A Study on the Nonlinear Stability of Fronts in the Ocean on a Sloping Continental Shelf①

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ABSTRACT

The baroclinic nonlinear stability of fronts in the ocean on a sloping continental shelf is studied, the model equations, called the frontal geostrophic model, developed by Cushman-Roisin et al.(1992) for describing the dynamics of surface density fronts in the ocean are developed and the two-layer frontal geostrophic model for fronts on a sloping continental shelf is first obtained. The nonlinear stability criteria for the fronts on a sloping bottom are obtained by using Arnol'd (1965, 1969) variational principle and a prior estimate method. It is shown that our result is better than the former works.

Key words: Fronts in the ocean, Frontal geostrophic model, Nonlinear stability

1. Introduction

Arnol'd (1965, 1969) variational principle and a prior estimate method is essentially a generalization of Lyapunov stability method for finite-dimensional dynamical systems in infinite-dimensional ones, and he studied the nonlinear stability of 2-dimensional incompressible ideal fluid motion by use of this method, and established two theorems which are Arnol'd's first theorem and Arnol'd's second theorem. Ever since the 1980's, many scientists have been working on this subject, Holm et al. (1985); McIntyre and Shepherd (1987); Zeng (1989); Mu et al. (1994); Li and Mu (1996a,b) and Li et al. (1998a,b) studied the nonlinear stability of many kinds of fluid by employing Arnol'd's method.

Density or temperature fronts and isolated eddies are common features in many coastal regions of the world oceans. Examples of the currents associated with density fronts include the Denmark Strait overflow (Smith, 1976), Antarctic Bottom Water formed in the Weddell Sea (Whitehead and Worthington, 1982) and deep water formation in the Adriatic Sea (Zoccolatti and Salusti, 1987), etc. The instability of these currents may lead to the formation of deep cold coherent eddies. These flows play an important role in the mesoscale physical and biological dynamics of the benthic boundary layer on continental shelves. In particular, it will be crucial to correctly parameterize many fluxes associated with density-driven currents and solitary eddies in order to appropriately ascertain the contribution of these flows to longer term climate variations.

There are two important kinematic characteristics which must be accounted for in a complete dynamical description of these flows. The first one centres on the fact that

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the isopycnal deflections associated with density currents and solitary cold eddies are not small in comparison with the scale height of the hydrostatic geopotential. This is an important fact because it implies that the space and time derivatives of the density field in the mass continuity equation cannot be neglected in comparison with the horizontal divergence of the velocity field. That is to say, at least in principle, a quasi–geostrophic model will be inappropriate as a dynamical theory. The second characteristic is based on the fact that these flows are strongly baroclinic. This fact is expressed theoretically through the Stern integral constraint (Mory, 1983, 1985), experiments have consistently shown (Mory et al., 1987; Whitehead et al., 1990) that the Eulerian velocity field in the overlying surrounding fluid is at least comparable (if not larger than) to the co–moving velocities in the eddy interior.

Notwithstanding these points, Smith (1976) was able to describe some aspects of the instability observed for the density–current associated with the Denmark Strait overflow with a baroclinic quasi–geostrophic model, and Griffiths et al. (1982) studied the stability of couple density fronts by use of a long wavelength perturbation analysis of the ageostrophic barotropic instability of a gravity current on a sloping bottom. But there are substantial differences between their results and the laboratory simulations of the instability of a buoyant coupled density front. It is very important to find some suitable dynamics to study the instability fronts in the ocean. In order to construct a theory which could address the deficiency of these earlier models, Whitehead et al. (1990) have developed a model that focus directly on the baroclinic and finite–amplitude isopycnal deflection aspects of physics. This model is a kind of simplified two–layer mixed model, i.e., for fronts in coastal regions, the upper layer is quasi–geostrophic, and the lower one is the simplified shallow water model. Swaters and Flierl (1991) discussed the linear instability of fronts in coastal regions by using this model, and Swaters (1993) studied the nonlinear stability of intermediate baroclinic flow on a sloping bottom.

