstratively, (0, 0) can be proved as a stable center by using sequential–function categorizing method.

b). As to equilibrium state $(v = 0;\ \eta = -f_0 / \beta)$, its characteristic equation is $\lambda^2 = f_0\left(\xi_0 - \frac{f_0}{2\beta}\frac{\partial\xi}{\partial y}\right)$. Therefore, if $\xi_0 > \frac{f_0}{2\beta}\frac{\partial\xi}{\partial y}$, then $\lambda_1 > 0$ and $\lambda_2 < 0$, the equilibrium state $(0, -f_0 / \beta)$ is an unstable saddle, whereas, if $\xi_0 < \frac{f_0}{2\beta}\frac{\partial\xi}{\partial y}$, then the characteristic roots are a couple of conjugate imaginary ones whose real parts are zero, decidedly, the equilibrium state $(0, -f_0 / \beta)$ can be proved as a stable center.

c). As to equilibrium state $(v = 0;\ \eta = -2\xi_0 / \frac{\partial\xi}{\partial y})$, its characteristic equation is $\lambda^2 = \xi_0\left(f_0 - 2\beta\xi_0 / \frac{\partial\xi}{\partial y}\right)$. In the following passages, we will sort out two conditions to analyse.

First, $\frac{\partial\xi}{\partial y} > 0$, if $\xi_0 < 0$, the characteristic roots are a couple of conjugate imaginary ones whose real parts are zero, so $\left(0, -2\xi_0 / \frac{\partial\xi}{\partial y}\right)$ can be proved as a stable center with sequential–function categorizing method, on the contrary, if $0 < \xi_0 < \frac{f_0}{2\beta}\frac{\partial\xi}{\partial y}$, then $\lambda_1 > 0$ and $\lambda_2 < 0$ and $\left(0, -2\xi_0 / \frac{\partial\xi}{\partial y}\right)$ is an unstable saddle, similarly, if $\xi_0 > \frac{f_0}{2\beta}\frac{\partial\xi}{\partial y}$, the characteristic roots are a couple of conjugate imaginary roots whose real parts are zero, so

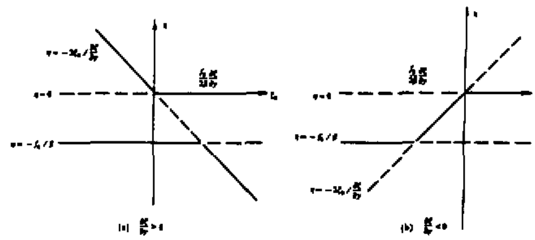

Fig.5. The variation of each equilibrium state's stability with ξ_0 in parabolic non–linear basic Zephyr flow, under the effect of Rossby parameter $\beta(\alpha = 0)$.

$\left(0, -2\xi_0 / \frac{\partial \xi}{\partial y}\right)$ becomes a stable center again.

Second, $\frac{\partial \xi}{\partial y} < 0$, if $\xi_0 < \frac{f_0}{2\beta}\frac{\partial \xi}{\partial y}$, then $\lambda_1 > 0$ and $\lambda_2 < 0$, and $\left(0, -2\xi_0 / \frac{\partial \xi}{\partial y}\right)$ is an unstable saddle; if $\frac{f_0}{2\beta}\frac{\partial \xi}{\partial y} < \xi_0 < 0$, the characteristic roots are a couple of conjugate imaginary ones whose real parts are zero, and $\left(0, -2\xi_0 / \frac{\partial \xi}{\partial y}\right)$ becomes a stable center; if $\xi_0 > 0$, then $\lambda_1 > 0$ and $\lambda_2 < 0$, $(0, -2\xi_0 / \frac{\partial \xi}{\partial y})$ becomes an unstable saddle again.

The stability's changing with ξ_0 of every equilibrium state is presented in Fig.5 based on the different sorts of $\frac{\partial \xi}{\partial y} > 0$ and $\frac{\partial \xi}{\partial y} < 0$.

