

Three-dimensional Global Scale Permanent-wave Solutions of the Nonlinear Quasigeostrophic Potential Vorticity Equation and Energy Dispersion

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ABSTRACT

The three-dimensional nonlinear quasi-geostrophic potential vorticity equation is reduced to a linear form in the stream function in spherical coordinates for the permanent wave solutions consisting of zonal wavenumbers from 0 to n and r_z vertical components with a given degree n . This equation is solved by treating the coefficient of the Coriolis parameter square in the equation as the eigenvalue both for sinusoidal and hyperbolic variations in vertical direction. It is found that these solutions can represent the observed long term flow patterns at the surface and aloft over the globe closely. In addition, the sinusoidal vertical solutions with large eigenvalue G are trapped in low latitude, and the scales of these trapped modes are longer than 10 deg. lat. even for the top layer of the ocean and hence they are much larger than that given by the equatorial β -plane solutions. Therefore such baroclinic disturbances in the ocean can easily interact with those in the atmosphere.

Solutions of the shallow water potential vorticity equation are treated in a similar manner but with the effective depth $H = RT/g$ taken as limited within a small range for the atmosphere.

The propagation of the flow energy of the wave packet consisting of more than one degree is found to be along the great circle around the globe both for barotropic and for baroclinic flows in the atmosphere.

Key words: Finite amplitude potential vorticity solution, General circulation, Global scale energy dispersion

1. INTRODUCTION

The study of the propagation of permanent waves for purely horizontal and nondivergent flows was initiated by Rossby (1939), who obtained solutions of the linearized vorticity equation under the influence of a variable Coriolis parameter f for an atmosphere of infinite extent in meridional direction, and Haurwitz (1940) extended the results to the nonlinear vorticity equation over the spherical earth. Later on, Ertel (1943) extended the results further by allowing superposition of solutions of different wavenumbers in zonal direction, which appears to be extraordinary in view of the nonlinearity of the vorticity equation.

The study was extended further by this author (Kuo, 1959) to three-dimensional baroclinic solutions of the potential vorticity equation for the stably stratified atmosphere, which showed that more realistic flow patterns can be obtained by superposition of such solutions with different zonal and vertical wavenumbers. The present paper is a continuation of this research, where I shall extend the three-dimensional baroclinic solutions of the problem more in a systematic manner. In addition, I shall investigate the dispersion relation for these three-dimensional global scale disturbances.

II. THE QUASI-GEOSTROPHIC POTENTIAL VORTICITY EQUATION FOR ADIABATIC AND INVISCID FLOW

The quasi-geostrophic potential vorticity equation for inviscid and adiabatic flow is

$$\frac{D(q' + f)}{Dt} = 0, \quad (1)$$

where q' is the relative potential vorticity and is given by (see Kuo, 1959):

$$q' = \nabla^2 \psi' - \frac{f}{R} \frac{\delta}{\delta p} \left[\frac{p}{\Gamma_0} \frac{\delta \psi'}{\delta p} \right], \quad (1a)$$

where $\Gamma_0 = T_0 \theta_0^{-1} (\delta \theta_0 / \delta p)$. In accordance with the equation of motion, we take the quasi-geostrophic approximation as given by

$$dp' = f d\psi'. \quad (1b)$$

so that we have

$$q' = \nabla^2 \psi' - \frac{f^2}{R} \frac{\delta}{\delta p} \left[\frac{p}{\Gamma_0} \frac{\delta \psi'}{\delta p} \right] \quad (1a^*)$$

and in (1) D/Dt operates on the ψ' parts of q' only.

For convenience, I use $Z = \ln(p_s / p)$ as the vertical coordinate, where p_s is the mean value of the sea level pressure. We then have

$$q' = \nabla^2 \psi' + f^2 (\psi'_{ZZ} - 2\lambda' \psi'_Z) / S, \quad (1c)$$

where $S = N^2 H^2$ is the stratification factor, $\lambda' = (1 + S_Z / S) / 2$ and the subscript Z denotes partial differentiation.

We set

$$\psi' = a^2 \psi e^{\lambda' Z}, \quad q' = q e^{\lambda' Z} \quad (1d)$$

and write q and D/Dt in spherical coordinates in terms of ψ , where we have

$$u = -a \psi'_\varphi, \quad v = a \psi'_\lambda / \cos \varphi, \\ \zeta = [v_\lambda - (u \cos \varphi)_\varphi] / a \cos \varphi = \nabla_s^2 \psi',$$

and therefore

$$\frac{Dq}{Dt} + 2\Omega \psi_\lambda = 0, \quad (2)$$

$$q = \nabla_s^2 \psi + \frac{4\Omega^2 a^2 \eta^2}{S} (\psi_{ZZ} - \lambda'^2 \psi), \quad (2a)$$

$$D/Dt = \frac{\partial}{\partial t} + \psi_\lambda \frac{\partial}{\partial \eta} - \psi_\eta \frac{\partial}{\partial \lambda}, \quad (2b)$$

$$\nabla_s^2 = \frac{\partial}{\partial \eta} (1 - \eta^2) \frac{\partial}{\partial \eta} + \frac{1}{1 - \eta^2} \frac{\partial^2}{\partial \lambda^2}, \quad (2c)$$

where λ is the longitude, φ is latitude and $\eta = \sin \varphi$. As has been mentioned above, D/Dt operates on the ψ parts of q only in (2), hence in spherical coordinates this equation is given by

$$\frac{D}{Dt} \nabla_s^2 \psi + \frac{4\Omega^2 a^2 \eta^2}{S} \frac{D}{Dt} (\psi_{ZZ} - \lambda'^2 \psi) + 2\Omega \psi_\lambda = 0. \quad (2^*)$$

III. GLOBAL SCALE BAROTROPIC AND BAROCLINIC PERMANENT-WAVE SOLUTIONS OF THE POTENTIAL VORTICITY EQUATION

We take the disturbance as moving with the angular speed α in the direction of increasing longitude λ as a whole so that we have $\delta / \delta t = -\alpha \delta / \delta \lambda$ and

$$\psi = \psi(\lambda - \alpha t, \eta, z), \quad q = q(\lambda - \alpha t, \eta, Z). \quad (3a, b)$$

Eq. (2*) then reduces to

$$\psi_{\lambda}(q_{\eta} + 2\Omega) - (\psi_{\eta} + \alpha)q_{\lambda} = 0. \quad (4)$$

We set

$$\psi = \psi^* - \alpha\eta, \quad (5a)$$

$$q = P(\psi^*) - 2\Omega\eta, \quad (5b)$$

where $P(\psi^*)$ is an arbitrary function of ψ^* . Eq. (4) then reduces to

$$\psi^*_{\lambda} P_{\eta} - \psi^*_{\eta} P_{\lambda} = 0, \quad (4^*)$$

which implies that the nonlinear terms of (4) cancel each other when P is any arbitrary function of ψ^* . For simplicity, we take $P = -\mu\psi^*$, where μ is a positive constant. The relations (2*) and (5b) then yield the following equation for ψ :

$$\nabla_s^2 \psi + 4\Omega^2 a^2 \eta^2 / S[\psi_{ZZ} - \lambda'^2 \psi] + \mu\psi = -(\mu\alpha + 2\Omega)\eta. \quad (6)$$

