Simplified Dynamic Models of Grass Field Ecosystem①

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ABSTRACT

Some simplified dynamic models of grass field ecosystem are developed and investigated. The maximum simplified one consists of two variables, living grass biomass and soil wetness. The analyses of such models show that there exists only desert regime without grasses if the precipitation p is less than a critical value p_c; the grass biomass continuously depends on p if the interaction between grass biomass and the soil wetness is weak, but the strong interaction results in the bifurcation of grass biomass in the vicinity of p_c: the grass biomass is rich as $p > p_c$, but it becomes desertification as $p < p_c$. Periodic solutions also exist in the model, if the seasonal cycle of model's parameters is introduced. An improved model consists of three variables, i.e. the living grass biomass x, the nonliving grass biomass accumulated on the ground surface y and the soil wetness z. The behaviours of such three variables model are more complicated. The initial values of y and z play a very important role.

Key words: Dynamic model, Grass field ecosystem, Bifurcation

I. INTRODUCTION

The Earth natural environment–ecosystem is always changing, regularly or catastrophically. Now, we are facing at a serious planetary scale problem – the "global change." The impact of climate change and the human activities on the environment–ecosystem is remarkable, and the feedback of degenerated environment–ecosystem to the climate change can even make the situation more and more serious. On the other hand, some engineerings which aim at the improvement of environment and living condition, such as hydraulic engineerings – the hydro–electrical power engineering and irrigation system, afforestation and so on, may also bring positive or negative effect on the environment. In order to protect the environment from disadvantageous climate condition and to have positive effect of the natural engineering it is important to understand the dynamics of the environment–ecosystem, especially the interactions between its components such as the atmosphere, ocean, land, and ecosystem, and to develop theory on natural cybernetics like putting the technical controlling practices in the industry on a scientific base.

In this paper an attempt is made to show the essential thing of a very simple ecosystem and its interaction with surrounding environment, i.e.

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namics of the ecosystem. The model developed in such a way can qualitatively indicate the conditions for mutual maintenance of grass and soil wetness or the desertification, and the criteria of minimum irrigation and maximum consuming. It may be useful in the understanding of the natural behaviours. Based on more practical considerations and parameters, further development of more complicated and practical models will give better understanding of the real natural behaviours and lead to the theory of practical natural cybernetics.

II. SIMPLIFIED DYNAMIC MODEL WITH TWO VARIABLES

Assume that (1), the sunlight, temperature, water supply (precipitation and irrigation) and the physical–chemical properties of the soil, as well as the consuming are all known, (2) the grass field consists of only single species, and (3) the coverage of soil surface by the grass depends only on the living grass biomass, we have an interactive dynamic system consisting of two variables, the (living) grass biomass per unit area x and the soil wetness y (Zeng Q.C. et al., 1993, 1994):

$$\dot{x} = g(x,y) - d(x,y) - c \ , \tag{1}$$

$$\dot{y} = (p+i) - w(x,y) - r(x,y) \ , \tag{2}$$

where g, d, w, and r are the growing rate, decaying rate, evaporation rate and run–off respectively, c the consuming rate, p and i the precipitation and irrigation respectively.

In general, g, d, w, and i should be determined by empirical functions. However, their basic behaviours can be understood by some general logical considerations, hence an analytical model which demonstrates the essence of grass field ecosystem can also be constructed.

The major characteristics of these functions are as follows (see, Zeng Qingcun et al., 1993 a,b), for example, (1) g increases with x and y if $x \leqslant x_c$ (x_c gives full coverage) and $y \leqslant y_c$ (saturated value for the grass), and $g \to 0$ as $x \to 0$ or $y \to 0$; (2) $d \to \infty$ as $y \to 0$, but $d \to 0$ as $x \to 0$; (3) $d(x_c, y_n) = g(x_c, y_n)$, where y_n is a "normal value" of y ; (4) w and r both increase with y if $y \leqslant y_s$ (the soil capacity for keeping the water, and can be grossly taken as ∞ in the mathematic formulation), and $w \to 0, r \to 0$ as $y \to 0$; (5) r increases with $p+i$; (6) w and r both decrease with x in most important range, especially near x_c ; and so on.