(2) *Now, let's fix* ξ_0 , *and take* $\frac{\partial \xi}{\partial y}$ *as the controlling parameter*

a). The stability's variation of equilibrium state (0,0) is similar to what we have discussed in (1).

b). As to equilibrium state $(0, -f_0 / \beta)$, its characteristic equation is $\lambda^2 = f_0\left(\xi_0 - \frac{f_0}{2\beta}\frac{\partial \xi}{\partial y}\right)$. If $\frac{\partial \xi}{\partial y} < \frac{2\beta}{f_0}\xi_0$, then $\lambda_1 > 0$ and $\lambda_2 < 0$, $(0, -f_0 / \beta)$ is an unstable saddle; and if $\frac{\partial \xi}{\partial y} > \frac{2\beta}{f_0}\xi_0$, the characteristic roots are a couple of conjugate imaginary ones whose real parts are zero, consequently, $(0, -f_0 / \beta)$ becomes a stable center.

c). As regards equilibrium state $\left(0, -2\xi_0 / \frac{\partial \xi}{\partial y}\right)$, its characteristic equation is $\lambda^2 = \xi_0\left(f_0 - 2\beta\xi_0 / \frac{\partial \xi}{\partial y}\right)$. Kinds of conditions should be discussed.

First, in the inertial stable region, i.e., $\xi_0 > 0$, if $\frac{\partial \xi}{\partial y} > 2\beta\xi_0 / f_0$, then $\lambda_1 > 0$ and $\lambda_2 < 0$, $\left(0, -2\xi_0 / \frac{\partial \xi}{\partial y}\right)$ is an unstable saddle, and if $0 < \frac{\partial \xi}{\partial y} < 2\beta\xi_0 / f_0$, then the characteristic roots are a couple of conjugate imaginary ones whose real parts are zero, so $\left(0, -2\xi_0 / \frac{\partial \xi}{\partial y}\right)$ is a stable center, and if $\frac{\partial \xi}{\partial y} < 0$, then $\lambda_1 > 0$ and $\lambda_2 < 0$, $\left(0, -2\xi_0 / \frac{\partial \xi}{\partial y}\right)$

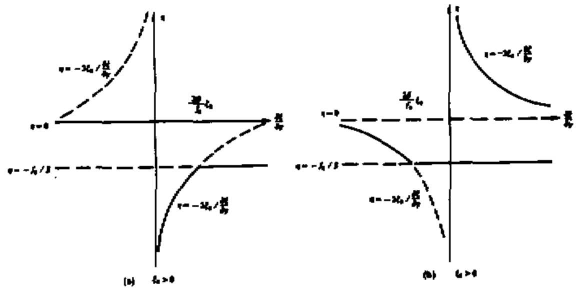

Fig.6. The variation of each equilibrium state's stability with $\frac{\partial \xi}{\partial y}$ in parabolic non–linear basic Zephyr flow, under the effect of Rossby parameter $\beta(\alpha = 0)$.

becomes an unstable saddle again.

Contradictorily, in the inertial unstable region, i.e., $\xi_0 < 0$, if $\frac{\partial \xi}{\partial y} > 0$, the characteristic roots are a couple of conjugate imaginary ones whose real parts are zero, so $(0, -2\xi_0 / \frac{\partial \xi}{\partial y})$ is a stable center, and if $2\beta\xi_0 / f_0 < \frac{\partial \xi}{\partial y} < 0$, then $\lambda_1 > 0$ and $\lambda_2 < 0$, so $\left(0, -2\xi_0 / \frac{\partial \xi}{\partial y}\right)$ is an unstable saddle, and if $\frac{\partial \xi}{\partial y} < 2\beta\xi_0 / f_0$, the characteristic roots are a couple of conjugate imaginary ones whose real parts are zero, so $\left(0, -2\xi_0 / \frac{\partial \xi}{\partial y}\right)$ becomes a stable center again.

