The Theoretical Model of Atmospheric Turbulence Spectrum in Surface Layer①

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ABSTRACT

It is shown that the slope of energy spectrum obtained from the velocity solution of Kdv–Burgers equation lies between $-5/3$ and -2 in the dilogarithmic coordinates paper. The spectrum is very close to one of Kolmogorov's isotropic turbulence and Frisch's intermittent turbulence in inertial region. In this paper, the Kdv–Burgers equation to describe atmospheric boundary layer turbulence is obtained. In the equation, the $1/R_e$ corresponds to dissipative coefficient ν, $R_i^{1/2}$ to dispersive coefficient β, then $(\nu/2\beta)^2$ corresponds to $1/R_e^2 \cdot R_i$. We prove that the wave number corresponding to maximum energy spectrum $S(k)$ decreases with the decrease of stability (i.e., the increase of $(\nu/2\beta)^2$ in eddy–containing region. And the spectrim amplitude decreases with the increase of $(\nu/2\beta)^2$ (i.e., the decrease of stability). These results are consistent with actual turbulence spectrum of atmospheric surface layer from turbulence data.

Key words: Atmospheric turbulence, Energy spectrum

I. TURBULENCE SPECTRUM OF SURFACE LAYER

The energy spectrum in inertial region for homogeneous isotropic turbulence was derived as

$$S(k) \sim k^{-5/3} \tag{1}$$

using dimensional analysis method by Kolmogorov in 1941 (Monin, 1971). Here k is wave

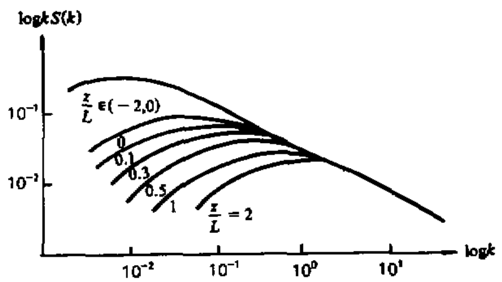

Fig. 1. The turbulence velocity u spectrum in atmospheric surface layer.

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number. The relation (1) is consistent with practical spectrum for atmospheric turbulent data in surface layer. But the behavior of energy spectrum for eddy–containing region depends upon atmospheric stability. If z / L is taken as stability parameter of atmospheric surface layer (L is Monin–Obukhov length), the wave number which is corresponding to maximum spectrum shifts to low–frequency with the decrease of z / L and the spectrum strength increases.

The actural turbulent velocity u spectrum is shown in Fig. 1 (Kaimal and Wyngard, 1972).

II. KDV–BURGERS EQUATION MODELING OF TURBULENCE

The turbulence can be described by Navier–Stokes equations. Because the equations are complex, the simple Kdv–Burgers equation which contains dissipative, dispersive and nonlinear factor in Navier–Stokes equations may be seen as the model of turbulence. That is

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + \beta\frac{\partial^3 u}{\partial x^3} = \nu\frac{\partial^2 u}{\partial x^2} \ . \tag{2}$$

where $\frac{\partial u}{\partial t}$ is a non–steady term, $u\frac{\partial u}{\partial x}$ a nonlinear term, $\nu\frac{\partial^2 u}{\partial x^2}$ the dissipative term, ν the coefficient of turbulence viscosity, the $\beta\frac{\partial^3 u}{\partial x^3}$ dispersive term, and β the coefficient of turbulence dispersion. We have shown (Liu Shida et al., 1992) that when $\nu^2 < 4\beta\sqrt{c^2 + 2A}$, the traveling wave of saddle–focus heteroclinic orbit may be represented as in Fig. 2.

According to the traveling wave in Fig. 2, the pattern of turbulence cascade process can be described as follows:

When a traveling wave travels along the $\xi = x - ct$ axis with the propagation speed c toward right (as shown in Fig. 2), the front field is first controlled by state u^2 and then the wave enters slowly into the part of solitary wave where the interaction between dispersion factor and nonlinear factor results in sufficient energy in eddy–containing region. After the amplitude of traveling wave achieves its maximum, the field is controlled by linear damped oscillation with dispersion. The amplitude and scale of wave are gradually decreased. It is the result of interaction between dispersive factor and dissipative factor. Finally the field is controlled by state u^2. That is the cascading process of turbulence eddy, an important property of turbulence.

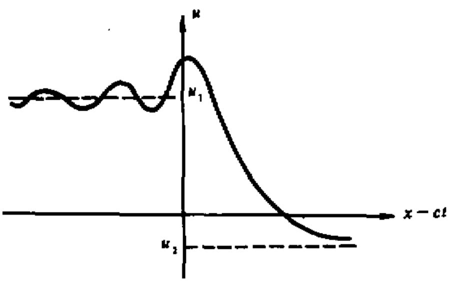

Fig. 2. The traveling wave solution of Kdv–Burgers equation.