One compact and analytical formulation of such functions satisfying all the requirements mentioned above has been constructed by us as follows:

$$g = \alpha(1-e^{-x})(1-e^{-y}) \ , \tag{3}$$

$$d = \beta(e^{x}-1)(1-e^{-y})^{-1} \ , \tag{4}$$

$$w = (1-e^{-y})[f + (1-f)\kappa_2(1+\kappa_0 e^{-x})^m] \ , \tag{5}$$

$$r = (p+i)\lambda(e^{y}-1)[f + (1-f)\kappa_3(1+\kappa_1 e^{-x})^n] \ , \tag{6}$$

where f is the coverage given by

$$f = e^{-x} \ , \tag{7}$$

c, p and i are the given (nondimensional) external parameters or functions of time t, and α, β, λ, κs, m and n are (nondimensional) internal parameters or functions of time t. Note, Equations (1)–(7) have been non–dimensionalized, and the nondimensional biomass which is associated with the full coverage is taken as $x=2$, while the normal value of y is 2. Therefore, the balance between g and d at $x=2$ and $y=2$ gives

$$\beta = \alpha(1-e^{-2})^2 e^{-2} \ . \tag{8}$$

Besides, it is better to introduce alternative parameters of interactions φ_0, φ_1 and

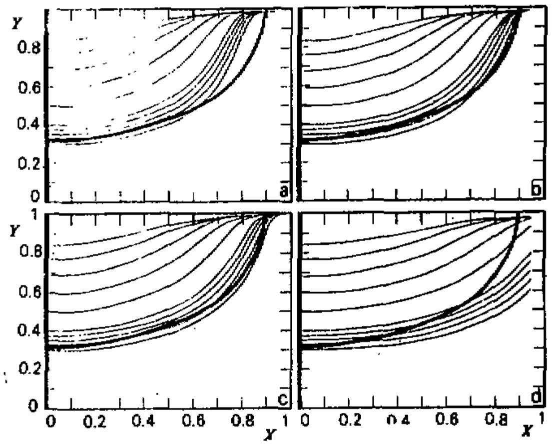

Fig. 1. Curves $\dot{X}=0$ and $\dot{Y}=0$ with $\psi=1/4$ but different p and φ_0. The heavy solid line (with two branches) is $\dot{X}=0$, and the thin solid lines are curves $\dot{Y}=0$ with $p=0.9, 0.8, 0.7, 0.6, 0.5, 0.4, 0.375, 0.325$ and 0.3 respectively from the top to the bottom. The equilibrium states are located at the common points of curves $\dot{X}=0$ and $\dot{Y}=0$, a — $\varphi_0=0.1$, b — $\varphi_0=0.2$, c — $\varphi_0=0.25$ and d — $\varphi_0=0.5$.

other parameter ψ (for replacing λ) in the following way,

$$\begin{cases}\kappa_2(1+\kappa_0)^m=1\ , \\ \kappa_2(1+\kappa_0 e^{-2})^m=\varphi_0\end{cases}, \tag{9}$$

$$\begin{cases}\kappa_3(1+\kappa_1)^n=1\ , \\ \kappa_3(1+\kappa_1 e^{-2})=\varphi_1\end{cases}, \tag{10}$$

$$\lambda=\psi/(e^3-1)\ . \tag{11}$$

In fact, φ_0 and φ_1 are the fraction of evaporation and the fraction of run–off respectively at $x=2$, and ψ is the fraction of run–off at $y=3$ and $x=0$.

After introduction of new variables X and Y as follows

$$X=1-e^{-x},\quad Y=1-e^{-y}\ , \tag{12}$$

the functions become algebraic, and we have

$$\frac{\dot{X}}{1-X}=\alpha XY-\beta\frac{X}{Y(1-X)}-c\ , \tag{13}$$

$$\frac{\dot{Y}}{1-Y}=(p+i)\left\{1-\lambda\frac{Y}{1-Y}\left[(1-X)+\kappa_3X(1+\kappa_1(1-X)^n\right]\right\} \\ -Y[(1-X)+\kappa_2X(1+\kappa_0(1-X)^m]\ . \tag{14}$$

If the seasonal variations are neglected, c, $p+i$, as well as α, β, φ_0, φ_1 and ψ are all constants. In such case possible ecological regimes (x_p, y_p) are the stable equilibrium states of the model.

Taking $i=c=0, \alpha=m=n=1, \varphi_0=\varphi_1$, and $\psi=1/4$ the curves $\dot{x}=0$ and $\dot{y}=0$ as well as the equilibrium states (the common points of $\dot{x}=0$ and $\dot{y}=0$) with different φ_0 and

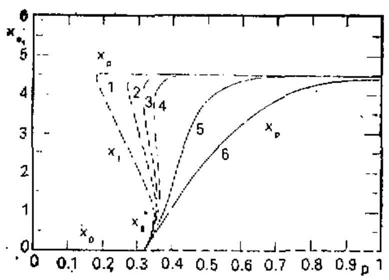

Fig. 2. Bifurcation diagram showing the dependence of x_e (the grass biomass of equilibrium state) on p for different φ_0. $\psi = 1/4$ is taken. The solid line is the stable state, and dashed line the unstable one. 1 — $\varphi_0 = 0.1$, 2 — $\varphi_0 = 0.2$, 3 — $\varphi_0 = 0.25$, 4 — $\varphi_0 = 0.3$, 5 — $\varphi_0 = 0.5$, and 6 — $\varphi_0 = 0.75$.