Finally, the aforementioned results are generalized in Fig.6 according to different sorts of $\xi_0 > 0$ and $\xi_0 < 0$.

From Fig.5 and Fig.6 we can see that with the interaction of Rossby parameter β and non–linear basic Zephyr flow, the stability's variation of air motion has evolved more complicatedly. Comparing above–menfioned results comprehensively with Wan et al. (1990)'s which was drawn out with the β–plane approximation and in linear basic Zephyr flow, we can get the following conclusions:

(1) In the linear and the non–linear basic Zephyr flows, because of the effect of Rossby parameter β , some equilibrium states always lost their stabilities in inertial stable region and some new stable equilibria are generated in inertial unstable region.

(2) As to the linear distribution of basic Zephyr flow, the stability of every equilibrium state transforms two times with the variation of the absolute vorticity of basic Zephyr flow at primary position, that means that the air motion generates two times of transcritical bifurcation. As to the quadric form non–linear basic Zephyr flow presented in this paper, the stability of each equilibrium state is controlled by parameters ξ_0 and $\frac{\partial \xi}{\partial y}$ simultaneously. Once $\frac{\partial \xi}{\partial y}$ is fixed, in the two different regions, i.e., $\frac{\partial \xi}{\partial y} > 0$ and $\frac{\partial \xi}{\partial y} < 0$, the stabilities of

equilibrium state $\left(0,-2\xi_0/\frac{\partial\xi}{\partial y}\right)$, respectively, generate two times of transcritical bifurcation with the variation of ξ_0, and its stability's changing is precisely contrary in these two different regions (see.Fig.5). This result approximately testifies the significance of non–linear distribution of basic Zephyr flow for air motion's stability variation. While ξ_0 is fixed, in the two different regions, i.e., the inertial stable region ($\xi_0>0$) and inertial unstable one ($\xi_0<0$), the stabilities of equilibrium state $(0,-f_0/\beta)$ transform only one time at $\frac{\partial\xi}{\partial y}=\frac{2\beta}{f_0}\xi_0$ with the variation of $\frac{\partial\xi}{\partial y}$; nevertheless, the stabilities of equlibrium state $\left(0,-2\xi_0/\frac{\partial\xi}{\partial y}\right)$, transform two times at $\frac{\partial\xi}{\partial y}=0$ and $\frac{\partial\xi}{\partial y}=\frac{2\beta}{f_0}\xi_0$ with the variation of $\frac{\partial\xi}{\partial y}$, and its stability's variation just opposite in the two different regions ($\xi_0>0$ and $\xi_0<0$). Noticeably, the bifurcation presented in Fig.6 is a kind of special ones with defect.

(3) It can be easily observed that no matter the basic Zephyr flow distributes linearly or non–linearly, the stability's variation of equilibrium state ($v=0$; $\eta=0$) is only influenced by ξ_0, and it accords with the common inertial stability criterion. Obviously, once the air package is disturbed at initial position, its movement is affected by neither Rossby parameter β nor the distribution of basic Zephyr flow, it is only decided by the absolute vorticity of basic Zephyr flow at primary position.

(4) From Fig.5 and Fig.6 we can see that an unstable equilibrium state always exists in two stable equilibrium states, and that a stable equilibrium state always exists in two unstable equilibrium states. This phenomenon just conforms to the theory of manifold's separatrix in non–linear differential equations. In the very following passage, the author will only take Fig.6a as an example to discuss minutely. The rest parts shown in Fig.5 and Fig.6, can be analysed similarly.

