Power and Cross–Spectra for the Turbulent Atmospheric Motion and Transports in the Domain of Wave Number Frequency Space: Theoretical Aspects

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ABSTRACT

The study of large–scale atmospheric turbulence and transport processes is of vital importance in the general circulation of the atmosphere. The governing equations of the power and cross–spectra for the atmospheric motion and transports in the domain of wave number frequency space have been derived. The contributions of the nonlinear interactions of the atmospheric waves in velocity and temperature fields to the conversion of kinetic and potential energies and to the meridional transports of angular momentum and sensible heat in the atmosphere have been discussed.

Key words: *Spectral analysis, Fourier transform, Power and cross–spectra, Atmospheric turbulence, Wave number frequency space, Kinetic and potential energies, Transports of angular momentum and sensible heat*

I. INTRODUCTION

Basically there are three approaches for the study of turbulence and transport processes of the large–scale atmosphere, They are: (i) wave number space only (ii) frequency space only and (iii) wave number frequency space only. But the most general way of studying the atmospheric turbulence and transport processes is, of course, to analyze the motion in the wave number frequency space. The mean motion in the atmosphere and movement of the large–scale atmospheric systems are primarily parallel to the latitude circles. These suggest that the large–scale atmospheric motion and transport processes may be examined in the longitude–time space at various altitudes and latitudes. One of the fruitful lines of studying large–scale atmospheric turbulence and transport processes is the analysis of power and cross–spectra of the turbulent motion and transports in the atmosphere. The power spectra, which deal primarily with kinetic, potential and internal energies are basic to the understanding of the mechanism of turbulence. The cross–spectra, which concern primarily to the transport and conversion of energies, are fundamental in the maintenance of the general circulation in the atmosphere.

Studies of the large–scale turbulence in the free atmosphere have mostly been confined to either space or time spectra of the motion. The Eulerian Space (longitude) spectra (Benton and Kahn, 1958; Eliasen 1958; Kao, 1954; Saltzman, 1958), the Eulerian time spectra (Van der Hoven, 1957; Chin, 1960; Shapiro and Ward, 1960) and the Lagrangian time spectra (Kao, 1962, 1965; Kao and Bullock, 1964) of the large–scale motion in the atmosphere have been analyzed. These investigations have provided a great deal of information regarding the contributions of the large–scale atmospheric motion due either to the longitude or time eddies. In the atmosphere, however, motion at a point, which is generally non–stationary, is affected by eddies of various sizes and periods. To gain an insight into the mechanism of the large–scale turbulence and transports, it is necessary to analyze the power and cross–spectra in the wave number frequency space.

The primary advantage of this method is that it permits analyses of the transicnt waves in trems of their length scale, phase speed and direction of motion. Thus, the relative importance of retrogressing waves may be considered in the analysis.

II. METHOD OF ANALYSIS

In order to make a spectral analysis of the large–scale atmospheric motion and transports in the frequency wave number space, we make use of the following Fourier transform

$$Q(k \pm n) = \frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi} q(\lambda,t)e^{-i(k\lambda \pm nt)}d\lambda dt \ , \quad (1)$$

where λ, t, k and n are the longitude, time, wave–number and frequency respectively. $q(\lambda,t)$ is a real, single valued function, which is piecewise differentiable in a normalized domain, $0 \leqslant \lambda,\ t \leqslant 2\pi$, Q is the complex coefficient of the Fourier transform. The positive and negative signs in front of n are designated to waves moving toward the west or in the direction of decreasing longitude and east or in the direction of increasing longitude respectively. The inverse transform of (1) gives $q(\lambda,\ t)$ expressed in terms of its complex coefficients as follows

$$q(\lambda,t) = \sum_{k=-\infty}^{+\infty}\int_{-\infty}^{\infty} Q(k,\pm n)e^{i(k\lambda \pm nt)}dn \ . \quad (2)$$

Consider the same conditions for another scalar function $\zeta(\lambda,t)$ with a Fourier transform $S(k,\ n)$. It can be shown that for functions $\zeta(\lambda,t)$ and $q(\lambda,t)$, we have

$$\frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi} \zeta(\lambda,t)q(\lambda,t)e^{-i(k\lambda \pm nt)}d\lambda dt = \sum_{j=-\infty}^{+\infty}\int_{-\infty}^{\infty} S(j,m) Q(k-j,n-m)dm \ ,$$

where

$$S(j,m) = \frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi} \zeta(\lambda,t)e^{-i(j\lambda \pm mt)}d\lambda dt \ . \quad (3)$$