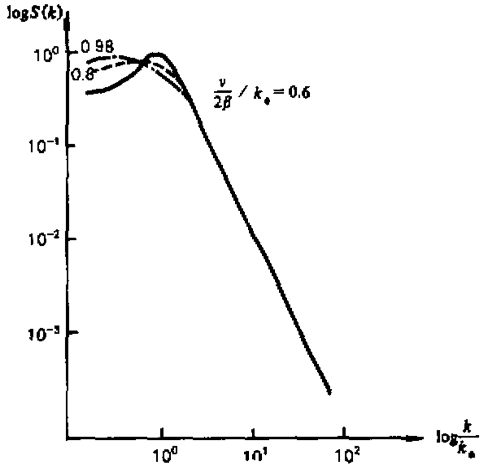

Fig. 3. The energy spectrum of turbulence by (3) in coordinate system ($\log S(k)$, $\log(k / k_0)$).

The energy spectrum of turbulence due to the pattern is

$$S(k)=\sqrt{\frac{2}{\pi}}\ \frac{(c^2+2A)}{2Lk_0^2}\ \cdot\frac{k_0^2(k_0^2-k_1^2+k^2)}{(k_0^2+k^2)^2-4k_1^2k^2}\ , \tag{3}$$

where $k_1=\sqrt{k_0^2-(\nu/2\beta)^2}$, $k_0^2=\frac{u_l-u_2}{2\beta}=\frac{\sqrt{c^2+2A}}{\beta}$, A is an integral constant. The energy of turbulence by (3) in dilogarithmic coordinates paper is shown in Fig. 3. The spectrum is similar to one by practical turbulence data.

From Fig. 3, we can see that the slope of energy spectrum lies between $-5/3$ to -2. The spectrum approaches one of the intermittent turbulence

$$S(k)\sim k^{-\frac{5}{3}-\frac{3-D}{3}} \tag{4}$$

by U. Frisch (1978). Here D is the Fractal dimension of intermittent turbulence. If D equals 3, Eq.(4) is reduced into the spectrum Eq.(1) of isotropic turbulence. If D is between 2 and 3, the slope of the energy spectrum is between -2 and $-5/3$.

The above result is true. If we take $\beta=0$ in the Kdv–Burgers equation, Eq.(2) is reduced into Burgers equation which was described as the model of turbulence by Burgers in 1948.

When $\beta=0$, we have:

$$k_0^2=\frac{u_1-u_2}{2\beta}\rightarrow\infty\ ,$$

$$k_1^2=\frac{2(u_1-u_2)\beta-\nu^2}{4\beta^2}\rightarrow\infty\ , \tag{5}$$

$$L\rightarrow\frac{2\nu}{u_1-u_2}\ .$$

Then the spectrum of (3) is reduced into

$$S(k) \to \frac{1}{k^2 - \nu^2} \sim k^{-2} \ . \tag{6}$$

That is just the spectrum of Burgers equation (Sagdeev et al., 1988).

III. THE CORRESPONDING MODE OF ATMOSPHERIC TURBULENCE

It is well known that the scale of turbulent eddy–containing region is corresponding to one of the internal–gravity waves in atmospheric boundary layer. Because the dispersive relation of internal gravity wave (Liu Shikuo et. al., 1991) is

$$\omega^2 = \frac{k^2 N^2}{k^2 + n^2} \ . \tag{7}$$

Here k and n are the wave number in x and y direction respectively, N the Brunt frequency.

From Eq.(7) we give

$$\omega = \frac{kN}{n\sqrt{1 + (k/n)^2}} \ . \tag{8}$$

If $k^2 < < n^2$ or $k < < n$, we give the expansion of Taylor series into the second term

$$\omega = \frac{kN}{n}\left(1 - \frac{1}{2}\frac{k^2}{n^2}\right) = kc_1 - \frac{1}{2}\frac{N}{n^3}k^3 \ . \tag{9}$$

Here $c_1 = \frac{N}{n} = \frac{NH}{nH} = \frac{c_a}{\pi}$ is the phase speed in x direction.

Because the dispersive relation of linear Kdv–equation

$$\frac{\partial u}{\partial t} + c_0 \frac{\partial u}{\partial x} + \beta \frac{\partial^3 u}{\partial x^3} \tag{10}$$

is

$$\omega = kc_0 - \beta k^3 \ . \tag{11}$$

Comparing the Eq.(9) with Eq.(11), we get the Kdv equation corresponding to internal–gravity wave

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + \frac{1}{2}\frac{N}{n^3}\frac{\partial^3 u}{\partial x^3} = 0 \ . \tag{12}$$

If the dispersive term $\nu\frac{\partial^2 u}{\partial x^2}$ is added into Kdv equation, we give Kdv–Burgers equation to describe atmospheric boundary layer turbulence, i.e.

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + \frac{1}{2}\frac{N}{n^3}\frac{\partial^3 u}{\partial x^3} = \nu\frac{\partial^2 u}{\partial x^2} \ . \tag{13}$$

We take H as length scale, U as velocity scale, H/U as time scale, $n = D/H$, then the dimensionless form of (13) is

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+\frac{1}{2\pi^3}R_i^{1/2}\frac{\partial^3 u}{\partial x^3}=\frac{1}{R_e}\frac{\partial^2 u}{\partial x^2}\ . \tag{14}$$

Here $R_e=\frac{UH}{\nu}$ and $R_i=\frac{N^2}{(\frac{U}{N})^2}$ are Reynolds number and Richardson number respectively.