In Fig.6a, if $\frac{\partial\xi}{\partial y}<0$, there has a stable equilibrium state (0,0) between the two unstable equilibrium states $\left(0,-2\xi_0/\frac{\partial\xi}{\partial y}\right)$ and $(0,-f_0/\beta)$. Therefore, for the primitive disturbance originating anywhere in the region $\left(v=0;-f_0/\beta<\eta<-2\xi_0/\frac{\partial\xi}{\partial y}\right)$, the system will eventually tend to the equilibrium state (0,0). These results can also be verified by the system's topological structure, i.e., the phase chart in plane (η,v) (Fig.7a). Conversely, if $0<\frac{\partial\xi}{\partial y}<\frac{2\beta}{f_0}\xi_0$, there has an unstable equilibrium state $(0,-f_0/\beta)$ between the two stable equilibrium states, (0,0) and $\left(0,-2\xi_0/\frac{\partial\xi}{\partial y}\right)$. So, as regards the primitive disturbance originating everywhere in the region ($v=0$; $-f_0/\beta<\eta<0$), the system will eventually tend to the equilibrium state (0,0), meanwhile, as regards the initial disturbance originating everywhere in the region $\left(v=0;\ -2\xi_0/\frac{\partial\xi}{\partial y}<\eta<-f_0/\beta\right)$, the system will finally tend to the equilibrium state $\left(0,-2\xi_0/\frac{\partial\xi}{\partial y}\right)$. The above–mentioned phenomenon can also be seen in the phase chart of plane (η,v) (Fig.7b). If $\frac{\partial\xi}{\partial y}>\frac{2\beta}{f_0}\xi_0$, the result is similar to the one of $\frac{\partial \xi}{\partial y} < 0$. From above conclusions we can realize that because of the influence of Rossby parameter β and non–linear basic Zephyr flow, there have more than one equilibrium states in the large–scale horizontal air motion, and that once the air package is disturbed, it can only tend to a certain stable equilibrium state instead of moving aimlessly.

(5) If the cubic form non–linear basic Zephyr flow is further considered, not only is the stability of each equilibrium dissimilar in different space domain which is determined by parameters $\left(\xi_0, \frac{\partial \xi}{\partial y}, \frac{\partial^2 \xi}{\partial y^2}\right)$, but also generates Hopf bifurcation at $\xi_0 = -9\left(\frac{\partial \xi}{\partial y}\right)^2 / \left(24\frac{\partial^2 \xi}{\partial y^2}\right)$, that is to say, the number of equilibrium states also changes at the two sides of the bifurcation point (For the limit of the paper's length, the demonstration is omitted). Reasonably, we can imagine that the more complicate the distribution of basic Zephyr flow is, the more complex the bifurcation and the catastrophe will appear in air motion.

3. *For the β–Plane Approximation, by Considering the Effect of Geostrophic Deviation at Initial Position, the Controlling Equations can be Expressed as Follows*:

$$\frac{d\eta}{dv} = v \quad , \tag{26$'$}$$

$$\frac{dv}{dt} = f_0 \alpha + (\beta\alpha - f_0 \xi_0)\eta - \left(\frac{1}{2} f_0 \frac{\partial \xi}{\partial y} + \beta \xi_0\right)\eta^2 - \frac{\beta}{2}\frac{\partial \xi}{\partial y}\eta^3 \quad . \tag{27$'$}$$

This dynamic system has three equilibrium states ($v = 0; \eta = -f_0 / \beta$), $\left(v = 0;\ \eta = -\xi_0 / \frac{\partial \xi}{\partial y} + \sqrt{\xi_0^2 + 2\alpha\frac{\partial \xi}{\partial y}} / \frac{\partial \xi}{\partial y}\right)$ and $\left(v = 0;\ \eta = -\xi / \frac{\partial \xi}{\partial y} - \sqrt{\xi_0^2 + 2\alpha\frac{\partial \xi}{\partial y}} / \frac{\partial \xi}{\partial y}\right)$. Comparing with abovementioned chapters' analysis, one can notice that no matter how the basic Zephyr flow distributes, the equilibrium state $(0, -f_0 / \beta)$ always exists as far as the β–plane approximation is involved in. Possibly, the effect of Rossby parameter β accounts for the phenomenon. Specially, the air motion controlled by dynamic system (Eqs.(26)$'$–(27)$'$) only motivates a kind of lag transcritical bifurcation with the variation of α instead of generating cusp catastrophe as the common cubic form non–linear dynamic system does. This speciality is the distinctive characteristic of large–scale horizontal air motion. For expounding clearly, a special example is put out in the following passages.