We represent ψ by the sum of the function $\psi'(\lambda - \alpha t, \eta, Z)$ which satisfies the homogeneous part of this equation, and the particular solution ψ_p given by

$$\psi_p = -\frac{(\mu\alpha + 2\Omega)\eta}{(\mu - 2)}. \quad (7a)$$

This solution represents a solid rotation with angular velocity

$$\bar{\omega} = \frac{(\mu\alpha + 2\Omega)}{(\mu - 2)}. \quad (7b)$$

Hence we have

$$\alpha = \bar{\omega} - \frac{2(\Omega + \bar{\omega})}{\mu}. \quad (7c)$$

The condition for the vertical variation of ψ' at the flat surface corresponding to the vanishing of the vertical velocity \dot{Z} is

$$\delta\psi' / \delta Z + \lambda'\psi' = 0. \quad (8)$$

Thus in the troposphere we can write ψ' as

$$\psi' = \psi^* (\lambda - \alpha t, \eta). R(Z), \quad (9)$$

with $R(Z)$ given by

$$R(Z) = \cos \gamma\pi Z - \frac{\lambda'}{\gamma\pi} \sin \gamma\pi Z, \quad (9a)$$

where γ is given by the following expression in terms of the eigenvalue G for the vertical mode r for a give zonal wavenumber m and total wavenumber square $\mu = n(n+1)$:

$$\gamma^2 \pi^2 = -\frac{S}{4\Omega^2 a^2} G - \lambda'^2. \quad (10)$$

Here γ can either be real or imaginary, with the former corresponding to sinusoidal variation with Z and positive G and the latter corresponding to exponential vertical variation and negative G . Thus, we write ψ as given by

$$\psi'(\lambda, \eta, Z) = \sum_{m=0}^n \sum_{r=0}^{r_n} A_{mr} \psi_{nr}^m(\eta) e^{\lambda Z} R_{nmr}(Z) \cos m(\lambda - \alpha t) - \frac{(2\Omega + \mu\alpha)\eta}{(\mu - 2)}, \quad (11)$$

with $R_{nm0} = 1$, $G_{m0} = 0$ for the barotropic nondivergent mode, and $R(Z)$ given by (9a) for the r th baroclinic modes for each m , while $\psi_{nr}^m(\eta)$ is given by the following equation

$$\frac{\partial}{\partial \eta} (1 - \eta^2) \frac{\partial \psi_{nr}^m}{\partial \eta} + \left[\mu - \frac{m^2}{1 - \eta^2} - G_{mr} \eta^2 \right] \psi_{nr}^m = 0. \quad (12)$$

Thus, for the barotropic and nondivergent mode ψ_{n0}^m is given by the associated Legendre's function $P_n^m(\eta)$. This solution is obtained by Ertel (1943) and it is more general than Haurwitz's (1940) Rossby wave solution as it includes all zonal wavenumbers from 0 to n with arbitrary amplitudes. The function P_n^m has $(n-m)/2$ zeros between $\eta=0$ and $\eta=1$ for even $(n-m)$, and $(n-m-1)/2$ zeros for odd $(n-m)$ in this range.

To obtain the solutions for the baroclinic modes, we multiply (12) by $(1-\eta^2)$ and set $\psi_{nr}^m = V(1-\eta^2)^{m/2}$. This equation then becomes

$$V_{yy} - 2m * \tanh y * V_y + [\mu - m(m+1) - G * \tanh^2 y] \text{sech}^2 y * V = 0, \quad (12^*)$$

where y is the Mercator coordinate and is given by

$$y = \frac{1}{2} \ln[(1+\eta)/(1-\eta)], \quad (12a)$$

$$\eta = \sin \phi = \tanh y.$$

We take fixed $\mu = n(n+1)$ and seek G as the eigenvalue of the problem for given n and m . The conditions for V are that it is even or odd at $y=0$ and vanishes as $y \rightarrow \infty$ except for $m=1$. Here the solutions of the problem are obtained numerically by the iterative reduction method developed by the author (see Kuo, 1978) with the $y \sim \infty$ condition replaced by $V \sim \exp(-\alpha'(y-Y))$ for $Y=5.0$ and with sufficient number of grid points to guarantee accuracy.

IV. EIGENVALUES AND EIGENFUNCTIONS OF THE SOLUTIONS OF (12)

Unlike the early work on this problem (Kuo, 1959) where μ is allowed to take non-integer values, here I shall restrict μ to integer values of the form $\mu = n(n+1)$ for positive integer n in order to include the nondivergent barotropic modes as part of the 3D solution. Further, both positive and negative values of G are considered as permissible, with the former representing sinusoidal and the latter transcendental vertical variations of $R(Z)$. The positive eigenvalues obtained for different values of μ and m are listed in Table 1A, which includes both the global modes denoted by r , and the odd and even modes confined in the equatorial region, which are denoted by O and E respectively. Only the eigenvalues of the one- and two-cell trapped modes are given for most of μ greater than 240.

Table 1. Eigenvalues G of the Global Baroclinic Modes of the 3D Potential Vorticity Equation for Different Values of Zonal Wavenumber m , Vertical Index r and Global Index $\mu = n(n+1)$. The eigen function ψ is even or odd according to $m+r$ is even or odd. Each m/r panel is for a given μ arranged in order of μ . A. For $G > 0$ and $\mu = 12, 20, 30, 56, 90, 110, 132, 182, 240, 306, 380, 462$, with trapped solutions represented by O and E for odd and even modes, and with the subscript stands for the number of cells in the meridional direction. Values of G greater than 1000 are written in decimal form with exponent E. For $\mu = 182, 240$ and 380 , only the G value for the one cell and two cells trapped modes are given.

