Baroclinic Instability in the Generalized Phillips' Model Part I: Two–layer Model

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ABSTRACT

By employing Arnol'd's method (energy–Casimir), this paper has studied the nonlinear stability of the two–layer generalized Phillips' model for which the top and bottom surfaces are either rigid or free, and obtained some nonlinear stability criteria. In addition, some linear stability criteria are obtained by normal mode method. The results reveal the influences of the free surface parameter on the stability of atmospheric and oceanic motions.

Key words: Nonlinear, Stability, Free surface

I. INTRODUCTION

The instability of atmospheric and oceanic motions is a classical and important problem. One universal way is normal mode method (Rayleigh, 1880; Kuo, 1949; Lin, 1955, etc.). But it is only linear theory. In recent years, Arnol'd's method (energy–Casmir) is successfully applied to investigate the nonlinear stability of various atmospheric motions. (McIntyre and Shepherd, 1987; Shepherd, 1988; Zeng, 1989; Mu and Shepherd, 1994; Mu et al, 1994, etc.).

To investigate the baroclinic instability of motions, Phillips (1954) presented a simple and illuminating two–layer model for which the velocities in each layer are constants, but the velocity shear is nonzero. Pedlosky (1963) showed the linear stability theory of this model. Mu et al.(1994) studied the nonlinear stability of the multilayer quasigeostrophic model, and gained the nonlinear stability criteria of the classical Phillips' model, which are applicable to finite–amplitude disturbances, and the structure of disturbances unnecessary being the single–wave form. But their work only studies the case that the top and bottom surfaces are rigid.

This paper studied the Phillips' model where the top and bottom surfaces are either rigid or free (hereafter called the generalized Phillips' model), and obtained its nonlinear stability criteria. And also we derived its linear stability criteria by normal mode method. The results reveal the nonlinear property of the shortwave cut–off phenomenon, the minimal critical shear phenomenon, etc. in the linear theory. Finally, we introduced a free surface parameter to discuss the influences of the free surface approximation on the stability of atmospheric and oceanic motions.

The paper is organized as follows: in Section 2, a generalized two–layer quasigeostrophic model is suggested; in Section 3 the linear stability of the model is studied, and the influences of the free surface parameter are discussed; the nonlinear stability criteria of this model are derived in Section 4. Finally, the main results of this paper are summerized in Section 5.

II. THE MODEL

The governing equations for the two–layer quasigeostrophic motion are the conservation of potential vorticity (Pedlosky, 1979; Zeng, 1989):

$$\frac{\partial P_i}{\partial t} + \partial(\Phi_i, P_i) = 0 \ , \qquad i = 1,2 \tag{(2.1)}$$

$$P_i = \nabla^2 \Phi_i - d_i^{-1} \sum_{j=1}^{2} T_{ij} \Phi_j + f_i(x,y) \ , \tag{2.2}$$

where Φ_i, P_i are the stream function and potential vorticity in the ith layer respectively, d_i the height of the ith layer, $\partial(f,g) = f_x g_y - f_y g_x$ the two–dimensional Jacobian, ∇^2 the two–dimensional Laplace operator, and

$$f_1 = f_2 = f_0 + \beta y \ , \tag{2.3}$$

where f_0 is (constant) Coriolis parameter, and

$$T = \begin{bmatrix} f_0^2(g_0^{-1} + g_1^{-1}) & -f_0^2(g_1^{-1}) \\ -f_0^2(g_1^{-1}) & f_0^2(g_1^{-1} + g_2^{-1}) \end{bmatrix} , \tag{2.4}$$

where g_i is the buoyancy jump across the interface between the ith and $(i+1)$th layer, and if the top (or bottom) surface is rigid, then $g_0^{-1} = 0$ ($g_2^{-1} = 0$); and when the top (or bottom) surface is free $g_0^{-1} > 0$ ($g_2^{-1} > 0$). For convenience, without loss of generality, we just consider the case that the top surface is free, i.e., $g_0^{-1} > 0$, while the bottom surface is rigid, i.e., $g_2^{-1} = 0$. And define

$$\alpha = g_0^{-1} / g_1^{-1} \tag{2.5}$$

as the free surface parameter.

