A Two–Step Shape–Preserving Advection Scheme①

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ABSTRACT

This paper proposes a new two–step non–oscillatory shape–preserving positive definite finite difference advection transport scheme, which merges the advantages of small dispersion error in the simple first–order upstream scheme and small dissipation error in the simple second–order Lax–Wendroff scheme and is completely different from most of present positive definite advection schemes which are based on revising the upstream scheme results. The proposed scheme is much less time consuming than present shape–preserving or non–oscillatory advection transport schemes and produces results which are comparable to the results obtained from the present more complicated schemes. Elementary tests are also presented to examine the behavior of the scheme.

Key words: Shape–preserving, Non–oscillation positive definite advection scheme

I. INTRODUCTION

In numerical modeling of atmospheric phenomena, it is often necessary to solve the advection equation for positive definite scalar functions. Processes associated with water in the atmosphere are particularly difficult to model accurately. There are very large horizontal and vertical spatial variations in the moisture field and very strong and small–scale sources and sinks of moisture associated with phase change.Yet because it is so important both in vapor and in cloud forms and because the heat release associated with phase change can be large, it is critical to model reasonably and accurately the processes which affect the distribution of water.

Accuracy is a primary requirement for any numerical algorithm. However, it is important to note that high formal accuracy as evaluated from a Taylor–series expansion of the algorithm does not always imply an accurate solution to a particular problem. There are important physical situations (shocks, and fronts for example) where the phenomena to be modelled are (quasi–) discontinuous, and the evaluation according to Taylor–series expansion error estimates is relatively meaningless. Moisture is quasi–discontinuous field as compared with other atmosphere fields, so accuracy for moisture field might have different implication. Using second–order or high–order–accuracy advection schemes to solve moisture advection straightforwardly can lead to significant erroneous dispersion and cause some difficulties because negative values arise in the solution.

Numerical methods with shape–preserving (SP) properties are designed to ensure that certain properties relating to the shape of the solution of continuous equations are preserved in the discrete solutions. There are a variety of names (i.e. positivity, monotonicity, and

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non–oscillatory) for properties which are in principle equivalent to the shape–preserving property. For a scalar field undergoing pure advection, new maxima or minima cannot develop, so extremes are limited to the value they had at a previous time. It follows that fields which are initially positive everywhere cannot become negative. The well shape–preserving scheme can reduce the dispersion errors, and also reduce the non–physical charactor of solution.

Care about the positiveness of the solution leads to the use of upstream difference or other low–order schemes (Soong and Ogura, 1973) which produce no dispersive "ripples" but suffer from excessive numerical diffusion. In the last twenty years, many schemes have been devised for a possible resolution of this dilemma. Especially, Smolarkiewicz(1983,1984,1986) used a predictor–corrector sequence to the upstream scheme so that the corrector reversed the effect of the implicit diffusion in the upstream predictor and Smolarkiewicz (1990) perfected a monotone version of the scheme which merges the flux–corrected transport methodology of Boris and Book (1976) and Zalesak(1979) with the prodictor–iterative–corrector scheme.

Notwithstanding the success of those methods, many modellers still employ the simpler first–order upstream scheme, which has excessive numerical diffusion, or second–order Lax–Wendroff conservative method, which introduces difficulties because of negative values, due to computational ease and cost (Takacs, 1985). Any useful numerical method must be simple, flexible, and inexpensive in the application to realistic problems. Almost all of present resultful positive definite advection schemes are complicated and computer–time expensive in different degrees.

This paper presents a two–step shape–preserving advection scheme (hereafter, TSPAS) which is completely different from other predictor–corrector schemes. As it will be shown later, the scheme is much less time consuming than other shape–preserving or non–oscillatory schemes and produces results which are comparable to the results obtained from other more complicated hybrid schemes.

In Section II the scheme and its development are presented. Section III contains the proof of the consistency, conservation, positivity, shape–preservation and stability of the scheme. Section IV presents the results of elementary tests. Section V concludes with a discussion.

II. THE DEVELOPMENT OF THE TSPAS

The equation to be solved is the continuity equation describing the advection of a nondiffusive quantity in a flow field, i.e.,

$$\frac{\partial F}{\partial t} + \nabla \cdot (\vec{V}F) = 0, \tag{1}$$

where $F(x, y, z, t)$ is the nondiffusive scalar quantity, $\vec{V} = (u, v, w)$ is the velocity vector, and x, y, z, t are the space and time independent variables. For simplicity the one–dimensional case of (1),

$$\frac{\partial F}{\partial t} + \frac{\partial uF}{\partial x} = 0 \tag{2}$$

will be discussed. As it will be shown later, the multi–dimensional case is a simple generalization of the one–dimensional results.