r/m	0	1	2	3	4	5	6	7	8
2	10.4								
3	22.16							$n=3$	
E ₁	170.0	142.3	76.6						
1	14.27	17.23							
2	24.68							$n=4$	
O ₁	53.47	47.04	28.60						
E ₁	448.00	403.00	285.00						
2	18.53	20.18	26.11						
3	30.62	41.89							
4	47.56							$n=5$	
O ₁	112.90	104.55	80.76	44.56	—				
E ₁	970.00	898.00	737.00	478.00	207.00				
2	26.83	27.64	30.60						
3	46.22	50.60							
4	60.60								
O ₂	83.47	77.98	62.32	36.50				$n=7$	
E ₂	144.42	138.00	119.50	90.10	49.60				
O ₁	372.00	358.00	317.60	254.30	174.90				
E ₁	.327E4	.316E4	.284E4	.230E4	.167E4				
2	35.00	35.66	37.60	41.01	47.20				
3	62.97	65.15	71.27	83.10					
4	83.14	91.25	109.00						
5	100.59	124.50							$n=9$
6	130.00								
O ₂	193.93	188.28	171.54	144.00	106.19				
E ₂	354.20	345.70	319.00	277.00	221.60				
O ₁	939.00	918.00	854.00	753.00	622.00				
E ₁	.838E4	.821E4	.770E4	.679E4	.564E4				
1	39.04	39.75	41.35	44.35	49.13	57.10			
2	71.51	72.94	77.83	87.00	103.00				
3	138.04	101.27	114.43	138.80					$n=10$
4	184.51	131.56	164.25						
O ₂	281.38	275.17	256.47	226.00	184.37	132.61	71.17		
E ₂	521.40	512.14	480.00	430.00	365.00	284.00	194.50	98.90	
O ₁	.140E4	.137E4	.129E4	.117E4	.100E4				
E ₁	.125E5	.123E5	.115E5	.105E5	.091E5				

Continued Table 1A

r/m	0	1	2	3	4	5	6	7	8
2	43.15	43.93	45.22	47.87	51.80	58.11	67.86		$n=11$
3	79.52	80.91	84.97	92.41	104.65	125.29			
4	150.70	181.86	122.45	140.58	172.00				
5	187.18	251.00	166.29	208.50					
6	256.25	389.62	234.95						
O ₂	396.91	389.62	368.59	334.25	287.62	229.75	162.01		
E ₂	743.38	730.28	693.00	635.00	556.00	463.41	356.00		
O ₁	.200E4	.197E4	.188E4	.173E4	.153E4				
E ₁	.180E5	.176E5	.170E5	.156E5	.138E5				
O ₂	734.00	724.50	697.70	653.70	593.70				$n=13$
E ₂	.139E4	.137E4	.133E4	.125E4	.114E4				
O ₁	.378E4	.374E4	.361E4	.341E4	.313E4				
E ₁	.343E5	.334E5	.323E5	.304E5	.283E5				
O ₂	.126E4	.124E4	.121E4	.115E4	.108E4				$n=15$
E ₂	.240E4	.238E4	.231E4	.228E4	.207E4				
O ₁	.656E4	.651E4	.634E4	.606E4	.570E4				
E ₁	.596E5	.585E5	.572E5	.543E5	.513E5				
2	69.70	71.20	72.00	73.50	75.50	78.20			$n=17$
3	130.10	131.50	133.50	137.00	141.90	148.90			
4	184.00	185.70	189.30	195.90	205.40	219.30			
5	230.60	233.30	240.10	252.00	269.70	295.00			
6	325.80	313.20	288.00	309.60	341.00				
O ₄	512.00	506.90	491.70	466.50	431.30				
E ₄	648.40	642.80	626.00	598.50	560.00				
O ₃	869.00	862.00	841.60	808.00	761.60				
E ₃	.126E4	.125E4	.122E4	.117E4	.111E4				
O ₂	.202E4	.201E4	.196E4	.189E4	.180E4				
E ₂	.389E4	.386E4	.377E4	.365E4	.347E4				
O ₁	.107E5	.106E5	.104E5	.100E5	.096E5				
E ₁	.966E5	.951E5	.935E5	.908E5	.858E5				
O ₂	.310E4	.308E4	.303E4	.294E4	.283E4				$n=19$
E ₂	.599E4	.590E4	.585E4	.569E4	.547E4				
O ₁	.164E5	.164E5	.161E5	.157E5	.151E5				
E ₁	.150E6	.149E6	.146E6	.141E6	.136E6				
4	90.20	92.80	93.50	94.60	96.20	98.20			
5	166.20	168.50	170.00	172.60	176.20	181.00			
6	235.90	238.00	240.70	245.20	257.70	260.60			
7	298.90	301.00	305.50	312.90	323.70	338.60			
8	439.80	407.20	364.70	377.00	394.60	338.70			

Continued Table 1A

r/m	0	1	2	3	4	5	6	7	8
O_5	738.70	733.60	718.50	693.50	658.60				$n=21$
E_5	884.30	878.80	862.90	836.50	800.00				
O_4	.110E4	.109E4	.107E4	.104E4	.100E4				
E_4	.141E4	.141E4	.138E4	.135E4	.130E4				
O_3	.192E4	.191E4	.188E4	.184E4	.177E4				
E_3	.281E4	.280E4	.276E4	.269E4	.260E4				
O_2	.457E4	.455E4	.448E4	.438E4	.424E4				
E_2	.884E4	.880E4	.868E4	.848E4	.820E4				
O_1	.244E5	.243E5	.239E5	.234E5	.227E5				
E_1	.222E6	.222E6	.218E6	.213E6	.208E6				

B. For $G^* = -G > 0$, $\mu = 12, 20, 30, 56, 90, 132$

r/m	0	1	2	3	4	5	6	7	8
0	16.70	17.41	20.50	25.55					
1	38.94	40.50	43.22	56.23					
2	68.00	67.36	74.50	78.10					
3	98.88	103.30	104.02	123.20					
4	140.40	137.37	149.20	149.45					
1	20.83	21.31	23.48	28.00	36.93				
2	47.35	48.07	51.65	56.99	74.00				
3	79.12	80.00	83.58	92.71	100.65				
4	116.82	116.94	122.68	128.13	151.54				
5	158.26	160.56	163.28	176.38	182.12				
1	55.47	56.00	58.78	64.00	72.50	93.00	—	—	—
2	91.50	92.16	96.00	101.57	114.10	125.55			
3	132.86	134.00	137.23	145.00	154.37	182.85			
4	180.30	179.86	185.00	191.22	208.00	216.50		—	
5	231.09	233.00	235.69	246.00	254.73	288.75	—	—	—
1	71.59	71.70	74.06	78.00	84.20	93.91	108.74	141.26	—
2	115.92	116.07	118.80	123.77	131.57	142.73	161.80	182.36	—
3	165.54	165.57	168.83	174.87	183.23	196.90	213.18	251.56	—
4	219.40	220.39	223.00	230.30	238.70	251.60	276.08	295.33	—
5	280.85	281.00	284.23	292.00	301.19	318.00	334.20	375.50	—
1	87.56	87.35	89.23	92.37	97.34	104.51	114.75	129.78	151.88
2	139.96	139.63	141.90	146.10	152.70	161.25	173.64	191.75	217.00
3	197.78	197.18	200.00	204.89	212.16	222.70	236.25	255.80	280.36
4	258.85	260.10	260.90	266.50	274.60	288.06	303.08	323.20	353.20
5	329.63	328.25	331.68	337.39	346.40	359.00	374.65	397.00	422.75
1	103.30	102.66	104.20	107.20	111.03	116.78	124.73	135.40	150.28
2	160.90	162.80	164.95	168.41	173.76	181.13	190.95	204.30	221.70
3	229.63	228.44	230.81	235.05	241.36	249.97	261.50	276.37	296.10
4	301.20	299.24	301.74	306.84	313.81	323.59	332.50	352.83	370.00
5	377.64	375.62	378.43	383.59	391.47	401.89	415.87	433.30	455.98

The negative eigenvalues obtained are given in Table 1B as $G^* = -G$. The eigen functions corresponding to these eigenvalues all extend to high latitudes and hence not trapped to low latitude.