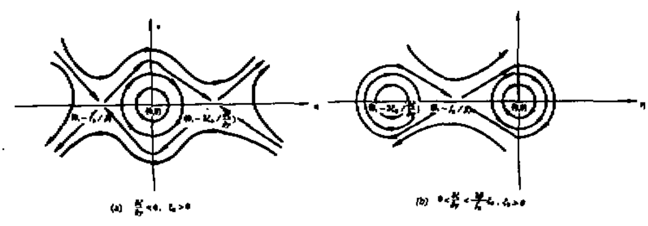

Fig.7. The phase chart in plane (η, v).

Let $f_0 = \beta = 1,\ \xi_0 = 2,\ \dfrac{\partial \xi}{\partial y} = 1$, from Eqs.(26)′–(27)′ we have

$$\frac{d\eta}{dt} = v \quad , \tag{32}$$

$$\frac{dv}{dt} = \alpha + (\alpha - 2)\eta - \frac{5}{2}\eta^2 - \frac{1}{2}\eta^3 \quad . \tag{33}$$

This system has equilibrium states $(0, -1)$, $(0, -2 + \sqrt{4 + 2\alpha})$ and $(0, -2 - \sqrt{4 + 2\alpha})$, its characteristic equation is $\lambda^2 = (\alpha - 2) - 5\eta - \frac{3}{2}\eta^2$. Therefore, if $\alpha < -\frac{13}{6}$ or $\eta > -\frac{5}{3} + \frac{1}{3}\sqrt{13 + 6\alpha}$ or $\eta < -\frac{5}{3} - \frac{1}{3}\sqrt{13 + 6\alpha}$, the characteristic roots are a couple of conjugate imaginary ones whose real parts are zero, so each equlilibrium state can be proved as a stable center by means of sequential–function categorizing method. If $-\frac{5}{3} - \frac{1}{3}\sqrt{6\alpha + 13} < \eta < -\frac{5}{3} + \frac{1}{3}\sqrt{6\alpha + 13}$, then $\lambda_1 > 0$ and $\lambda_2 < 0$, each equilibrium state is an unstable saddle (Fig.8)

From Fig.8 one can observe that with the α's increasing from negative to positive, the motion motivates bifurcation at $\alpha = -\frac{3}{2}$ and the equilibrium state $(0, -1)$ transforms to equilibrium state $(0, -2 + \sqrt{4 + 2\alpha})$. Dissimilarly, with the α's decreasing from positive to negative, the equilibrium state $(0, -2 - \sqrt{4 + 2\alpha})$ leaps over state point A to state point B, i.e., $(0, -1)$, this is the very lag transcritical bifurcation.

V. CATASTROPHE

In this section, the author will briefly analyse the catastrophe phenomenon of the large–scale horizontal air motion in cubic form non–linear basic Zephyr flow with β–plane approximation.

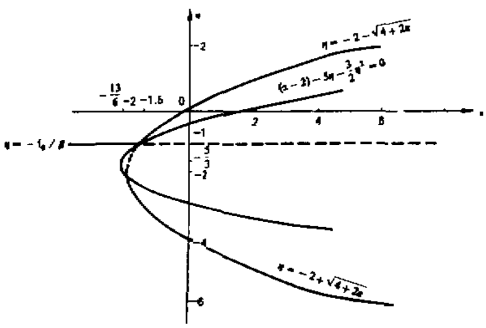

Fig.8. The lag transcritical bifurcation $(f_0 = \beta = 1,\ \xi_0 = 2,\ \frac{\partial \xi}{\partial y} = 1)$. The auxiliary curve $(\alpha - 2) - 5\eta - \frac{3}{2}\eta^2 = 0$ is used to determine the stability region.