Considering a transport shape–preserving rule (hereafter, TSPR): New extremes for a scalar field undergoing pure advection should be limited to the values they had at a previous time, which is used to define the implication of shape–preservation in this paper, and the necessary finite–difference accuracy for most good continuous field, we propose a hybrid advection transport methodology which uses high–order–accuracy scheme in regions where the transported flow variable is smooth and uses upstream scheme in the vicinity of a few mesh points where the shape–preserving rule is broken by the high–order–accuracy scheme.

Considering computation ease and simple, second–order Lax–Wendroff conservation scheme (L–W–S),

$$F_i^* = F_i^n - \frac{\Delta t}{2\Delta x}[u_{i+\frac{1}{2}}^n(F_{i+1}^n + F_i^n) - u_{i-\frac{1}{2}}^n(F_i^n + F_{i-1}^n)] + \frac{\Delta t}{2\Delta x}[|u^2|_{i+\frac{1}{2}}^n \frac{\Delta t}{\Delta x}(F_{i+1}^n - F_i^n) - |u^2|_{i-\frac{1}{2}}^n \frac{\Delta t}{\Delta x}(F_i^n - F_{i-1}^n)], \tag{3}$$

is chosen as the high–order–accuracy scheme.

Based on the scheme (3) and upstream scheme (U–S)

$$F_i^{n+1} = F_i^n - \frac{\Delta t}{2\Delta x}[u_{i+\frac{1}{2}}^n(F_{i+1}^n + F_i^n) - u_{i-\frac{1}{2}}^n(F_i^n + F_{i-1}^n)] + \frac{\Delta t}{2\Delta x}[|u|_{i+\frac{1}{2}}^n(F_{i+1}^n - F_i^n) - |u|_{i-\frac{1}{2}}^n(F_i^n - F_{i-1}^n)], \tag{4}$$

we propose the following two–step scheme (TSPAS)

$$F_i^* = F_i^n - \beta\frac{\Delta t}{2\Delta x}[u_{i+\frac{1}{2}}^n(F_{i+1}^n + F_i^n) - u_{i-\frac{1}{2}}^n(F_i^n + F_{i-1}^n)] + \beta\frac{\Delta t}{2\Delta x}[|u^2|_{i+\frac{1}{2}}^n \frac{\Delta t}{\Delta x}(F_{i+1}^n - F_i^n) - |u^2|_{i-\frac{1}{2}}^n \frac{\Delta t}{\Delta x}(F_i^n - F_{i-1}^n)], \tag{5}$$

$$F_i^{n+1} = F_i^n - \frac{\Delta t}{2\Delta x}[u_{i+\frac{1}{2}}^n(F_{i+1}^n + F_i^n) - u_{i-\frac{1}{2}}^n(F_i^n + F_{i-1}^n)] + \frac{\Delta t}{2\Delta x}[|\tilde{u}|_{i+\frac{1}{2}}^n(F_{i+1}^n - F_i^n) - |\tilde{u}|_{i-\frac{1}{2}}^n(F_i^n - F_{i-1}^n)], \tag{6}$$

where $\beta \geqslant 1$ and

$$|\tilde{u}|_{i+\frac{1}{2}}^n = C_{i+\frac{1}{2}}|u|_{i+\frac{1}{2}}^n, \tag{7}$$

$$|\tilde{u}|_{i-\frac{1}{2}}^n = C_{i-\frac{1}{2}}|u|_{i-\frac{1}{2}}^n, \tag{8}$$

$$C_{i+\frac{1}{2}} = C_{i+\frac{1}{2}}^* + (1 - C_{i+\frac{1}{2}}^*)\frac{|u|_{i+\frac{1}{2}}^n \Delta t}{\Delta x}, \tag{9}$$

$$C_{i-\frac{1}{2}} = C_{i-\frac{1}{2}}^* + (1 - C_{i-\frac{1}{2}}^*)\frac{|u|_{i-\frac{1}{2}}^n \Delta t}{\Delta x}, \tag{10}$$

$$C_{i+\frac{1}{2}}^* = 0.5\ (\frac{|A_i| + A_i}{|A_i| + \varepsilon} + \frac{|A_{i+1}| + A_{i+1}}{|A_{i+1}| + \varepsilon})\ - 0.25\frac{(|A_i| + A_i)(|A_{i+1}| + A_{i+1})}{|A_i||A_{i+1}| + \varepsilon}, \tag{11}$$