The variations of the eigen functions for $\mu=132$, $n=11$, $m=0,2,4,6,8$ and $G=0$ are represented in the top panel of Fig. 1, and the middle panel represents that of the baroclinic solutions with positive G and $r=1$, while the eigen functions of the $G<0$, $r=1$ modes are

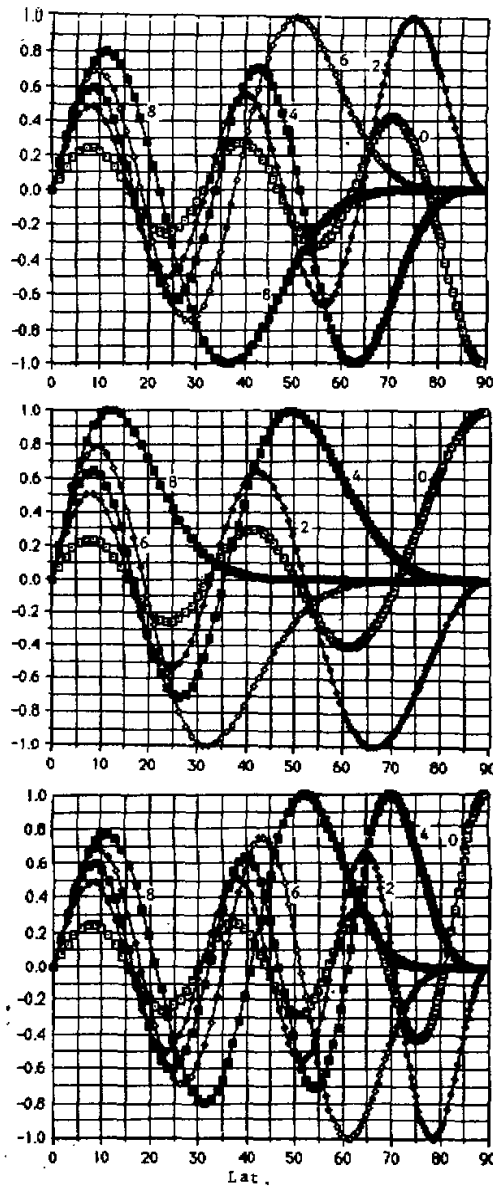


Fig. 1. Eigenfunctions ψ_m for $\mu=132$, $m=0,2,4,6,8$. Top: $r=0$, $G_0=0$; middle: $r=1$, $G>0$; bottom: $r=1$, $G<0$.

illustrated in the bottom panel of this diagram. These functions are all normalized with the maximum amplitude of the function set to unity.

The eigen functions of the odd and even one-cell trapped modes for $\mu = 56$, $m = 0$ to 4 are illustrated in Fig. 2. It is seen that these functions are almost the same for different values of m . This is also true for other values of μ . On the other hand, the width of the trapped cell is larger for smaller μ , as can be seen from Figs 3 and 4, which show the eigen functions of the odd and even one- and two-cell trapped modes for $m = 0$, $\mu = 30, 90, 182, 306$ and 462. Notice that the width of all these cells are larger than 10 deg. lat. even for the largest μ , which are considered as representative of the flow in the tropical ocean. Thus, according to the potential

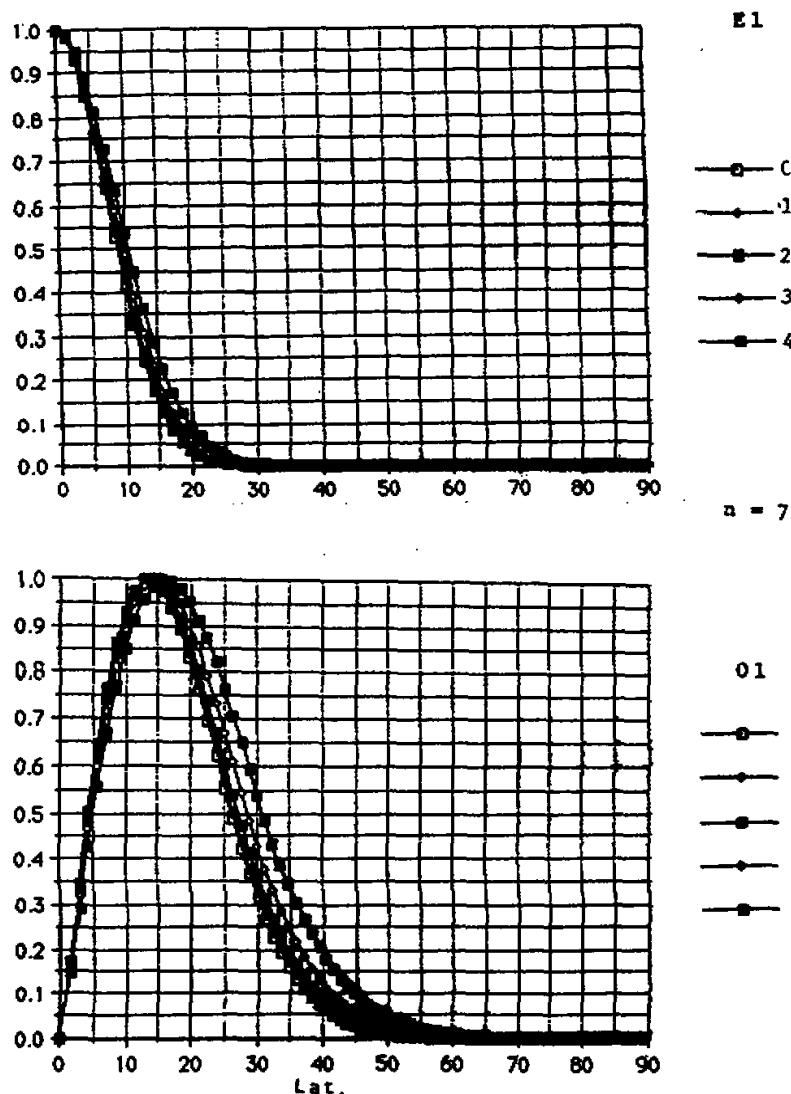


Fig. 2. Eigenfunctions of even and odd one cell trapped solutions for $\mu = 56$, $m = 0, 1, 2, 3, 4$.