The horizontal domain D under consideration is a periodic channel

$$D = \{ -\pi L \leqslant x \leqslant \pi L, -Y \leqslant y \leqslant Y \} \tag{2.6}$$

The boundary conditions are the usual ones of no normal flow and conservation of circulation in each layer, namely

$$\left. \frac{\partial \Phi_i}{\partial x} \right|_{y=-Y,Y} = 0, \quad \frac{d}{dt} \left\{ \int_{-\pi L}^{\pi L} \frac{\partial \Phi_i}{\partial y} \, dx |_{y=-Y,Y} \right\} = 0 \ , \quad i = 1,2 \tag{2.7}$$

Suppose that $(\Phi_i, P_i) = (\Psi_i, Q_i)$ is a steady solution to (2.1)–(2.4). For the generalized Phillips model, $U_i = -\frac{\partial \Psi_i}{\partial y}$, $i = 1,2$ are constants. A disturbance superimposed on the steady basic state is defined according to

$$\Phi_i = -U_i y + \psi_i, \ P_i = Q_i + q_i \tag{2.8a}$$

with

$$q_i = \nabla^2 \psi_i - d_i^{-1} \sum_{j=1}^{2} T_{ij} \psi_j \ , \qquad i = 1,2 \tag{2.8b}$$

III. LINEAR THEORY (NORMAL MODE METHOD)

Substituting (2.8) into (2.1)–(2.4) and neglecting the quadratic terms in (2.1) yield

$$\frac{\partial q_i}{\partial t}+U_i\frac{\partial q_i}{\partial x}+\frac{\partial \psi_i}{\partial x}\cdot\frac{\partial \pi_i}{\partial y}=0,\quad i=1,2 \tag{3.1}$$

$$q_1=\nabla^2\psi_1-F_1((1+\alpha)\psi_1-\psi_2)\ , \tag{3.2a}$$

$$q_2=\nabla^2\psi_2-F_2(\psi_2-\psi_1) \tag{3.2b}$$

$$\frac{\partial \pi_1}{\partial y}=\beta+F_1((1+\alpha)U_1-U_2) \tag{3.3a}$$

$$\frac{\partial \pi_2}{\partial y}=\beta-F_2(U_1-U_2) \tag{3.3b}$$

where

$$F_i=d_i^{-1}f_0^2(g_1^{-1})\ ,\qquad i=1,2\ . \tag{3.3c}$$

The boundary conditions of the disturbance are

$$\frac{\partial \psi_i}{\partial x}=0\ ,\qquad y=\pm Y \tag{3.4a}$$

$$\frac{\partial}{\partial t}\int_{-\pi L}^{\pi L}\frac{\partial \psi_i}{\partial y}dx=0\ ,\qquad y=\pm Y\ ,\qquad i=1,2 \tag{3.4b}$$

Normal mode solutions to (3.1) and (3.4) may be sought in the form,

$$\psi_i=A_i\cos(l_jy)e^{ik(x-ct)} \tag{3.5}$$

where A_i is the disturbance amplitude in the ith layer (constant), and

$$l_j=(j+\frac{1}{2})\pi/Y\ ,\qquad j=0,1,\cdots \tag{3.6}$$

Substituting (3.5) into (3.1), we get two coupled algebraic equations for A_1 and A_2, i.e.

$$A_1[(c-U_1)[K^2+F_1(1+\alpha)]+\beta+F_1((1+\alpha)U_1-U_2)]-A_2(c-U_1)F_1=0 \tag{3.7a}$$

$$A_2[(c-U_2)(K^2+F_2)+\beta-F_2(U_1-U_2)]-A_1(c-U_2)F_2=0\ , \tag{3.7b}$$

where K is the total wavenumber

$$K^2=k^2+l_j^2\ . \tag{3.8}$$

Nontrivial solutions for A_1 and A_2 are possible only if the determinant of the coefficients of A_1 and A_2 in (3.7) vanishes. This condition leads directly to a quadratic equation for c:

$$a_0\cdot c^2+a_1\cdot c+a_2=0\ , \tag{3.9}$$

where $c=c_r+ic_i$, and

$$a_0=K^4+((1+\alpha)F_1+F_2)K^2+\alpha F_1F_2\ , \tag{3.10a}$$

$$a_1=-(U_1+U_2)K^4+(2\beta-(2+\alpha)F_1U_2-2U_1F_2)K^2 + \beta(F_1(1+\alpha)+F_2)-\alpha F_1F_2U_1\ , \tag{3.10b}$$

$$a_2=U_1U_2K^4+(\beta(U_s-2U_1)+F_1U_s(U_s-2U_1) + U_1^2(F_1+F_2))K^2+\beta^2+\beta(F_1U_s-U_1(F_1+F_2))\ , \tag{3.10c}$$

where $U_s=U_1-U_2$ is the velocity shear.

The solutions to (3.9) are

$$c = -\frac{a_1}{2a_0} \pm \frac{\sqrt{a_1^2 - 4a_0 \cdot a_2}}{2a_0} \tag{3.11}$$

It follows that the sufficient and necessary condition for instability is $c_i \neq 0$ i.e.

$$a_1^2 - 4a_0 \cdot a_2 < 0 \tag{3.12a}$$

or equivalently,

$$\begin{aligned} & U_s^2 K^4 (K^4 - 4F_1 F_2) + 2U_s K^4 (F_1 - F_2)\beta + \beta^2 (F_1 + F_2)^2 - 2\alpha F_1 U_1 U_s K^6 \\ & + (\alpha^2 F_1^2 U_2^2 - 2\alpha F_1 F_2 U_1 U_s + 2\alpha\beta F_1 U_s) K^4 \\ & - 2\alpha F_1 U_2 (\alpha\beta F_1 + \beta(F_1 - F_2) - \alpha U_1 F_1 F_2 - 2F_1 F_2 U_s) K^2 + \alpha^2 F_1^2 F_2^2 U_1^2 \\ & - 2F_1 F_2 [\alpha^2 F_1 U_1 + \alpha U_1 (F_1 - F_2) - 2\alpha F_1 U_2]\beta \\ & + (\alpha^2 F_1^2 + 2\alpha F_1 (F_1 - F_2))\beta^2 < 0 \end{aligned} \tag{3.12b}$$

By defining nondimensional variables as follows (an asterisk denotes a dimensional quantity):

$$\begin{aligned} & (x^*, y^*) = L(x, y) , \quad (u^*, v^*) = U(u, v) , \\ & \beta^* = (U / L^2)\beta , \quad F^* = (1 / L^2) F , \\ & U_1^* = UU_1 , \quad U_2^* = UU_2 , \quad U_s^* = UU_s . \end{aligned} \tag{3.13}$$

We can derive the nondimensional expression of (3.12b), that is the same as the inequality (3.12b) in form. In the special case that $F_1 = F_2 = F$, we neglect terms $O(\alpha^2)$. Then, (3.12b) reduces to

$$\begin{aligned} & U_s^2 K^4 (K^4 - 4F^2) + 4F^2 \beta^2 - 2\alpha F U_1 U_s K^6 - 2\alpha F (F U_1 U_s + \beta U_s) K^4 \\ & + 4\alpha F^3 U_2 U_s K^2 - 2F^3 \alpha (U_s - U_1)\beta < 0 . \end{aligned} \tag{3.14}$$

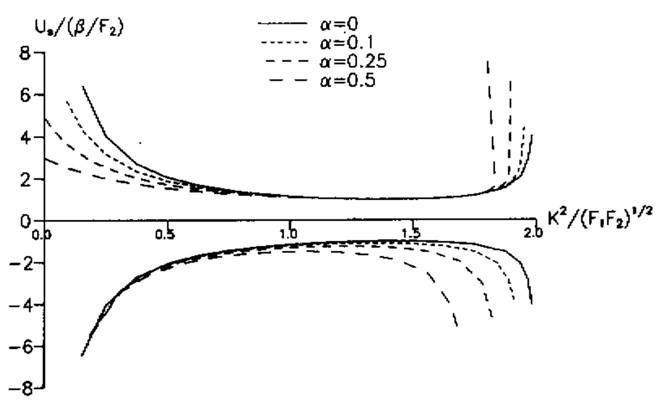

Fig. 1. The curves of marginal stability when $F_2 = F_1$ with various α.