$$C_{i-\frac{1}{2}}^* = 0.5\ (\frac{|A_i| + A_i}{|A_i| + \varepsilon} + \frac{|A_{i-1}| + A_{i-1}}{|A_{i-1}| + \varepsilon})\ - 0.25\frac{(|A_i| + A_i)(|A_{i-1}| + A_{i-1})}{|A_i||A_{i-1}| + \varepsilon}, \tag{12}$$

$$A_i = (F_i^* - F_{i\,max}^n)(F_i^* - F_{i\,min}^n), \tag{13}$$

$$F_{i\,max}^n = max(F_{i-1}^n, F_i^n, F_{i+1}^n), \tag{14}$$

$$F_{i\,min}^n = min(F_{i-1}^n, F_i^n, F_{i+1}^n), \tag{15}$$

here ε is a small value, e.g., 10^{-15}. $A_i \leqslant 0$ means that first predicting value at ith point satisfies the TSPR, i.e.

$$F^n_{i\,min} \leqslant F^*_i \leqslant F^n_{i\,max} \ .$$

$A_i \geqslant 0$ means that first predicting value at ith point violates the TSPR, i.e.

$$F^*_i \leqslant F^n_{i\,min} \quad or \quad F^*_i \geqslant F^n_{i\,max} \ .$$

Formally, the coefficient formulas (7)—(15) are very complex. In fact, Eq.(5) is equal to the scheme (3) but with larger time increment, and Eq.(6) is a formal upstream scheme, or a hybrid scheme which is same as upstream scheme (4) at $A_i > 0$ or $A_{i+1} > 0$ and $A_{i-1} > 0$, and same as Lax–Wendroff scheme (3) at $A_i \leqslant 0, A_{i+1} \leqslant 0$ and $A_{i-1} \leqslant 0$, and a combination of scheme (3) and (4) at $A_i \leqslant 0$ but $A_{i+1} > 0$ or $A_{i-1} > 0$.

The designing methodology of present positive definite advection schemes is always based on correcting or revising the upstream scheme results. The proposed schemes (TSPAS) (5) and (6), which use upstream scheme only for a few mesh points where the TSPR is broken, are a simple two–step procedures and are completely different from the Smolarkiewicz scheme and other two–step predictor–corrector schemes. The scheme has second–order–accuracy except "first–order–accuracy" near same quasi–discontinuous or sharp gradient points. The consistency, conservation, positivity, shape–preservation and stability of the proposed scheme will be proved in next section.

III. CONSISTENCY, CONSERVATION, POSITIVITY, SHAPE–PRESERVATION AND STABILITY

For transport problems, it is convenient to characterize the differences between different numerical schemes in terms of the following desirable properties of transport algorithms: accuracy, stability, consistency, conservation, positivity, shape–preservation and computational rationality.

The proposed scheme has quasi–second–order accuracy and is sufficiently simple as mentioned in last section.

The consistency of the scheme can be obtained because Eq.(6) approximates with the second–order accuracy equation

$$\frac{\partial F}{\partial t} + \frac{\partial}{\partial x}(uF) = \frac{\partial}{\partial x}(Kimp\frac{\partial F}{\partial x}), \tag{16}$$

where $Kimp = 0.5\ (\Delta x|\tilde{u}| - \Delta t u^2)$. When $\Delta x, \Delta t \to 0$, $Kimp \to 0$, Eq.(16)$\to$ Eq.(2), which means that the scheme has consistency.

It is easy to understand that for the scalar quantity transport problem, the positivity and mass conservation also imply stability because the total quantity $\sum_i F_i \Delta x$ can be defined as a norm function.

From Eq.(6), it can be shown

$$\sum_i^I F_i^{n+1}\Delta x = \sum_i^I F_i^n \Delta x + 0.5\Delta t[u^n_{I+\frac{1}{2}}(F^n_{I+1} + F^n_I) - u^n_{\frac{1}{2}}(F^n_1 + F^n_0)]$$
$$+ 0.5\Delta t[|\tilde{u}|^n_{I+\frac{1}{2}}(F^n_{I+1} - F^n_I) - |\tilde{u}|^n_{\frac{1}{2}}(F^n_1 - F^n_0)], \tag{17}$$

where I is the total grid number. When the last two terms of Eq.(17) are zero at certain boundary conditions (cyclic boundary condition for example), the scheme exists conservation property

$$\sum_{i}^{I} F_i^{n+1}\Delta x = \sum_{i}^{I} F_i^n \Delta x \ . \tag{18}$$

The positivity and shape–preservation of the scheme can be shown as follows:

(1) $A_i < 0, A_{i+1} < 0, A_{i-1} \leqslant 0$, then $C_{i+\frac{1}{2}} = \dfrac{|u|_{i+\frac{1}{2}}^n \Delta t}{\Delta x}$, $C_{i-\frac{1}{2}} = \dfrac{|u|_{i-\frac{1}{2}}^n \Delta t}{\Delta x}$,

$$\beta F_i^{n+1} = (\beta-1)F_i^n + F_i^* ,$$

$$F_i^{n+1} = \frac{(\beta-1)F_i^n + F_i^*}{1+(\beta-1)} ,$$

$$0 \leqslant F_{i\ \min}^n \leqslant F_i^{n+1} \leqslant F_{i\ \max}^n ,$$

(2) $A_i > 0$, or $A_{i+1} > 0$ and $A_{i-1} > 0$, then $C_{i+\frac{1}{2}} = C_{i-\frac{1}{2}} = 1$.

The scheme is equal to upstream scheme, positivity and shape–preservation exist under the following condition,

$$\max_i\left(\frac{|u|_{i+\frac{1}{2}}^n \Delta t}{\Delta x}\right) \leqslant \alpha, \tag{19}$$

where $\alpha = 1$ except $\alpha = \dfrac{1}{2}$ at appearing spatial discontinuous velocity ($u_{i+\frac{1}{2}} \cdot u_{i-\frac{1}{2}} < 0$ for example).

(3) $A_i \leqslant 0, A_{i+1} \leqslant 0, A_{i-1} > 0$, then $C_{i+\frac{1}{2}} = \dfrac{|u|_{i+\frac{1}{2}}^n \Delta t}{\Delta x}$, $C_{i-\frac{1}{2}} = 1$,

$$\beta F_i^{n+1} = (\beta-1)F_i^n + F_i^* - \frac{\beta}{2}\frac{|u|_{i-\frac{1}{2}}^n \Delta t}{\Delta x}\left(1 - \frac{|u|_{i-\frac{1}{2}}^n \Delta t}{\Delta x}\right)(F_i^n - F_{i-1}^n)$$

$$= \left[(\beta-1) - \frac{\beta}{2}\frac{|u|_{i-\frac{1}{2}}^n \Delta t}{\Delta x}\left(1 - \frac{|u|_{i-\frac{1}{2}}^n \Delta t}{\Delta x}\right)\right]F_i^n$$

$$+ \frac{\beta}{2}\frac{|u|_{i-\frac{1}{2}}^n \Delta t}{\Delta x}\left(1 - \frac{|u|_{i-\frac{1}{2}}^n \Delta t}{\Delta x}\right)F_{i-1}^n + F_i^* \ .$$

Let

$$\gamma_{i-\frac{1}{2}} = \frac{|u|_{i-\frac{1}{2}}^n \Delta t}{\Delta x}\left(1 - \frac{|u|_{i-\frac{1}{2}}^n \Delta t}{\Delta x}\right),$$

then

$$F_i^{n+1} = \frac{1}{\beta}\left[F_i^* + \left(\beta - 1 - \frac{\beta}{2}\gamma_{i-\frac{1}{2}}\right)F_i^n + \frac{\beta}{2}\gamma_{i-\frac{1}{2}}F_{i-1}^n\right] \ .$$

If

$$\beta - 1 - \frac{\beta}{2}\gamma_{i-\frac{1}{2}} \geqslant 0 \quad i.e. \quad \beta \geqslant \frac{2}{2-\gamma_{i-\frac{1}{2}}},$$

then

$$0 \leqslant F^n_{i\ min} \leqslant F^{n+1}_i \leqslant F^n_{i\ max} \ .$$

(4) $A_i \leqslant 0, A_{i-1} \leqslant 0, A_{i+1} > 0$, then $C_{i-\frac{1}{2}} = \frac{|u|^n_{i-\frac{1}{2}}\Delta t}{\Delta x}$, $C_{i+\frac{1}{2}} = 1$.

This situation is similar to (3). If

$$\beta \geqslant \frac{2}{2-\gamma_{i+\frac{1}{2}}}, \quad \text{then} \quad 0 \leqslant F^n_{i\ \min} \leqslant F^{n+1}_i \leqslant F^n_{i\ \max} \ .$$

It is obviously that non–oscillation shape–preservation, positivity and stability exist under condition (19) and

$$\beta \geqslant \max\left(\frac{2}{2-\gamma_{i+\frac{1}{2}}}, \frac{2}{2-\gamma_{i-\frac{1}{2}}}\right) \ . \tag{20}$$

In general, in order to obtain minimized dissipation errors, we replace formula (20) by

$$\beta = \max\left(\frac{2}{2-\gamma_{i+\frac{1}{2}}}, \frac{2}{2-\gamma_{i-\frac{1}{2}}}\right) \qquad (20)'$$

in the scheme (6).