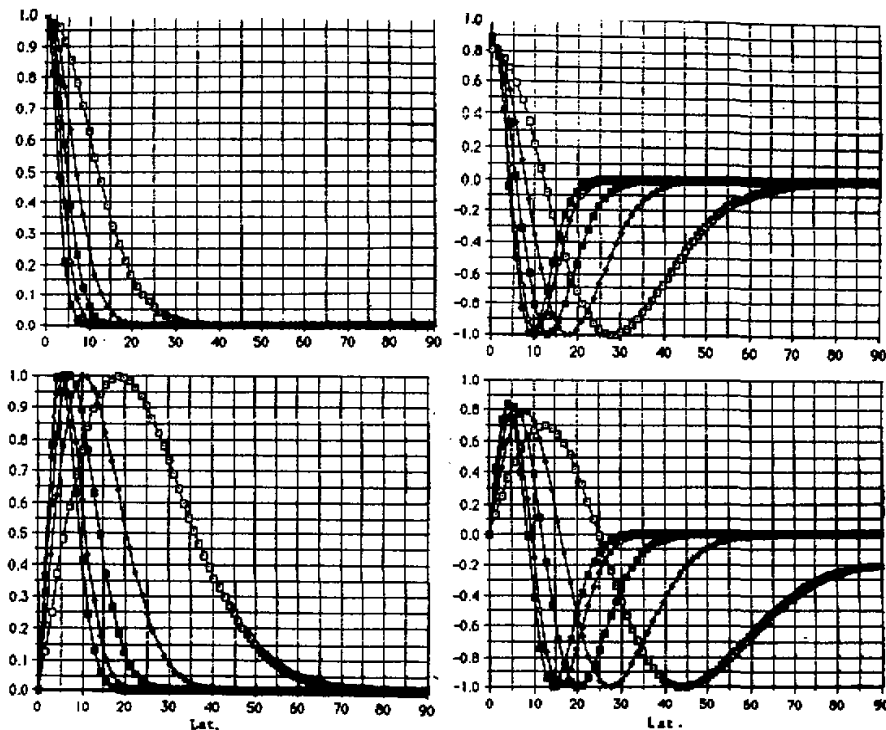


Fig. 3. Eigenfunctions of even and odd trapped solutions for $m=0$, $\mu=30, 90, 182$ and 462 . a, one cell; b, two cell.

vorticity equation, the scale of the baroclinic disturbances in the tropical oceans in meridional direction is similar to those in the atmosphere, and much larger than that given by the equatorial β -plane solution (see Kuo, 1989). Therefore such baroclinic disturbances in the two media are quite easy to influence each other in the tropical region as their scales of variations are similar.

The reason for the presents of the trapped baroclinic modes of the potential vorticity equation can be seen from Eq. (12), which shows that the influence of the baroclinic term $G\eta^2\psi$ with positive G is to reduce the effective value of $\mu - m(m+1)$ and hence large G will make the eigen function confined to low latitude.

The vertical functions $\exp(\lambda'Z) * R(Z)$ for $\mu=132$, $m=0$, $r=3, 5, 7, 2$, $G > 0$ are illustrated in the lower panel in Fig. 4 while the upper panel gives that for $\mu \approx 20$, $m=0$, with negative G .

We mention that the potential vorticity equation for the shallow water system, which corresponds to that of the large scale flow with V independent of height but possesses divergence, is also represented by Eq. (12) except here we have $G_{mr} = 4\Omega^2 a^2 / C^2$ and $C^2 = gH$, hence G is a constant for a given C . Thus, for this case it is natural to take $\mu = \mu(n, m) (\neq n(n+1))$ as the eigenvalue. However, in order to generate a more general solution involving more zonal wavenumber components as in (11) (with $R(Z)\exp(\lambda'Z) = 1$), we set $\mu = n(n+1)$

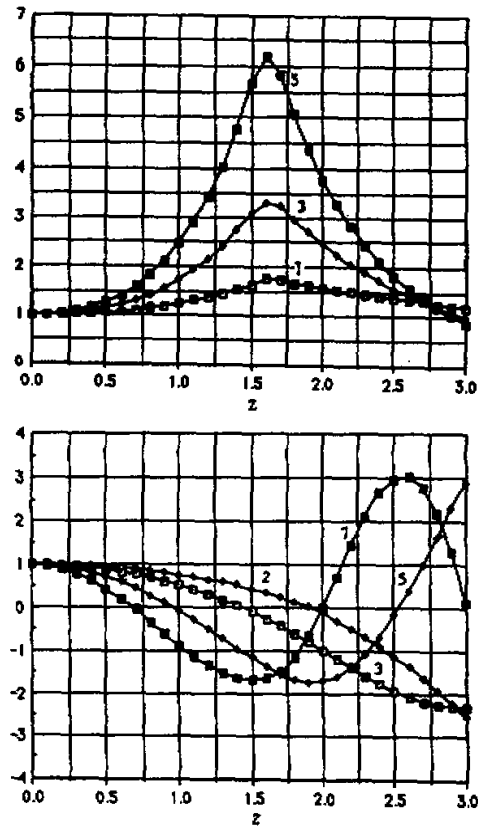


Fig. 4. Variations of vertical function with Z for $m=0$. top: $\mu=20$, $r=1, 3, 5$, $G<0$; bottom; $\mu=132$, $r=3, 5, 7$ and 2 , $G>0$.

$+\Delta\mu$ and treat G_m as the eigenvalue, but only for a small range of G corresponding to a small range of gH . These eigenvalues are given in Table 1C for the atmosphere with $gH=RT$.

C. Eigenvalue $G = 4\Omega^2 a^2 / C^2$ of shallow water potential vorticity equation for atmosphere with $9.7 < G < 14.0$ and fixed μ . With $\mu = n(n+1) + 5$ for $n > 3$ and $\mu = 17.5$ for $n = 3$. The function $\psi(n, m, \phi)$ is even for even $n+m$ and odd for odd $n+m$.

μ	$n \setminus m$	0	1	2	3	4	5
17.5	3	10.056	12.138				
25	4	10.000	10.208	12.642			
35	5	9.705	10.800	11.472			
47	6	9.723	10.144	10.800	12.819		
77	8	10.000	10.300	10.800	11.584	13.000	
95	9	10.049	10.400	10.794	11.650	12.500	14.000
115	10	10.200	10.650	11.100	11.421	12.064	13.410