Letting $k,\ n \to 0$, we have the generalized Parseval's formula,

$$\frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi} \zeta(\lambda,t)q(\lambda,t)d\lambda dt = \sum_{j=-\infty}^{+\infty}\int_{-\infty}^{\infty} S(j,m)Q(-j,-m)dm$$
$$= \sum_{k=-\infty}^{+\infty}\int_{-\infty}^{\infty} S(k,n)Q^*(k,n)dn \ . \quad (4)$$

It can be further shown that

$$\frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi}\zeta(\lambda,t)q(\lambda,t)d\lambda dt=\frac{1}{4}\sum_{k=-\infty}^{+\infty}\int_{-\infty}^{+\infty}[S(k,n)Q(-k,-n)$$
$$+S(-k,-n)Q(k,n)$$
$$+S(k,-n)Q(-k,n)$$
$$+S(-k,n)Q(k,-n)]dn\ . \qquad (5)$$

In view of the integrand of the right hand side of the above equation being an even function, Eq.(5) may be written as

$$\frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi}\zeta(\lambda,t)q(\lambda,t)d\lambda dt=\int_0^{\infty}[S_r(0,n)Q_r(0,n)$$
$$+S_i(0,n)Q_i(0,n)+S_r(0,-n)Q_r(0,-n)$$
$$+S_i(0,-n)Q_i(0,-n)]dn$$
$$+2\sum_{k=1}^{\infty}\int_0^{\infty}[S_r(k,n)Q_r(k,n)$$
$$+S_i(k,n)Q_i(k,n)+S_r(k,-n)Q_r(k,-n)$$
$$+S_i(k,-n)Q_i(k,-n)]dn\ . \qquad (6)$$

Denote the cross–spectrum of $\zeta(\lambda,t)$ and $q(\lambda,t)$ due to eddies of wave number k and frequency n moving toward the direction of increasing longitude or toward east and decreasing longitude or toward west, respectively, by

$$E_{\zeta q}(0,\mp n)=S_r(0,\mp n)Q_r(0,\mp n)+S_i(0,\mp n)Q_i(0,\mp n)\ ,$$
$$E_{\zeta q}(k,\mp n)=2[S_r(k,\mp n)Q_r(k,\mp n)+S_i(k,\mp n)Q_i(k,\mp n)]\ ,$$
$$\text{for}\quad k\neq 0\ .$$

The contribution of $\zeta(\lambda,t)$ and $q(\lambda,t)$ integrated over a latitude circle and over a normalized time interval 2π may then be expressed in terms of the sum of $E_{\zeta q}(k,n)$ and $E_{\zeta q}(k,-n)$ integrated over the frequency and wave number domain.

Thus,

$$\frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi}\zeta(\lambda,t)q(\lambda,t)d\lambda dt=\sum_{k=0}^{\infty}\int_0^{\infty}[E_{\zeta q}(k,n)+E_{\zeta q}(k,-n)]dn\ . \qquad (8)$$

The above equation may also be expressed as

$$\frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi}\zeta(\lambda,t)q(\lambda,t)d\lambda dt=\sum_{k=0}^{\infty}\{E_{\zeta q}(k,+)+E_{\zeta q}(k,-)\}\ , \qquad (9)$$

where $E_{\zeta q}(k,\mp)=\int_0^{\infty}E_{\zeta q}(k,\mp n)dn$ is the cross–spectrum of $\zeta(\lambda,t)$ and $q(\lambda,t)$ due to eddies of wave number k and all frequencies, moving, respectively in the direction of increasing and decreasing longitude, or

$$\frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi}\zeta(\lambda,t)q(\lambda,t)d\lambda dt=\int_0^{\infty}\{E_{\zeta q}(+n)+E_{\zeta q}(-n)\}dn\ , \tag{10}$$

where $E_{\zeta q}(\mp n)=\sum_{k=0}^{\infty}E_{\zeta q}(k,\mp n)$ is the cross–spectrum due to eddies of frequency n and all wave numbers, moving, respectively in the direction of increasing and decreasing longitude.