The shape–preserving property and inexpensive computation of the scheme will be proved further in the following section by elementary tests.

IV. ELEMENTARY TESTS

To examine the behavior of the proposed scheme a series of elementary tests and simulation tests for real atmospheric problems have been done. Results for a one–dimensional step profile translation and a two–dimensional solid body rotation will be shown only in this section due to limited space.

One of the most common ways of measuring the relative merit of a given numerical scheme for advection is to analyse the scheme's dissipation and dispersion properties. In this paper, three kinds of error measurements were taken to represent the total error, the dissipation error, and the dispersion error. The total error was defined to be the mean square error for the experiment given by

$$E_{TOT} = \frac{1}{M}\sum_i (F_T - F_D)^2, \tag{21}$$

where F_T is the true solution, F_D is the finite–difference solution, and M is the total number of grid–points. Because of arithmetic mean $\overline{F}_D = \overline{F}_T$ for conservation scheme, Formula (21) can be rewritten as

$$\begin{aligned} E_{TOT} &= \frac{1}{M}\sum_i [(F_T - \overline{F}_T) - (F_D - \overline{F}_D)]^2 \\ &= [\sigma(F_T) - \sigma(F_D)]^2 + 2(1-\rho_{T,D})\sigma(F_T)\sigma(F_D), \end{aligned} \tag{22}$$

where

$$\sigma(F)=\sqrt{\frac{1}{M}\sum_{i}(F-\bar{F})^{2}}\,,$$

and
$$\rho_{T,D}=\frac{\frac{1}{M}\sum_{i}(F_T-\bar{F}_T)(F_D-\bar{F}_D)}{\sigma(F_T)\sigma(F_D)}$$

are the variance of F and the correlation coefficient between F_T and F_D.

The dissipation and dispersion error are defined as

$$E_{DISS}=[\sigma(F_T)-\sigma(F_D)]^2, \tag{23}$$

$$E_{DISP}=2(1-\rho_{T,D})\sigma(F_T)\sigma(F_D), \tag{24}$$

respectively.

(1) *A step profile testing*

Figs. 1 and 2 show the results for advecting one–step initial profile through two and ten translations of the 70–grid–point domain using cyclic boundary conditions for the case of one–dimensional uniform flow (u=const) with $\Delta x=1.0$, $u=1.0$, $\Delta t=0.7$ (One translation is equal to 200 time–steps). In each figure, the initial field and three numerical results for upstream scheme(4), L–W scheme (3) and the proposed scheme (6) are shown.

Figs. 1 and 2 indicate that the proposed scheme has well shape–preserving property with non–oscillation, positivity and small numerical diffusion, and Table 1 indicates that the proposed scheme merges the advantages of small dispersion error in upstream scheme and small dissipation error in L–W scheme and has the smallest total errors.

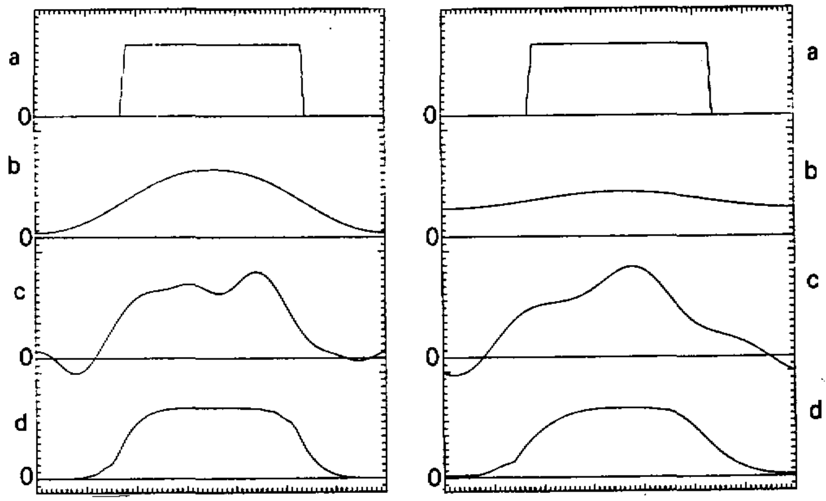

Fig. 1. (LEFT) and Fig. 2. (RIGHT). The results for advecting one–step initial profile through two and ten translations. (a) Initial field; (b) U–S results; (c) L–W–S results; (d) TSPAS results. The increment for each vertical scale and horizontal scale are 0.1 and 1, respectively.

Table 1 shows the total errors, the dissipation and dispersion errors of the three numerical results for the step initial profile after two, five and ten translations.