V. COMPOSITE SOLUTIONS OF THE POTENTIAL VORTICITY EQUATION

We have obtained solutions of the potential vorticity equation for a number of different values of μ , each including a number of zonal wavenumber m -components with chosen coefficients, which we consider as representative of the long term mean flow or some commonly occurring large scale flow pattern in the upper troposphere. The flow fields given by these solutions at the tropopause level for $\mu = 12, 20, 30$ and 132 are illustrated in Figs. 5a to 9 and they are given by the following expression

$$\psi(x, y) = \bar{\psi}_n(\varphi) + 4.32 \sum_m \psi_{nmr}(\varphi) \sin(m\lambda), \quad (13)$$

where $\bar{\psi}_n(\varphi)$ is the stream function for the mean zonal flow. Specifically we have

$$\bar{\psi}_{12} = 7.55\psi_{3,0,0} + 32.25\psi_{3,0,1} + 19.1\psi_{3,0,3} - 6.6\sin\varphi, \quad m=3, \quad (13a)$$

$$G_{00} = 0, \quad G_{01} = -38.94 \quad G_{03} = -98.88, \quad G_{31} = -25.55.$$

$$\bar{\psi}_{20} = 23.075\psi_{4,0,1} + 42.94\psi_{4,0,3} + 24.18\psi_{4,0,5} - 4.55\sin\varphi, \quad m=4,1 \quad (13b)$$

$$G_{01} = -20.83, \quad G_{03} = -79.12 \quad G_{05} = -158.26, \quad G_{41} = -36.93,$$

$$G_{11} = -48.07$$

$$\bar{\psi}_{30} = 14.282\psi_{5,0,0} + 40.635\psi_{5,0,1} + 27.8\psi_{5,0,3} - 5.0\sin\varphi, \quad m=4, \quad (13c)$$

$$G_{00} = 0, \quad G_{01} = -55.47 \quad G_{03} = -132.86, \quad G_{41} = -72.50.$$

$$\bar{\psi}_{132} = 10.32\psi_{11,0,0} + 3.266\psi_{11,0,3} - 22.24\psi_{11,0,5} + 6.5936\psi_{11,0,7} - 7.0295\psi_{11,0,2} - 4.0\sin\varphi, \quad m=4,8, \quad (13d)$$

$$G_{00} = 0, \quad G_{03} = 79.52 \quad G_{05} = 187.18, \quad G_{07} = 396.91, \quad G_{02} = 43.15,$$

$$G_{41} = 104.65, \quad G_{81} = 470.04,$$

where the values of ψ_{nm} are based on the normalized μ_{nm} . Here Fig. 5b is for the simple $n=3$, $m=1$, $r=0$ solution to illustrate the flow which brings cold air directly from the other side of the pole to low latitude in winter.

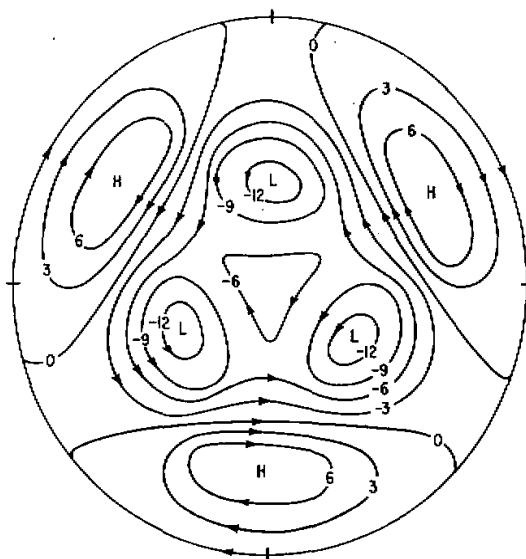


Fig. 5a. Streamline pattern given by composite solution with $m=0$ and 3 , $\mu=12$.

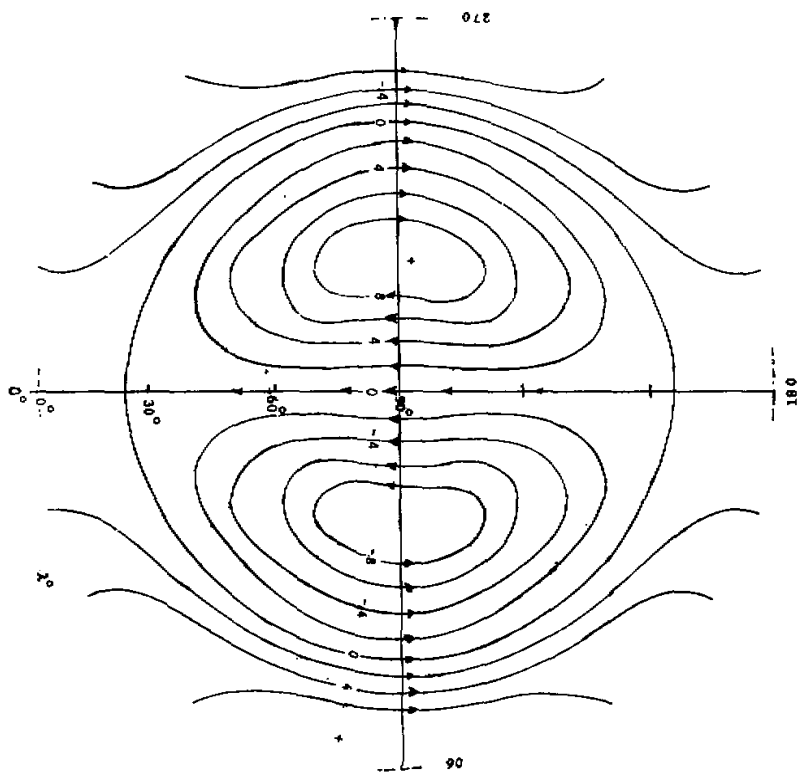


Fig. 5b. Cross pole flow given by $\mu = 12$, $m = 1$, $r = 0$ solution.

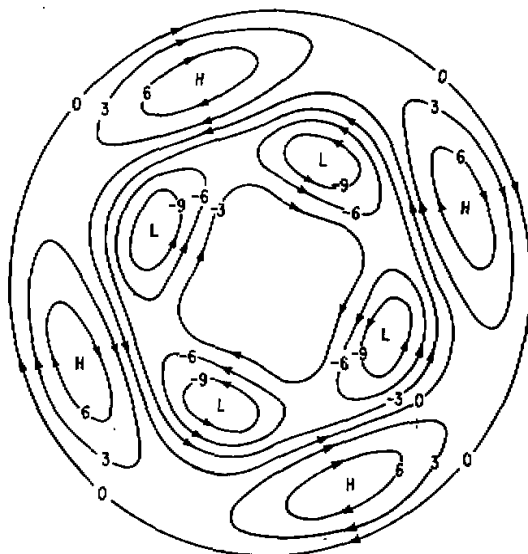


Fig. 6. Streamline pattern of $\mu = 20$, $m = 0$ and 4 composite solution.

To show that this type of solutions of the potential vorticity equation can actually represent the long term mean flow in the upper troposphere, we reproduced the observed mean flow in the upper troposphere, we reproduced the observed mean flow field obtained by Palmen and Newton (1969) in Fig. 10a and that of the more disturbed state in Fig. 10b. On comparing the flow pattern in Fig. 6 with that in Fig. 10a we see that they are quite similar to each other. Further, the flow pattern in Fig. 7 is also quite similar to part of that in Fig. 10b.