For the analysis of power spectrum of the scalar quantity $q(\lambda,t)$, the quantities $\zeta(\lambda,t)$ and $S(k,\pm n)$ in the Eqs.(3) to (10) should be replaced by $q(\lambda,\ t)$ and $Q(k,\pm n)$ respectively.

The resulting equation of power spectrum analysis of the function $q(\lambda,t)$ is given by

$$\frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi}q^2(\lambda,t)d\lambda dt=\sum_{k=0}^{\infty}\int_0^{\infty}[E_{qq}(k,n)+E_{qq}(k,-n)]dn\ , \tag{11}$$

where $E_{qq}(0,\pm n)=|Q(0,\pm n)|^2$,

$$E_{qq}(k,\pm n)=2|Q(k,\pm n)|^2 \qquad \text{for} \qquad k\neq 0\ . \tag{12}$$

III. EQUATIONS FOR THE LARGE–SCALE ATMOSPHERIC MOTION AND TRANSPORTS IN SPHERICAL COORDINATE (λ, φ, p) SYSTEM

In the longitude λ, latitude φ, pressure p coordinate system, the equations of motion, the hydrostatic equation, the continuity equation and the energy equation may be written as:

$$\frac{\partial u}{\partial t}+\frac{u}{a\cos\varphi}\frac{\partial u}{\partial\lambda}+\frac{v}{a}\frac{\partial u}{\partial\varphi}+w\frac{\partial u}{\partial p}-(f+\frac{u\tan\varphi}{a})v=-\frac{g}{a\cos\varphi}\frac{\partial z}{\partial\lambda}+F_1\ , \tag{13}$$

$$\frac{\partial v}{\partial t}+\frac{u}{a\cos\varphi}\frac{\partial v}{\partial\lambda}+\frac{v}{a}\frac{\partial v}{\partial\varphi}+w\frac{\partial v}{\partial p}+(f+\frac{u\tan\varphi}{a})u=-\frac{g}{a}\frac{\partial z}{\partial\varphi}+F_2\ , \tag{14}$$

$$\frac{\partial Z}{\partial P}+\frac{RT}{gP}=0\ , \tag{15}$$

$$\frac{\partial w}{\partial P}+\frac{1}{a\cos\varphi}\left(\frac{\partial u}{\partial\lambda}+\frac{\partial v\cos\varphi}{\partial\varphi}\right)=0\ , \tag{16}$$

$$c_p\left(\frac{\partial T}{\partial t}+\frac{u}{a\cos\varphi}\frac{\partial T}{\partial\lambda}+\frac{v}{a}\frac{\partial T}{\partial\varphi}+w\frac{\partial T}{\partial P}\right)=R\frac{wT}{P}-h\ , \tag{17}$$

where a is the radial distance from the centre of the earth, f is the Coriolis parameter, g the acceleration due to gravity, z the height of the isobaric surfaces, w is the individual rate of change of pressure, T the temperature, R the gas constant, h the heat addition per unit mass, u and v the longitudinal and meridional components of the wind. For large–scale atmospheric motion, F_1 and F_2 represent the sum of molecular frictional force and the Reynold's stress force due to eddies of high frequencies.

In the study of the maintenance of the general circulation in the atmosphere, we are particularly interested in the local rate of change of the kinetic and internal energies, and the rates of the meridional flux of sensible heat and angular momentum. With the use of Eqs.(13)

to (17) they can be shown to be

$$\frac{\partial}{\partial t}\frac{[u^2+v^2]}{2} = -\frac{1}{a\cos\varphi}\left\{\frac{\partial}{\partial\lambda}[u(u^2+v^2)]+\frac{\partial}{\partial\varphi}[u(u^2+v^2\}\cos\varphi]\right\}$$
$$-\frac{\partial}{\partial p}[w(u^2+v^2)]-\frac{g}{a}\left[\frac{u}{\cos\varphi}\frac{\partial z}{\partial\lambda}+v\frac{\partial z}{\partial\varphi}\right]+(uF_1+vF_2)\ , \tag{18}$$

$$\frac{\partial}{\partial t}(c_v T)=\frac{-c_v}{a\cos\varphi}\left\{\frac{\partial}{\partial\lambda}(uT)+\frac{\partial}{\partial\varphi}(vT\cos\varphi)\right\}+\frac{c_v}{c_p}\alpha w-\frac{c_v h}{c_p}-c_v\frac{\partial(wT)}{\partial p}\ , \tag{19}$$