It is very important to develop a suitable model to describe the fronts in a sloping bottom. Cushman–Roisin (1986) frist developed a one–layer frontal geostrophic model for describing the dynamics of surface density fronts in the ocean. His main idea is, from the primitive equation, similar to the procedure of establishment of the quasi–geostrophic model to obtain the frontal geostrophic model, but in the derivation of the frontal geostrophic equations the Rossby number is made small with requirement of the length scale being larger than the radius of deformation due to the characteristic of the finite interfacial displacements. Both formalisms of frontal geostrophic model and quasi–geostrophic model are based on the assumption of a small Rossby number so that the velocity field can be split into an easily calculable geostrophic component and an even much smaller ageostrophic correction. Due to the baroclinic of fronts in the ocean, Cushman–Roisin et al. (1992) established the two–layer frontal geostrophic model on the basis of the one–layer one. This model overcomes the disadvantages of the former models, and is suitable for both the baroclinic and the finite interfacial displacements of fronts in the ocean. Thus it plays an important role in studying the occurring, evolving, destroying and bursting of fronts in the ocean.

Following the former studies, we first obtain the two–layer frontal geostrophic equations for density fronts in the coastal regions in considering that the bottom topography is a sloping continental shelf in Section 2. In Section 3 we study the nonlinear stability of surface fronts

in the ocean and fronts on a sloping bottom using the frontal geostrophic models by employing Arnol'd's method, the nonlinear stability criteria for the fronts in a sloping are obtained. It is shown that for fronts in a sloping bottom our result is better than the former ones. We will summarize the main conclusion in Section 4.

2. Derivation of the governing equations

2.1 *The main idea*

Suppose that the interfacial displacement scale is δH (not necessarily equal to the mean depth layer, H) and the motion length scale is L (not necessarily the deformation radius, L_R), then, the f-plane geostrophy requirement provides that the velocity scale is

$$U = \frac{g'\delta H}{f_0 L}, \tag{1}$$

and it follows that the Rossby number is

$$\epsilon = \frac{U}{f_0 L} = \frac{g'\delta H}{f_0^2 L^2} = \left(\frac{\delta H}{H}\right)\left(\frac{L_R}{L}\right)^2, \tag{2}$$

thus the Rossby number depends on two scales that characterize the motion: δH and L.

In the case of $\delta H \ll H$ and thus $\epsilon \ll 1$, the dynamics is quasi-geostrophic. On the other hand, frontal motions with finite displacements ($\delta H = H$) are ageostrophic unless the length scale is large compared to the deformation radius ($L^2 \gg L_R^2$ and thus $\epsilon \ll 1$), in which case the dynamics is frontal geostrophic.

In fact, it turns out that the Rossby number is then the square of the ratio of deformation radius over length scale and is small enough for practical purposes when the length scale is three or more times the deformation radius.

Overall, it is recognized that the quasi-geostrophic and frontal geostrophic formalisms are established by parallel mathematical developments but are based on opposite and complementary physical processes.

2.2 *The model for density fronts in a sloping continental shelf*

In order to keep some degree of clarity in the derivations, the analysis starts with a series of assumptions aimed at limiting the number of parameters. These are two layers, f-plane, a sloping bottom, rigid lid, and the upper layer at least as deep as the lower layer (Fig.1). The equations which govern an oceanic system with one interfacial degree of freedom are those of the two-layer shallow water model on f-plane:

$$\vec{V}_{it} + (\vec{V}_i \cdot \nabla)\vec{V}_i + f_0 \vec{k} \times \vec{V}_i = -\frac{1}{\rho_i}\nabla p_i, \quad i = 1,2, \tag{3}$$

$$h_{it} + \nabla \cdot (h_i \vec{V}_i) = 0, \quad i = 1,2, \tag{4}$$

where f_0 is the Coriolis parameter (constant), $\vec{V}_i = (u_i, v_i)$, $i = 1,2$ is the horizontal velocity vector, ρ_1 and ρ_2 are the densities in the upper layer and the lower layer, respectively. $\nabla = (\partial_x, \partial_y)$ with (x, y) being the horizontal coordinates, and t is time, subscripts (x, y, t) indicate the partial derivatives with respect to (x, y, t), $\vec{k}$ is the unit vertical vector, and h_i is the ith layer depth. And by the assumptions, the pressure p_i can be written as