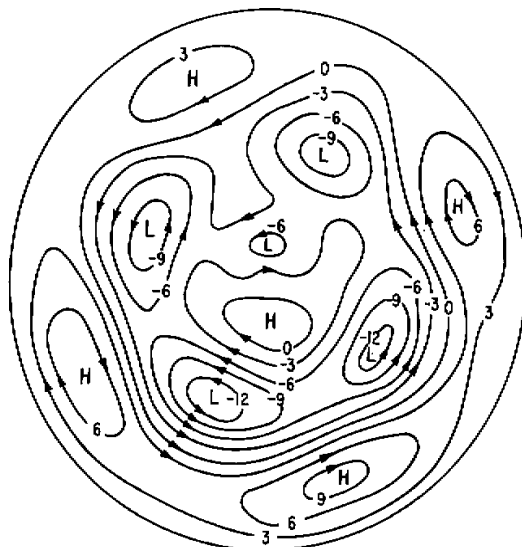


Fig. 7 Same as in Fig. 6 but with $m = 0.4$ and 1.

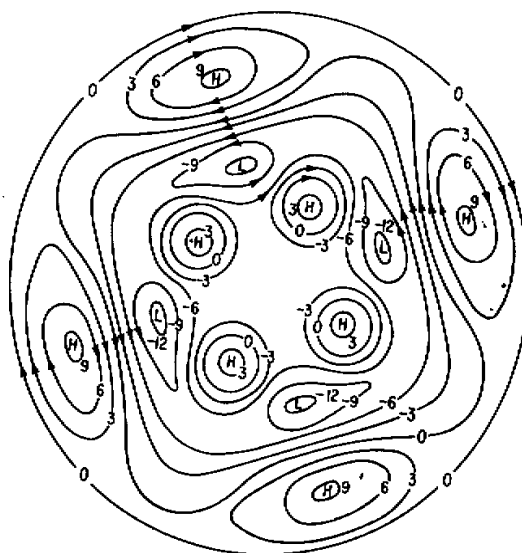


Fig. 8. Streamline pattern of $\mu = 30$, $m = 4$ solution.

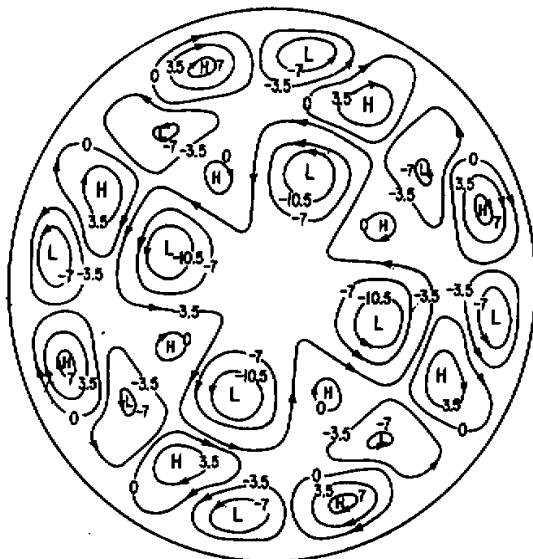


Fig. 9. Streamline pattern of $\mu = 132$, $m = 0.4$ and 8 solution.

In addition, Fig. 11 shows the flow field given by the following solution of the potential vorticity equation

$$\psi(x, y) = 6.82\psi_{11,0,3} + 10.0(\psi_{11,1} - \psi_{11,2})\sin(4\lambda) \quad (14)$$

with $G_{03} = 79.52$, $G_{41} = 104.65$, $G_{42} = 287.62$.

Here the coefficients are based on the normalized ψ_{nm} , and the second term in (14) is taken as occupying only a half of the quadrant. This solution represents a permanent vortex embedded in a shearing mean zonal flow, similar to that of the redspout in the Jovian atmosphere.

We also obtained new versions of the composite mean zonal velocity US for $\mu = 12, 20, 30$ and 56 which are somewhat closer to the observed mean zonal velocity field than that given by $\bar{\psi}_n$ in (13a-c). They are given by the following expressions in terms of the normalized u_{nm} :

$$US_{12} = 26.0u_{3,1} - 18.5u_{3,4} + 7.5u_{3,2} + 2.5u_{3,3} + 3.3 \cos\varphi$$

$$G_{01} = 0, \quad G_{02} = -38.94, \quad G_{03} = -98.88, \quad G_{04} = 22.16. \quad (15a)$$

$$US_{20} = 2.5u_{4,1} + 2.5u_{4,2} + 0.75u_{4,3} + 20.0u_{4,4} - 18.5u_{4,5} + 0.875 \cos\varphi$$

$$G_{01} = -20.8, \quad G_{02} = -79.12, \quad G_{03} = -158.26, \quad G_{04} = 14.27, \quad G_{05} = 53.47. \quad (15b)$$

$$US_{30} = 7.143u_{5,1} + 7.5u_{5,2} + 2.5u_{5,3} + 10.0u_{5,4} - 10.0u_{5,5} + 2.5 \cos\varphi$$

$$G_{01} = 0, \quad G_{02} = -55.47, \quad G_{03} = -132.86, \quad G_{04} = 30.62, \quad G_{05} = 112.9. \quad (15c)$$

$$US_{56} = 10u_{7,1} - 3.294u_{7,2} - 4.568u_{7,3} + 15u_{7,4} + 5.766u_{7,5} + 7.25 \cos\varphi$$

$$G_{01} = 46.22, \quad G_{02} = 83.47, \quad G_{03} = 372.0, \quad G_{04} = 0, \quad G_{05} = -71.59. \quad (15d)$$

The variations of these mean zonal velocities at the surface and the tropopause level with latitude are represented in Figs. 12a and b by the curves labelled u_{30} and u_{10} . It is seen that the mean zonal velocities in Fig. 12a are close to the observed annual mean zonal velocity distribution at these levels, except the easterly flow in the tropics is somewhat too high at the

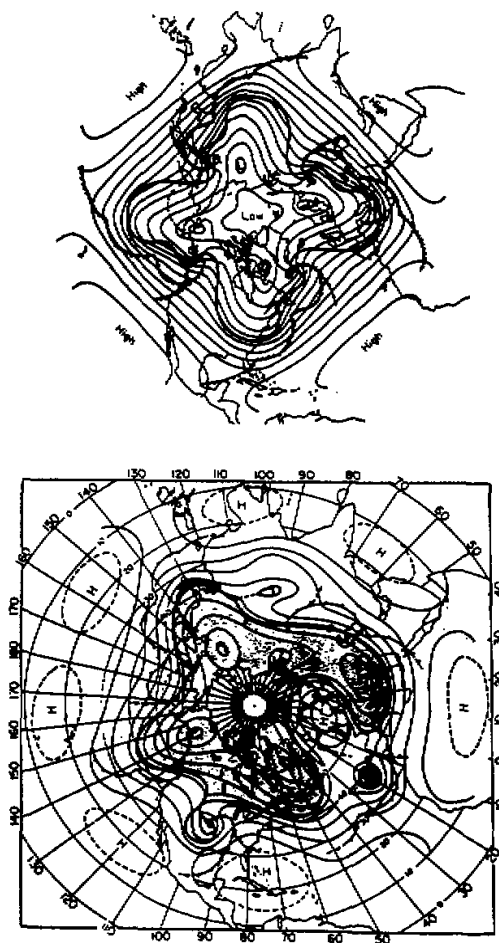


Fig. 10. 500 hPa flow pattern given by Palmen from observational data. top: Circumpolar four wave pattern; bottom: highly disturbed situation.

tropopause level and the vertical shear of the westerly around 60° is too low. Evidently, more realistic mean zonal flow distributions can be obtained by solutions including more different μ values. However, then the solutions will not be strictly steady when other zonal wavenumber components are included.