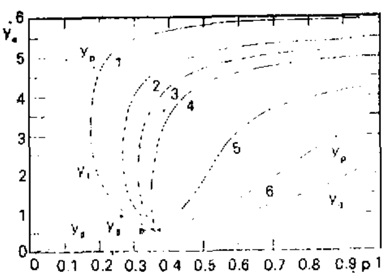

Fig. 3. The same as in Fig. 2, but for y_e (the soil wetness of equilibrium state).

p are given in Fig. 1. It can be seen that in general there exists a critical value p_c of p depending on φ_0, the possible ecological regime is associated with a grass field in which $x_p > 0$ if $p > p_c$; p_c ; decreases with φ_0 (the larger the interaction, the smaller the critical precipitation needed for the existence of a grass field); and that the grass biomass x_p continuously increases with p as φ_0 is large, but a bifurcation exists in the vicinity of $p = p_c$ as φ_0 is small. These features can more clearly be seen in the bifurcation diagrams (Figs. 2 and 3). This means that a grass field with rich biomass can be maintained even in the arid to semi-arid area (where the precipitation condition is marginal), provided that, (1) the grass species is favourable, (2) the initial x and y are large enough, and (3) the consumers disturb the ecosystem only slightly.

Similar calculations but with $c \neq 0$ can determine the maximum consuming supported by the grass field under given p and i, and the minimum irrigation for the maintenance of grass field under a given $c \neq 0$.

III. GRASS FIELD ECOSYSTEM WITH SEASONAL CYCLE

Taking seasonal cycle of the parameters into account in Eqs.(1)–(7), we have more realistic model of grass field ecosystem (Zeng Xiaodong et al., 1993).

It is not difficult to prove that periodic solution of Eqs.(1) and (2) exists if the seasonal variation of all the parameters of the model is small enough. In fact, let

$$\gamma = \gamma_0 + \varepsilon \sum_{k=1}^{\infty} \gamma_k e^{ik\omega t} , \tag{15}$$

where γ_k ($k = 0, 1, 2, \cdots$) are constants, $\{\gamma_k\} \in l_2$, and $\varepsilon > 0$ is a small parameter. γ stands for every one of the external and internal parameters of the model. The expansion of solution into power series of ε

$$\begin{cases} x = x_0 + \sum_{j=1}^{\infty} \varepsilon^j x_j , \\ y = y_0 + \sum_{j=1}^{\infty} \varepsilon^j y_j , \end{cases} \tag{16}$$

yields the following set of equations

$$\begin{cases} \dot{x}_j = \alpha_0(G_{x0}x_j + G_{y0}y_j) - \beta_0(D_{x0}x_j + D_{y0}y_j) + F_j(t) \ , \\ \dot{y}_j = (p_0 + i_0)\lambda_0(V_{x0}x_j + V_{y0}y_j) - (W_{x0}x_j + W_{y0}y_j) + H_j(t) \ , \end{cases} \tag{17}$$

where G_{x0} is the value of $\partial G / \partial x$ at the point $(x = x_0, y = y_0)$ etc.; (x_0, y_0) is the equilibrium state with $\gamma = \gamma_0$; $F_j(t)$ and $H_j(t)$ are periodic functions of t and depend on γ_k and $(x_{j'}, y_{j'}), j' = 0, 1, 2, \cdots, j-1; G \equiv \alpha^{-1}g, D \equiv \beta^{-1}d, W \equiv w$, and $V \equiv -r / [\lambda(p+i)]$. Because every equilibrium state of our model without seasonal cycle is not a center, every eigenvalue of the following matrix

$$M \equiv \begin{vmatrix} \alpha_0 G_{x0} - \beta_0 D_{x0} & \alpha_0 G_{y0} - \beta_0 D_{y0} \\ \lambda_0(p_0 + i_0)V_{x0} - W_{x0} & \lambda_0(p_0 + i_0)V_{y0} - W_{y0} \end{vmatrix} \tag{18}$$

is not pure imaginary, therefore there exists a solution (x_j, y_j) with period ω. The expansion is convergent provided ε is small enough, and the solution obtained by such expansion is around the point (x_0, y_0).