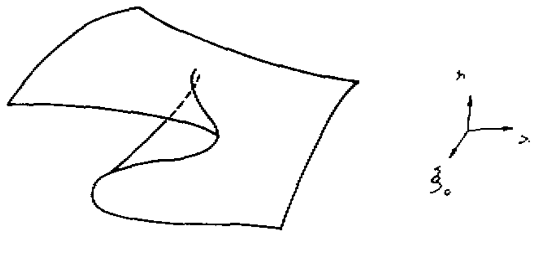

Fig.9. The curved surface $\eta^3 - \xi_0\eta + \alpha = 0$.

Let $\bar{u} = \bar{u}(y_0) + \frac{\partial \bar{u}}{\partial y}\eta + \frac{1}{2}\frac{\partial^2 \bar{u}}{\partial y^2}\eta^2 + \frac{1}{6}\frac{\partial^3 \bar{u}}{\partial y^3}\eta^3$, we can demonstrate the controlling equations as follows:

$$\frac{d\eta}{dt} = v \quad , \tag{34}$$

$$\frac{dv}{dt} = f_0\alpha + (\beta\alpha - f_0\xi_0)\eta - \left[\frac{1}{2}f_0\frac{\partial \xi}{\partial y} + \beta\xi_0\right]\eta^2 + \left[\frac{f_0}{6}\frac{\partial^3 \bar{u}}{\partial y^3} - \frac{\beta}{2}\frac{\partial \xi}{\partial y}\right]\eta^3 + \frac{\beta}{6}\frac{\partial^3 \bar{u}}{\partial y^3}\eta^4 \ . \tag{35}$$

Apparently, this dynamic system still contains the equilibrium state ($v = 0$; $\eta = -f_0 / \beta$). The other equilibrium states are given out by the following equation

$$\frac{1}{6}\beta\frac{\partial^3 \bar{u}}{\partial y^3}\eta^3 - \frac{\beta}{2}\frac{\partial \xi}{\partial y}\eta^2 - \beta\xi_0\eta + \beta\alpha = 0 \quad . \tag{36}$$

Let $\frac{\partial^3 \bar{u}}{\partial y^3} = 6$, $\frac{\partial \xi}{\partial y} = 0$, then Eq.(36) is simplified as

$$\eta^3 - \xi_0\eta + \alpha = 0 \quad .$$

Therefore, the collection of equilibrium states forms a curved surface in the parameter plane (ξ_0, α) (Fig.9)

On this curved surface, the stability of each equilibrium state is different from the cusp catastrophe determined by dynamic system $\dot{x} = -x^3 - bx - a$ which has ever been researched by many scientists (see, Liu, 1989), it must be decided by the characteristic roots of characteristic equation $\lambda^2 = \beta\alpha - f_0\xi_0 - 2\beta\xi_0\eta + 3f_0\eta^2 + 4\beta\eta^3$ in Eqs.(34)–(35). Therefore, the stability's variation of this system is more complex, the standard cusp catastrophe in three–dimensional space domain does not exist, and the common cusp catastrophe in two–dimensional plane does also not exist while ξ_0 is fixed. In fact, this catastrophe is a kind of degenerate swallow–tail catastrophe determined by quartic form non–linear dynamic system.

VI. CONCLUSIVE REMARKS

The variety and complexity of weather phenomena is due to the numerous factors which continuously affect the atmospheric movement and due to the complicated non–linear properties of atmospheric equations. Since Lorenz researched the phenomena of bifurcation, catastrophe and chaos of Rayleigh–Benard convection, using a non–linear model of cubic form truncated spectrum. The theory of non–linear differential equations has been gradually and widely used to explain and analyse many complex phenomena of air motion. Scientists have been recognizing the non–linear properties of atmospheric motion more and more deeply. Though the mathematical model used in this paper is still not perfect enough, anyhow, the complexity of large–scale horizontal air motion has basically been revealed. In a certain degree, the traditional idea of categorizing air motion's stability based on inertial stability criterion is also altered.

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