VI. PROPAGATION OF FLOW ENERGY OF WAVE PACKET CONSISTING OF MORE THAN ONE DEGREE n AND ZONAL WAVENUMBER m

It should be pointed out that for the rapidly changing fields we must use solutions which include more different values of μ as there is no nonlinear interaction between the various components of the system for a single μ . With more than one μ , nonlinear interactions will then take place between the various Fourier components of different μ .

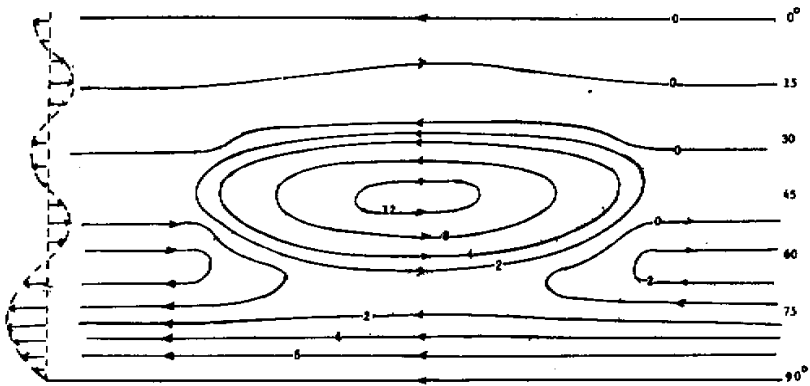


Fig. 11. Composite solution for $\mu = 132$, $m = 0$, $G_{01} = 79.52$ with isolated vortex.

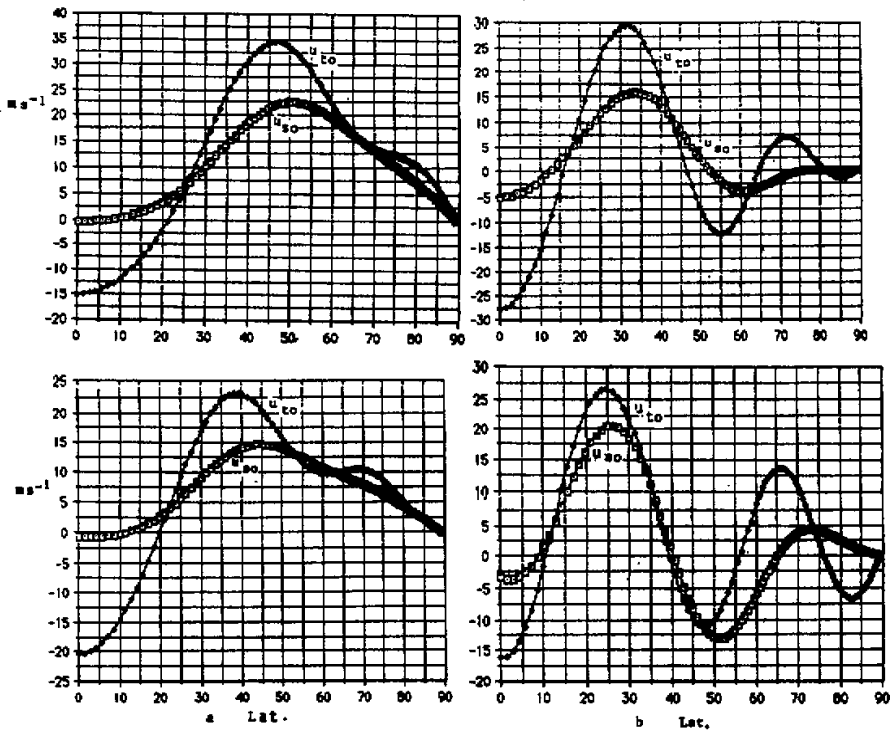


Fig. 12. Mean zonal velocity given by composite solution, a, top: $\mu = 12$, bottom: $\mu = 20$. b, top: $\mu = 30$, bottom: $\mu = 56$.

Instead of actually integrating the potential vorticity equation as an initial value problem, let us consider the propagation of the energy of a wave packet consisting of zonal wavenumbers around m and degrees around n , with corresponding $\mu = n(n+1)$. From the solutions obtained above we find that the frequency σ for the Fourier component m is given by

$$\sigma = \alpha m = \frac{[(\mu - 2)\bar{\omega} - 2\Omega]m}{\mu}, \quad (16)$$

where $\bar{\omega}$ is the solid rotation part of the zonal flow. Now, in the present system, the zonal wavenumber is m and the wavenumber l in meridional direction is $n - m'$, where $m' = m$ for barotropic flow and $m' = m \pm r/2$ or $m' = m \pm (r+1)/2$ for baroclinic flow, with the former for even modes with even m or odd modes with odd m , and the latter for odd modes with even m and even modes with odd m , where r is the eigenvalue index. Here the plus sign is for positive G and minus sign for minus G . Therefore we have

$$\mu = n(n+1) = (l+m')(l+m'+1) = l^2 + m'^2 + 2m'l + l + m'. \quad (17)$$

From (16) we find that the group angular velocities in zonal and meridional directions are given by

$$\alpha_{\lambda g} = \delta\sigma / \delta m = \alpha + m \delta\alpha / \delta\mu \cdot \delta\mu / \delta m, \quad (18a)$$

$$\alpha_{\varphi g} = \delta\sigma / \delta l = m \delta\alpha / \delta\mu \cdot \delta\mu / \delta l. \quad (18b)$$

Further, we have

$$\delta\mu / \delta m = 2m' + 2l + 1 = 2n + 1 = \delta\mu / \delta l. \quad (18c)$$

Therefore the angular group velocity vector relative to the mean angular velocity α is given by

$$\bar{\alpha}_g = (2n+1)\delta\alpha / \delta\mu(\bar{i} + \bar{j}). \quad (19)$$

Hence the flow-energy is propagating along the great circles around the globe relative to the mean angular velocity α . This property has been shown by Hoskins, Simons and Andrews (1977) from theoretical barotropic Rossby wave analysis and demonstrated by Horel and Wallace (1981) by observational data. Here we have shown that it holds for both the barotropic and the baroclinic solutions of the potential vorticity equation.

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