The numerical results validate this proof. Fig. 4 shows the existence of such periodic solution with $\varepsilon = 0.2, \alpha_1 = -\beta_1 = p_1 = 1, p_k = 0$ if $k \geqslant 2$, $c = i = 0, \alpha_0 = m = n = 1$, $\varphi_0 = \varphi_1 = \varphi = 0.25$. The integration curve approaches the periodic solution starting from an initial point located outside the curve of periodic solution.

The periodic solution also exists if the seasonal variation of parameters is not small. Fig. 5 elucidates an example similar to Fig. 4 but with $\varepsilon = 1$. The interesting thing is that the periodic solution is no longer around the equilibrium point (x_0, y_0).

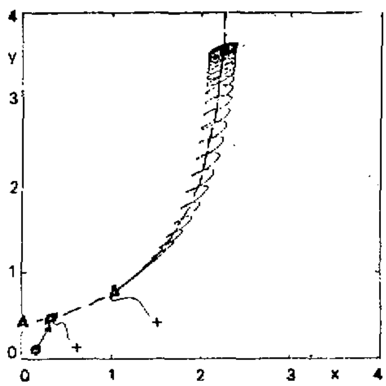

Fig. 4. Existence of periodic solution (x,y) with small seasonal cycle of parameters α, β, and p, where $\alpha = \alpha_0(1 + \varepsilon\cos\omega t)$, $\beta = \beta_0(1 - \varepsilon\cos\omega t)$, $p = p_0(1 + \varepsilon\cos\omega t)$, $\varepsilon = 0.2$, $p_0 = 0.35$ and $\varphi_0 = 0.25$, $\psi = 1/4$. The integral curve is started from its initial point denoted by +. The dashed line is the curve $\dot{x} = 0$ of the associated model with no seasonal cycle of the parameters, and its stable and unstable equilibrium states are denoted by ● and ▲ respectively.

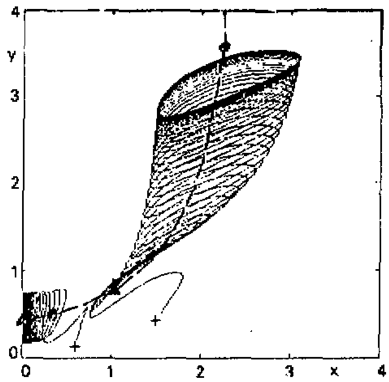

Fig. 5. The same as in Fig. 4, but for $\varepsilon = 1$.

Note, that $i > 0$ is equivalent to an increase of p, and Figs. 4 and 5 give a hint about the question: for how many years should the irrigation be maintained in order to have an established grass field with rich biomass in an arid to semi–arid area if the grass is planted.

IV. THREE VARIABLES MODEL

The major shortage of the two variables model is the assumption (3) listed in Section II, i.e., "the coverage of soil surface depends only on the living grass biomass." In fact, the non–living grass biomass can continuously be accumulated on the ground surface independently on the change in the living grass biomass, and the covering by the nonliving grasses reduces the evaporation very much effectively than by the living ones. Therefore, an important improvement of the dynamic model might be obtained by the introduction of a new variable. Therefore, the interactive dynamic model consists of three variables, the living grass biomass x, the nonliving grass biomass accumulated on the ground surface y, and the soil wetness z. The dynamic equations are as follows

$$\dot{x} = g(x,z) - d(x,z) - c \ , \tag{19}$$

$$\dot{y} = d(x,z) - l(x,y,,z) - b \ , \tag{20}$$

$$\dot{z} = (p+i) - w(x,y,z) - r(x,y,z) \ , \tag{21}$$

where the physical meanings of parameters and functions c, p, i, g, d, w and r are the same as in the two variables model but w and r now depend also on the nonliving grass biomass y; l is the loosing rate of accumulated biomass of nonliving grasses, and b the consuming rate of y by the non–natural way (for example, artificial covering or burning).

As a well known fact in the theory of nonlinear dynamics with more than two variables, the three variables model's behaviors can be very complicated, there exist not only equilibrium state, periodic solution and bifurcation, but also chaos and irregular solution of other types. Our preliminary results in the construction and investigation of grass field ecological model with three variables have shown that such complexity indeed exists, especially when the seasonal cycle of the parameters is also taken into account, and that the initial values of y and z play very important role. This is in agreement with the practice: before the planting of grasses one should first make the soil wet enough and cover the soil surface by thick enough nonliving vegetation.

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