Table 1. Comparison of Numerical Errors for a One–step Profile Translation

E \ T, S \ K	2nd translation			5th translation			10th translation		
	E_{TOT}	E_{DISS}	E_{DISP}	E_{TOT}	E_{DISS}	E_{DISP}	E_{TOT}	E_{DISS}	E_{DISP}
U–S	6.19×10^{-2}	3.20×10^{-2}	2.98×10^{-2}	1.13×10^{-1}	9.40×10^{-2}	1.92×10^{-2}	1.82×10^{-1}	1.74×10^{-1}	8.29×10^{-3}
L–W–S	3.63×10^{-2}	6.30×10^{-4}	3.56×10^{-2}	4.70×10^{-2}	1.01×10^{-3}	4.60×10^{-2}	8.23×10^{-2}	1.50×10^{-3}	8.10×10^{-2}
TSPAS	2.15×10^{-2}	4.20×10^{-3}	1.73×10^{-2}	3.09×10^{-2}	7.40×10^{-3}	2.35×10^{-2}	3.81×10^{-2}	1.34×10^{-2}	2.48×10^{-2}

notes: T = Translations, K = Kinds, E = Errors and S = Schemes

(2) *Two–dimensional solid body rotation testing*

In order to further demonstrate the shape–preserving property and small diffusion of the proposed scheme and have a comparison with Smolarkiewicz scheme and other positive definite schemes, the two–dimensional solid body rotation test was chosen as Smolarkiewicz(1983,1984,1986,1990). The grid space is $(100\Delta x$ by $100\Delta y)$, $\Delta x = \Delta y = 1$, $\Delta t = 0.1$, and the constant angular velocity $\omega = 0.1$. The velocity components are $u = -\omega(y - y_o)$ and $v = \omega(x - x_o)$ where $(x_o, y_o) = (50\Delta x, 50\Delta y)$. In this circumstance, one full rotation around the point (x_o, y_o) is equivalent to about 628 iterations (i.e., time steps). The initial condition was assumed in a form of a cone with base radius $15\Delta x = 15\Delta y$ and maximum value 4.0 in point $(X_m, Y_m) = (75\Delta x, 50\Delta y)$ (Fig. 3a). In all cases the same boundary conditions were used. The first spatial partical derivative in the normal direction was assumed to vanish at the outflow boundary. The undisturbed initial value of the field was assumed to exist at the inflow boundary.

The multi–dimensional scheme of the proposed scheme is a simple generalization of the one–dimensional formulae (5) and (6), which may be used in the time–splitting or the combined form optionally. In the time splitting form the stability and consistency of the scheme are a consequence of the stability and consistency of the one–dimensional scheme. However, because of the data transfer, this method is more time–consuming and expensive than the combined scheme, so the latter was chosen in this section. In two–dimensional space, when the scheme is applied in combined form, the scheme may be written as

$$
\begin{aligned}
F^{*}_{i,j} = F^{n}_{i,j} &- \beta\frac{\Delta t}{2\Delta x}[u^{n}_{i+\frac{1}{2},j}(F^{n}_{i+1,j} + F^{n}_{i,j}) - u^{n}_{i-\frac{1}{2},j}(F^{n}_{i,j} + F^{n}_{i-1,j})] \\
&+ \beta\frac{\Delta t}{2\Delta x}[|u^{2}|^{n}_{i+\frac{1}{2},j}\frac{\Delta t}{\Delta x}(F^{n}_{i+1,j} - F^{n}_{i,j}) - |u^{2}|^{n}_{i-\frac{1}{2},j}\frac{\Delta t}{\Delta x}(F^{n}_{i,j} - F^{n}_{i-1,j})] \\
&- \beta\frac{\Delta t}{2\Delta y}[v^{n}_{i,j+\frac{1}{2}}(F^{n}_{i,j+1} + F^{n}_{i,j}) - v^{n}_{i,j-\frac{1}{2}}(F^{n}_{i,j} + F^{n}_{i,j-1})] \\
&+ \beta\frac{\Delta t}{2\Delta y}[|v^{2}|^{n}_{i,j+\frac{1}{2}}\frac{\Delta t}{\Delta y}(F^{n}_{i,j+1} - F^{n}_{i,j}) - |v^{2}|^{n}_{i,j-\frac{1}{2}}\frac{\Delta t}{\Delta y}(F^{n}_{i,j} - F^{n}_{i,j-1})], \qquad (25) \\
F^{n+1}_{i,j} = F^{n}_{i,j} &- \frac{\Delta t}{2\Delta x}[u^{n}_{i+\frac{1}{2},j}(F^{n}_{i+1,j} + F^{n}_{i,j}) - u^{n}_{i-\frac{1}{2},j}(F^{n}_{i,j} + F^{n}_{i-1,j})] \\
&+ \frac{\Delta t}{2\Delta x}[|\tilde{u}|^{n}_{i+\frac{1}{2},j}(F^{n}_{i+1,j} - F^{n}_{i,j}) - |\tilde{u}|^{n}_{i-\frac{1}{2},j}(F^{n}_{i,j} - F^{n}_{i-1,j})]
\end{aligned}
$$