$$\frac{\partial}{\partial t}(c_p vT)=\frac{-c_p}{a\cos\varphi}\left\{\frac{\partial}{\partial\lambda}(uvT)+\frac{\partial}{\partial\varphi}(v^2T\cos\varphi)\right\}-c_p\frac{\partial}{\partial p}(wvT)$$
$$+\frac{R}{P}wvT-\left(f+\frac{u\tan\varphi}{a}\right)c_p uT-\frac{g}{a}c_p T\frac{\partial z}{\partial\varphi}-vh+c_p TF_2\ , \tag{20}$$

$$\frac{\partial}{\partial t}(vua\cos\varphi)=-\left\{\frac{\partial}{\partial\lambda}(u^2v)+\frac{\partial}{\partial\varphi}(v^2u\cos\varphi)\right\}-a\cos\varphi\frac{\partial}{\partial p}(wvu)$$
$$-a\cos\varphi\left\{\left(f+\frac{u\tan\varphi}{a}\right)(u^2-v^2)\right.$$
$$\left.+\frac{g}{a}\left(\frac{v}{\cos\varphi}\frac{\partial z}{\partial\lambda}+u\frac{\partial z}{\partial\varphi}\right)-(vF_1+uF_2)\right\}\ . \tag{21}$$

The Eqs.(18) to (21) will be transformed to the frequency wave number space in the following section.

IV. GOVERNING EQUATIONS FOR THE LARGE-SCALE ATMOSPHERIC MOTION AND TRANSPORTS IN THE WAVE NUMBER FREQUENCY SPACE

To transform the governing equations for the large-scale atmospheric motion and transports to the wave number frequency space, we introduce the following notations for the Fourier coefficients of the quantities used in this study.

$q(\lambda, t, p, \varphi)$	u	v	w	z	T	h	F_1	F_2
$Q(k, n, p, \varphi)$	U	V	W	Z	θ	H	G_1	G_2

One of the objectives of this study is to analyze the contribution of the large-scale atmospheric motion to the kinetic energy, the rate of the meridional transports of sensible heat and angular momentum and the available potential energy in the atmosphere. To do so, spectra of $\frac{1}{2}(u^2+v^2)$, vT, $vua\cos\varphi$ and wT need to be computed.

They can, respectively, be shown to be

$$E_{\frac{1}{2}(u^2+v^2)}(k,\pm n)=\pm\frac{i}{n}\sum_{j=-\infty}^{\infty}\int_{-\infty}^{\infty}\left\{\frac{i}{a\cos\varphi}jU(k-j,\pm n\mp m)\right.$$
$$[U(j,\pm m)U(-k,\mp n)+V(j,\pm m)V(-k,\mp n)]$$
$$+\frac{1}{a}V(k-j,\pm n\mp m)\{U_\varphi(j,\pm m)U(-k,\mp n)$$

$$
\begin{aligned}
&+V_\varphi(j,\pm m)V(-k,\mp n)]+W(k-j,\pm n\mp m)\\
&[U_p(j,\pm m)U(-k,\mp n)+V_p(j,\pm m)V(-k,\mp n)]\\
&+\frac{\tan\varphi}{a}U(j,\pm m)[V(-k,\mp n)U(k-j,\pm n\mp m)\\
&-U(-k,\mp n)V(k-j,\pm n\mp m)]\Big\}dm\pm\frac{1}{n}\Big\{\frac{g}{a}\\
&[iZ_\varphi(k,\pm n)V(-k,\mp n)-kZ(k,\pm n)U(-k\mp n)]\\
&+if[U(k,\pm n)V(-k,\mp n)-U(-k,\mp n)V(k,\pm n)]\\
&-i[U(-k,\mp n)G_1(k,\pm n)+V(-k,\mp n)G_2(k,\pm n)]\Big\}\ ,
\end{aligned}\tag{22}
$$