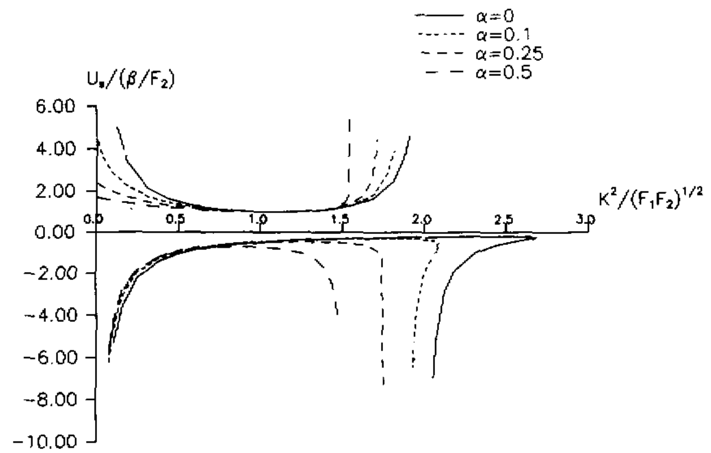

Fig. 2. The curves of marginal stability when $F_2 / F_1 = 0.2$ with various α.

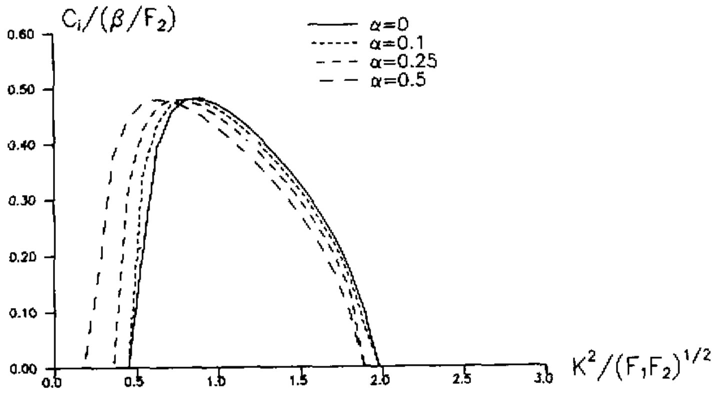

Fig. 3. The imaginary part of c as a function of wave number in the case $F_1 = F_2$ for $U_s = 2\beta / F_2$ various α.

Fig.1 shows U_c, the critical value of U_s required for instability, as a function of K^2. The minimum critical shear is as follows

$$(U_c)_{\min} = \beta / F$$

or

$$(U_c)_{\min} = -\beta / F - \alpha U_1 \tag{3.15}$$

Fig.2 shows the marginally stable curves for various α, when $F_1 / F_2 = 5$. From Fig.2 we can see that when the free surface parameter α increases, regardless of the sign of the vertical shear U_s, the unstable domain moves in the direction of small wavenumber (i.e. long wave direction). When the shear $U_s > 0$, because of the rigid condition in bottom surface, the minimal critical shear value does not change; when the shear $U_s < 0$, the absolute value of the

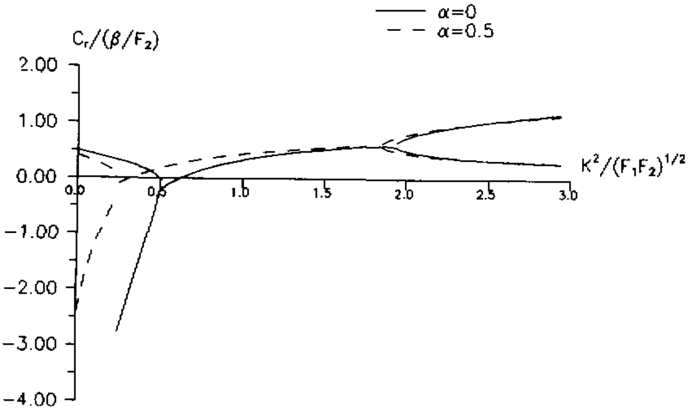

Fig. 4. The real part of c as a function of wave number when $U_s = 2\beta / F_2$ with $\alpha = 0$ and $\alpha = 0.5$

minimal critical shear increases with the increase of α, which is also clear from the formula.