$$-\frac{\Delta t}{2\Delta y}[v^n_{i,j+\frac{1}{2}}(F^n_{i,j+1}+F^n_{i,j})-v^n_{i,j-\frac{1}{2}}(F^n_{i,j}+F^n_{i,j-1})]$$
$$+\frac{\Delta t}{2\Delta y}[|\tilde{v}|^n_{i,j+\frac{1}{2}}(F^n_{i,j+1}-F^n_{i,j})-|\tilde{v}|^n_{i,j-\frac{1}{2}}(F^n_{i,j}-F^n_{i,j-1})], \tag{26}$$

where the coefficient formulas are very similar to (7)—(15) (neglected), the shape–preservation, consistency and stability conditions are also the sufficient stability conditions of two–dimensional upstream scheme which are

$$\max\left(\frac{|u|^n_{i+\frac{1}{2},j}\Delta t}{\Delta x},\ \frac{|v|^n_{i,j+\frac{1}{2}}\Delta t}{\Delta y}\right)\leqslant\alpha\leqslant 1 \tag{27}$$

and

$$\beta=\frac{2}{2-2\gamma_{i,j}}\ , \tag{28}$$

where

$$\gamma_{i,j}=\max\Big(\frac{|u|^n_{i-\frac{1}{2},j}\Delta t}{\Delta x}(1-\frac{|u|^n_{i-\frac{1}{2},j}\Delta t}{\Delta x}),\frac{|u|^n_{i+\frac{1}{2},j}\Delta t}{\Delta x}(1-\frac{|u|^n_{i+\frac{1}{2},j}\Delta t}{\Delta x}),$$
$$\frac{|v|^n_{i,j+\frac{1}{2}}\Delta t}{\Delta y}(1-\frac{|u|^n_{i,j+\frac{1}{2}}\Delta t}{\Delta y}),\frac{|v|^n_{i,j-\frac{1}{2}}\Delta t}{\Delta y}(1-\frac{|u|^n_{i,j-\frac{1}{2}}\Delta t}{\Delta y})\Big)\ .$$

It is worth noting that L–W scheme in the two–dimensional combined form has neglected cross–derivative term

$$\frac{(\Delta t)^2\overline{uv}}{4\Delta x\Delta y}(F^n_{i+1,j+1}+F^n_{i-1,j-1}-F^n_{i+1,j-1}-F^n_{i-1,j+1}),$$
$$\bar{u}=0.25(u_{i+\frac{1}{2},j}+u_{i-\frac{1}{2},j}+u_{i,j+\frac{1}{2}}+u_{i,j-\frac{1}{2}}),$$
$$\bar{v}=0.25(v_{i+\frac{1}{2},j}+v_{i-\frac{1}{2},j}+v_{i,j+\frac{1}{2}}+v_{i,j-\frac{1}{2}}), \tag{29}$$

and as will be shown later, introducing the cross term will lead to better shape–preserving solution.

In addition, because the values between adjacent time steps have small oscillation due to numerical errors we modify

$$F^n_{i,j\,\min}=\min(F^n_{i,j},F^n_{i+1,j},F^n_{i-1,j},F^n_{i,j+1},F^n_{i,j-1}), \tag{30}$$
$$F^n_{i,j\,\max}=\max(F^n_{i,j},F^n_{i+1,j},F^n_{i-1,j},F^n_{i,j+1},F^n_{i,j-1}), \tag{31}$$

to

$$F^n_{i,j\,\min}=\min(F^n_{i,j},F^n_{i+1,j},F^n_{i-1,j},F^n_{i,j+1},F^n_{i,j-1},F^{n-1}_{i,j\,\min}), \tag{32}$$
$$F^n_{i,j\,\max}=\max(F^n_{i,j},F^n_{i+1,j},F^n_{i-1,j},F^n_{i,j+1},F^n_{i,j-1},F^{n-1}_{i,j\,\max}), \tag{33}$$

in order to acquire minimized numerical diffusion.