The equation (14) is just the Kdv–Burgers equation model for atmospheric boundary layer turbulence. The equation (14) is compared to (2), we see that the dissipative coefficient ν is corresponding to $1/R_e$, and the dispersive coefficient β is corresponding to $R_i^{1/2}$. Then we see that $(\frac{\nu}{2\beta})^2$ is corresponding to $\frac{1}{R_i R_e^2}=-\frac{R_a}{P_r}$. Here $R_a=-P_r R_e^2 R_i$ is Rayleigh number, P_r is Prandtl number.

From here it is concluded that the increase of $\frac{\nu}{2\beta}$ means the decrease of R_i. The stability parameter z/L is proportional to R_i in atmospheric surface layer. Hence the increase of $\frac{\nu}{2\beta}$ means just the decrease of stability z/L.

In the following we discuss the wave number k_m which corresponds to maximum $S(k)$.

From Eq.(3), we derive the $S(k)$ with respect to k and make $\frac{dS(k)}{dk}$ as zero. Then the k_m corresponding to maximum $S(k)$ satisfies the equation

$$k_m[(k_m^2)^2+2(k_0^2-k_1^2)k_m^2+(k_0^4+4k_1^4-6k_0^2k_1^2)]=0\ . \tag{15}$$

From Eq.(15) we obtain

$$k_m^2=(k_1^2-k_0^2)+\sqrt{k_1^2(4k_0^2-3k_1^2)}\ . \tag{16}$$

Here we consider only the root with $k_m^2>0$ in Eq.(16). The k_m^2 as a function of stability $(\frac{\nu}{2\beta})^2$ and the corresponding $S(k_m)$ are shown in Table 1.

Table 1. k_m^2 $S(k_m)$ Change with $(\frac{\nu}{2\beta})^2$

$(\frac{\nu}{2\beta})^2$	0.04	0.16	0.25	0.36	0.49	0.64	0.81	0.85	0.90	0.93	0.96	0.98	0.99
k_m^2	1.04	1.11	1.15	1.15	1.12	1.03	0.80	0.73	0.61	0.52	0.39	0.28	0.20
$S(k_m)$	6.43	1.76	1.19	0.90	0.73	0.63	0.61	0.62	0.64	0.67	0.72	0.78	0.83

From Table 1 we see that the k_m is almost constant when $(\frac{\nu}{2\beta})^2$ is less than 0.5. But the k_m decreases obviously when $(\frac{\nu}{2\beta})^2$ is greater than 0.5. In the case of $(\frac{\nu}{2\beta})^2>0.8$, the result is consistent with that in Fig. 1. The k_m/k_0 changes with $\frac{\nu}{2\beta}$ that as in Fig. 1.

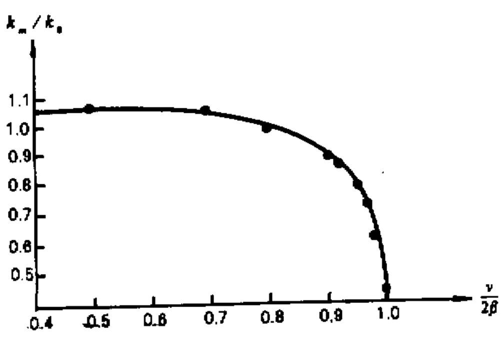

Fig. 4. The k_m / k_0 as a function of $\frac{\nu}{2\beta}$.

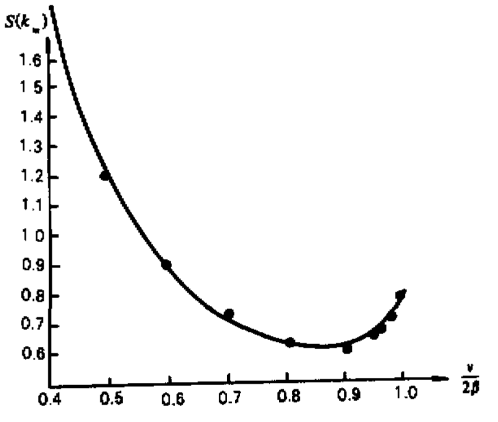

Fig. 5. The spectrum amplitude as a function of $\frac{\nu}{2\beta}$.

The k_m / k_0 as a function of $\frac{\nu}{2\beta}$ is shown in Fig. 4.

From Table 1, we also see that the spectrum amplitude decreases with the increase of $\frac{\nu}{2\beta}$ until 0.9. But the amplitude increases when $\frac{\nu}{2\beta} > 0.9$ implies that the small stability (or z / L) is necessary. The z / L is less than 0.2 in Fig. 1. The spectrum amplitude as a function of $\frac{\nu}{2\beta}$ is shown in Fig. 5.

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