In the case of $F_1 = F_2$ and $U_s = 2\beta / F$ the imaginary part of c as a function of wave number for various α is shown in Fig.3, which demonstrates that with the increase of α, the unstable domain moves in the direction of small wave number (i.e. long wave direction), and that the most unstable wavelength increase with the increase of α. Especially, for the typical large−scale atmospheric motions, $f_0 \sim 10^{-4}\text{s}^{-1}$, $H \sim 10^4$ m, $g_1 \sim 6$ ms^{-2}, $L \sim 10^6$ m, the most unstable wavelengths for different α are as follows

FSP	$\alpha = 0$	$\alpha = 0.1$	$\alpha = 0.25$	$\alpha = 0.5$
MUW (km)	5635	6281	6702	6969

FSP——Free Surface Parameter, MUW——Most Unstable Wavelength

Fig.4 shows c_r as a function of K^2 for various α, for $F_1 = F_2$, which exhibits the effect of the free surface parameter on the phase velocity.

The classical Phillips model (i. e. the top and bottom surfaces are all rigid) corresponds to the case $\alpha = 0$; and in this case the expression (3.12b) reduces to (7.11.31) of Pedlosky (1979).

In the case that the bottom surface is free, following the above discussions, we can get similar results. The detailed expressions are omitted for simplicity.

IV. NONLINEAR STABILITY CRITERIA

To investigate the nonlinear stability of the generalized Phillips′ model, we first establish a nonlinear stability criterion. Assume that there exists a constant γ and functions $\Psi_i^\gamma(\cdot)$, such that

$$\Psi_i + \gamma y = \Psi_i^\gamma(Q_i) \ , \qquad i = 1,2 \tag{4.1}$$

Corresponding to the hypothesis of Arnol′d′s second theorem, we suppose that there exist constants C_{1i} and C_{2i}, $i = 1,2$, such that

$$0 < C_{1i} \leqslant -\frac{d\Psi_i^\gamma}{dQ_i} \leqslant C_{2i} < \infty \ , \qquad i = 1,2 \ . \tag{4.2}$$

Noting conservation of the zonal momentum

$$M(t) = \iint_D \left\{ \sum_{i=1}^{2} d_i y P_i \right\} dxdy \ , \tag{4.3}$$

then similar to the proof for Criterion 3.1 of Liu and Mu (1994), we can show

Criterion 4.1. Suppose that the basic state (Ψ_i, Q_i) satisfies (4.1) and (4.2) and that

$$M = \begin{bmatrix} C_{11} - \dfrac{\lambda + F_1}{R} & -\dfrac{\sqrt{F_1 F_2}}{R} \\ -\dfrac{\sqrt{F_1 F_2}}{R} & C_{12} - \dfrac{\lambda + (1+\alpha) F_2}{R} \end{bmatrix} \tag{4.4}$$

is positive definite, then it is nonlinearly stable.

The above notation $R = \lambda^2 + (F_1 + F_2 + \alpha F_1)\lambda + \alpha F_1 F_2$ and λ is the lowest positive eigenvalue of the following problem

$$\begin{cases} \nabla^2 u + \lambda u = 0 \ , & \text{in } D \\ \left.\dfrac{\partial u}{\partial x}\right|_{y = \pm Y,} & \displaystyle\int_{-\pi L}^{\pi L} \frac{\partial u}{\partial y} dx = 0 \end{cases} \tag{4.5}$$

which can be easily solved as $\lambda = \pi / Y^2$.

Obviously, M is positive definite if and only if

$$C_{11} - \frac{\lambda + F_2}{R} > 0 \ , \tag{4.6a}$$

$$C_{12} - \frac{\lambda + (1+\alpha) F_1}{R} > 0 \ , \tag{4.6b}$$

$$(C_{11} - \frac{\lambda + F_2}{R})(C_{12} - \frac{\lambda + (1+\alpha) F_1}{R}) > \frac{F_1 F_2}{R} \tag{4.6c}$$

For the generalized Phillips' model, because U_1 and U_2 are constants, it is clear that

$$\Psi_1 + \gamma y = (\gamma - U_1) y \ , \tag{4.7a}$$

$$\Psi_2 + \gamma y = (\gamma - U_2) y \ , \tag{4.7b}$$

$$Q_1 = [\beta + F_1 (U_s + \alpha U_1)] y + f_0 \ , \tag{4.8a}$$

$$Q_2 = [\beta - F_2 U_s] y + f_0 \ . \tag{4.8b}$$

So we consider the stability of the generalized Phillips' model in the following four cases.