Fig. 3 shows the initial condition for this experiment and the numerical solution of the proposed scheme after six full rotations (i.e. 3770 iterations). Fig. 3a is the initial condition, Fig. 3b is the result without cross term and with no modified extreme limitations (30) and (31), Fig. 3c is the same as Fig. 3b except including cross term (29) and Fig. 3d is the same as

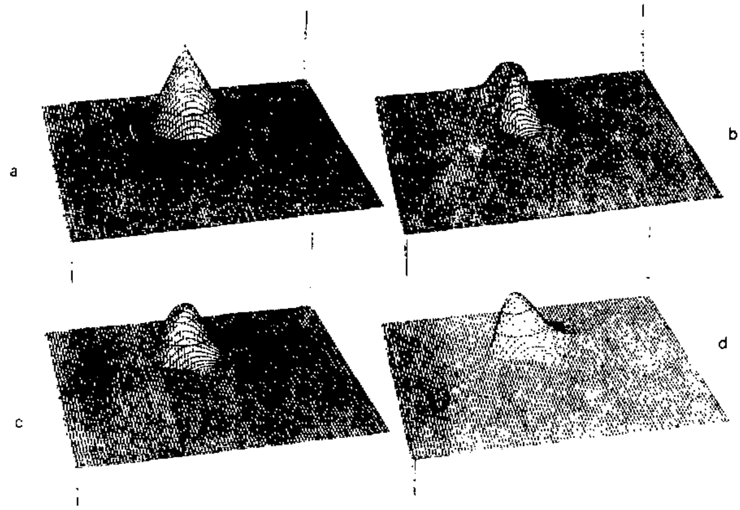

Fig. 3. Initial condition and numerical solutions for three versions of TSPAS using solid–body rotation. The reference spikes in the upper–right and the lower–left corners represent the initial– and minus half of the initial–height of the cone: (a) initial field; (b) without cross term and with no modified extreme limitation; (c) the same as (b) except including cross term; (d) the same as (c) except with modified extreme limitation.

Fig. 3c except with modified extreme limitations (32) and (33). The maximal values of the presented solutions are 2.45, 2.5 and 2.8, respectively.

Table 2. shows the numerical error comparison and maximal and minimal values after six full rotations for three schemes as in Table 1. The conclusion is also the same as that in Table 1. The maximal values of the TSPAS and L–W–S are very close, but the TSPAS is positivity and non–oscillation, i.e. better shape–conservation.

Furthermore, it has been numerically proved that the solution of the proposed scheme with a large constant background value is the same as that in Fig. 3 with a zero background (Figures not shown), which is easy to know from the TSPAS formulas and its shape–preserving processes.

Table 2. Numerical Comparison for Solid Body Rotation

	E_{TOT}	E_{DISS}	E_{DISP}	MAX.	MIN.
U–S	1.52×10^{-1}	1.27×10^{-1}	2.55×10^{-2}	0.27	0.00
L–W–S	7.68×10^{-2}	5.37×10^{-3}	7.14×10^{-2}	2.95	–0.67
TSPAS	6.24×10^{-2}	1.45×10^{-2}	4.80×10^{-2}	2.80	0.00

V. DISCUSSIONS AND CONCLUSIONS

1. A two–step non–oscillatory shap–preserving positive definite advection transport algorithm was presented. The TSPAS merges the advantages of small dispersion error in upstream scheme and small dissipation error in high–order–accuracy scheme.

2. In comparison to the flux corrected transport (FCT) methodology, originated by Boris and Book and generalized by Zalesak, and the nonoscillatory option of the multidimensional positive definite advection transport algorithm (MPDATA) developed by Smolarkiewicy, the proposed scheme (TSPAS) produces comparable results with much less computer–time consumption. The computer–time consumption of the FCT scheme is about eight times of that required by upstream scheme, and the MPDATA scheme is more than ten times required, but the computer–time consumption of the proposed scheme (TSPAS) is only about four times of that required by upstream scheme. All maximal values of solid body after six full rotations by the TSPAS, FCT and MPDATA scheme are about 70—80 percent of the initial maximal value. In addition, in comparison to the total–variation–diminishing (TVD) scheme developed by Harten and other positive definite schemes (including those without non–oscillatory property), the proposed scheme (TSPAS) produces much better result with almost the same computer–time consumption. The computer–time consumption of the TVD scheme is about three times that required by upstream scheme, which is the least time consumption scheme compared with other positive definite advection scheme except upstream schemes, but the maximum of the solid body after six full rotations is only 40 percent of the initial maximal value.

3. The TSPAS can be generalized, the L–W scheme can be Replaced by other high–order–accuracy conservation schemes as mentioned in Section II.

4. The TSPAS is also applicable to non–quantity field advection, because the scheme is the monotonicity preservation but only sign preservation.

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