$$
\begin{aligned}
E_{\gamma T}(k,\pm n)=\pm\frac{i}{n}\sum_{j=-\infty}^{\infty}\int_{-\infty}^{\infty}&\Big\{\frac{i}{a\cos\varphi}U(k-j,\pm n\mp m)\\
&[jV(j,\pm m)\theta(-k,\mp n)+jV(-k,\mp n)]\theta(j,\pm m)\\
&+\frac{\tan\varphi}{a}\theta(-k,\mp n)U(j,\pm m)]+\frac{1}{a}V(k-j,\pm n\mp m)\\
&[V_\varphi(j,\pm m)\theta(-k,\mp n)+V(-k,\mp n)\theta_\varphi(j,\pm m)]\\
&+W(k-j,\pm n\mp m)[V_p(j,\pm m)\theta(-k,\mp n)\\
&+V(-k,\mp n)\theta_p(j,\pm m)-\frac{R}{c_pP}V(-k,\mp n)\\
&\theta(j,\pm m)]\Big\}dm\pm\frac{i}{n}\Big\{\theta(-k,\mp n)\\
&\left[\frac{g}{a}Z_\varphi(k,\pm n)+fU(k,\pm n)-G_2(k,\pm n)\right]\\
&-\frac{1}{c_p}V(-k,\mp n)H(k,\pm n)\Big\}\ ,
\end{aligned}\tag{23}
$$

$$
\begin{aligned}
E_{\gamma u}(k,\pm n)=\pm\frac{i}{n}\sum_{j=-\infty}^{\infty}\int_{-\infty}^{\infty}&\Big\{\frac{i}{a\cos\varphi}jU(k-j,\pm n\mp m)\\
&[V(j,\pm m)U(-k,\mp n)+V(-k,\mp n)U(j,\pm m)]\\
&+\frac{1}{a}V(k-j,\pm n\mp m)[V_p(j,\pm m)U(-k,\mp n)\\
&+U_p(j,\pm m)V(-k,\mp n)]+\frac{\tan\varphi}{a}U(j,\pm m)\\
&[U(-k,\mp n)U(k-j,\pm n\mp m)-V(-k,\mp n)\\
&V(k-j,\pm n\mp m)]\Big\}dm\pm\frac{i}{n}\Big\{\frac{g}{a}\left[\frac{i}{\cos\varphi}kZ(k,\pm n)\right.\\
&V(-k,\mp n)+Z_\varphi(k,\pm n)U(-k,\mp n)]+f[U(k,\pm n)U(-k,\mp n)\\
&-V(k,\pm n)V(-k,\mp n)]-G_1(k\pm n)V(-k,\mp n)\\
&+G_2(k,\pm n)U(-k,\mp n)\Big\}\ ,
\end{aligned}\tag{24}
$$

$$
\begin{aligned}
E_{wT}(k,\pm n) = \pm\frac{i}{n}\sum_{j=-\infty}^{\infty}\int_{-\infty}^{\infty} & \left\{\frac{i}{a\cos\varphi} j\Big[W(k,\pm n)\right.\\
& \theta(-j,\mp m)U(-k+j,\mp n\pm m) + W(-k,\mp n)\\
& \theta(j,\pm m)U(k-j,\pm n\mp m)\Big] + \frac{1}{a}[W(-k,\mp n)\\
& \theta_{\varphi}(j,\pm m)V(k-j,\pm n\mp m) - W(k,\pm n)\\
& \theta_{\varphi}(-j,\mp m)V(-k+j,\mp n\pm m)]\\
& + W(-k,\mp m)W(k-j,\pm n\mp m)[\theta_p(j,\pm m)\\
& -\frac{R}{c_p P}\theta(j,\pm m)] - W(k,\pm m)W(-k+j,\mp n\pm m)\\
& \left.[\theta_p(-j,\mp m) - \frac{R}{c_p P}\theta(-j,\mp m)]\right\} dm\\
& \mp\frac{i}{c_p n}\{W(-k,\mp n)H(k,\pm n) - W(k,\pm n)H(-k,\mp n)\} \ . \qquad (25)
\end{aligned}
$$

V. CONCLUSIONS

The primary advantage of this method is that it permits analyses of the moving waves in terms of their wave lengths, phase speeds and direction of motion. The wave number and frequency spectra become special cases of the wave–number frequency spectra. The relative importance of the forward and backward moving waves may be evaluated. The linear and nonlinear effects of the velocity and temperature fields on the kinetic and internal energies and the meridional transports of sensible heat and angular momentum in the wave number frequency space may be studied globally or in either hemispheres.

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