Case 1. $\quad \beta + F_1 (U_s + \alpha U_1) = 0$.

In this case, we choose $\gamma = U_1$. Then, the basic state (ψ_i, Q_i) satisfies (4.2) with

$$C_{11}=c\ ,\qquad C_{12}=\frac{1}{F_1+F_2+\alpha F_1 U_1/U_s}\ , \tag{4.9}$$

where c is an arbitrary constant, (4.6) becomes

$$\frac{1}{F_1+F_2+\alpha F_1 U_1/U_s}-\frac{\lambda+(1+\alpha)F_1}{R}>0\ , \tag{4.10a}$$

which is equivalent to

$$\lambda^2-\frac{\alpha F_1 U_1}{U_s}\lambda-F_1F_2-F_2-\alpha F_2^2-(1+\alpha)F_1^2\cdot\frac{\alpha U_1}{U_s}>0\ . \tag{4.10b}$$

On the other hand, when c is chosen to be sufficiently large, (4.6a) and (4.6c) may be satisfied. Hence, by Criterion 4.1, when (4.10) holds, the basic state is nonlinearly stable.

Case 2. $\beta-F_2U_s=0$.

Similar to Case1,we choose $\gamma=U_2$ and

$$C_{11}=\frac{U_s}{\beta+F_1U_s}=\frac{1}{F_1+F_2}\ ,\qquad C_{12}=c \tag{4.11}$$

Then, one sufficient condition for the nonlinear stability of the basic state is

$$\lambda^2-F_2(F_1+F_2)-\frac{\alpha F_1U_2}{U_s}\lambda-\frac{\alpha F_1F_2U_1}{U_s}>0\ , \tag{4.12}$$

Case 3. $(\beta+F_1(U_s+\alpha U_1))(\beta-F_2U_s)>0$ i.e

$$-\frac{\beta}{F_1}-\alpha U_1<U_s<\frac{\beta}{F_2} \tag{4.13}$$

By (4.7) and (4.8), for any $\gamma=\min(U_1,U_2)$, the basic state satisfies (4.2). We can choose $\gamma\rightarrow-\infty$, such that C_{11} and C_{12} may be arbitrarily large, which makes (4.6a–c) be satisfied. By Criterion 4.1, (4.13) is a sufficient condition for the nonlinear stability of the basic state.

In this case, the potential vorticity gradients of the basic state dQ_1/dy and dQ_2/dy being the same sign, it follows from the finite–amplitude generalized Charney–Stern theorem (c.f. Shepherd, 1988) that the basic state is nonlinearly stable. This fact explains why no restriction on λ akin to (4.10) and (4.12). Finally, (4.13) is the same as the expression (3.15) in form, and reveals the nonlinear property of the minimal critical shear phenomenon.

Case 4. $(\beta+F_1(U_s+\alpha U_1))(\beta-F_2U_s)<0$ i.e

$$U_s<-\frac{\beta}{F_1}-\alpha_1U_1\qquad\text{or}\qquad U_s>\frac{\beta}{F_2}\ . \tag{4.14a,b}$$

The basic state satisfies (4.2) with

$$C_{11}=\frac{U_1-\gamma}{\beta+F_1(U_s+\alpha U_1)}\ ,\qquad C_{12}=\frac{U_2-\gamma}{\beta-F_2U_s} \tag{4.15}$$

where γ is chosen to satisfy $\min(U_1,U_2)<\gamma<\max(U_1,U_2)$. Substituting (4.15) into (4.6), and eliminating γ in (4.6a) and (4.6b) yield

$$\lambda^2+(\alpha F_1-\frac{\alpha F_1 U_1}{U_s})\lambda+\frac{\beta}{U_s}(F_1-F_2+\alpha F_1)$$
$$-2F_1F_2-F_1F_2\frac{\alpha U_1}{U_s}>0\ . \tag{4.16}$$

For (4.6c), expressing the left hand side in terms of powers of γ yields a quadratic inequality for γ. There exists γ such that the inequality holds if and only if its discriminant is positive, i.e.

$$\begin{aligned}&U_s^2\lambda^2(\lambda^2-4F_1F_2)+2U_s\beta(F_1-F_2)\lambda+(F_1+F_2)^2\beta^2\\&\quad-2\alpha F_1U_1U_s\lambda^3+(\alpha^2F_1^2U_2^2-2\alpha F_1F_2U_1U_s+2\alpha\beta F_1U_s)\lambda^2\\&\quad-2\alpha F_1U_2(\alpha\beta F_1+\beta(F_1-F_2)-\alpha U_1F_1F_2-2F_1F_2U_s)\lambda\\&\quad+\alpha^2F_1^2F_2^2U_1^2-2F_1F_2[\alpha^2F_1U_1+\alpha U_1(F_1-F_2)-2\alpha F_1U_s]\beta\\&\quad+(\alpha^2F_1^2+\alpha F_1(F_1-F_2))\beta^2<0\ .\end{aligned}$$

Then, by Criterion 4.1, (4.16) and (4.17) are one sufficient condition for the nonlinear stability of the basic state.

To sum up, the basic state for the generalized Phillips′ model is nonlinearly stable if one of the following conditions is satisfied:

(i) $U_s=-\beta/F_1-\alpha U_1$ and (4.10) holds

(ii) $U_s=\beta/F_2$ and (4.12) holds

(iii) $-\beta/F_1-\alpha U_1<U_s<\beta/F_2$

(iv) $U_s<-\dfrac{\beta}{F_1}-\alpha U_1$ or $U_s>\beta/F_2$ and (4.16) and (4.17) holds.

If we set $K^2=\lambda$ in (3.12b), then (3.12b) is exactly the opposite of (4.17). Since λ is just the minimum wavenumber, (4.17) corresponds to the marginally stable curve, and reveals the nonlinear property of the shortwave cut–off phenomenon.

Although the linear and nonlinear stability criteria of the generalized Phillips′ model are exactly the same in form, their theoretical contents are different. The former are derived from the linearized equation and applicable to infinitesimal disturbances, while the latter are derived from the fully nonlinear equation and applicable to finite–amplitude disturbances.

Finally, we discuss the influence of the free surface parameter on the instability of atmospheric and oceanic motions.

For oceans, according to the typical density–depth profile in large–scale oceanic motions (Pedlosky, 1979), $\alpha=O(10^{-3})$. From the stability criteria (i)–(iv), it is clear that terms with α are three orders of magnitudes less than ones without α. So, for large–scale oceanic motions the influences of the free surface parameter may be ignored; but for atmospheric motions, $\alpha=O(10^{-1})$, it shows that the influences of the free surface parameter can not be ignored here (Lindzen et al. (1968) obtained similar conclusion by studying the oscillations in atmosphere).

V. SUMMARY

By applying the results of Mu et al.(1994) and Liu (1994) and considering the conservation of the zonal momentum (or impulse), the nonlinear stability has been studied for the two–layer generalized Phillips′ model, and nonlinear criteria have been obtained. Meanwhile, the linear stability criteria have also been derived by classical normal mode method for this model. By comparing the above results the nonlinear property of the shortwave cut–off phenomenon and the minimum critical shear phenomenon have been revealed. The influences of free surface parameter α have been discussed. It is shown that when the free surface parameter α increases, then (i) the unstable domain moves in the direction of small wavenumber (i.e. longwave direction); (ii) the absolute value of the minimum critical shear inceases; (iii) the maximum growth rate diminishes. And the effect of the free surface parameter α on the phase velocity has been given in Fig. 4.

For the typical large–scale oceans the influences of the free surface parameter α may be ignored with $\alpha = O(10^{-3})$; but for the large–scale atmosphere, it can not be ignored with $\alpha = O(10^{-1})$.

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