$$p_1 = \rho_1 g\pi, \quad p_2 = \rho_1 \pi + \rho_2 g'(\eta - \alpha^* y), \tag{5}$$

where π is the pressure in the upper layer and η the upward interfacial displacement, $g' = g(\rho_2 - \rho_1)/\rho_2$ is the reduced gravity (stable stratification), and α^* is the slope parameter of the sloping bottom.

$$h_1 = H - H_2 - \eta, \quad h_2 = H_2 + \eta - \alpha^* y, \tag{6}$$

where H is the total depth of fluid, H_2 is the lower layer depth.

Suppose that L is the horizontal length scale, $L_R = \frac{\sqrt{g'H_2}}{f_0}$ is the internal deformation radius, and the scales of p_1, η, p_2 and t are P, ΔH, $g'\Delta H$ and T, respectively, then the scale of velocity (u_1, v_1) is $P/f_0 L$, that of (u_2, v_2) is $g'\Delta H/f_0 L$. Thus, we can define the dimensionless numbers of the layer-thickness ratio and the slope parameter as follows:

$$\delta = \frac{H_2}{H}, \tag{7}$$

$$\alpha = \frac{\alpha^* L}{H}, \tag{8}$$

the dimensionless numbers naturally resulting from the scaling of the equations are

$$s = \frac{g'H_2}{f_0^2 L^2} = \left(\frac{L_R}{L}\right)^2, \tag{9}$$

$$\epsilon = \frac{g'\Delta H}{f_0^2 L^2} = \left(\frac{\Delta H}{H_2}\right)\left(\frac{L_R}{L}\right)^2, \tag{10}$$

and the Rossby number, ϵ, can be considered as a measure of the motion amplitude, ΔH, once L is specified. Meanwhile, the scales of time and pressure T and P generate two last dimensionless numbers:

$$\omega = \frac{1}{f_0 T}, \quad \gamma = \frac{P}{g'\Delta H}. \tag{11}$$

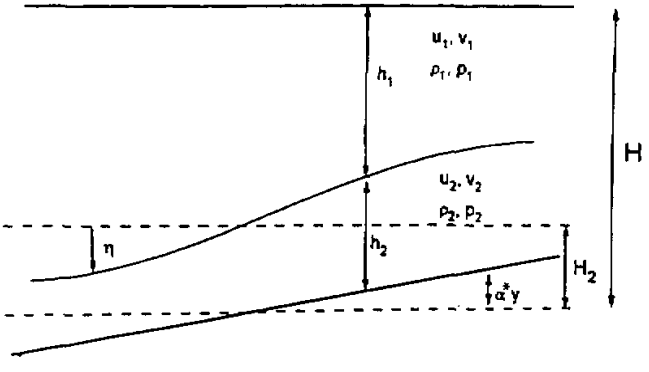

Fig. 1. Geometry of the two-layer model for fronts on a sloping bottom.

For fronts in the coastal regions, due to the above-mentioned assumptions, we have the relationships of these dimensionless numbers as follows:

$$s=\epsilon,\ \alpha=\delta,\ \omega=\epsilon^2,\ \gamma=\epsilon,\ \delta\leqslant\epsilon\leqslant\delta^{1/2}\ll 1. \tag{12}$$

By use of the relationships in Eq.(12), we transforme Eqs. (3) and (4) into nondimensionalized ones and the symbols of the variables remain unchanged, thus we have

$$\epsilon^2\vec{V}_{1t}+\epsilon^2(\vec{V}_1\cdot\nabla)\vec{V}_1+\vec{k}\times\vec{V}_1=-\nabla p_1 \tag{13}$$

$$-\epsilon^2\eta_t-\epsilon^2\nabla(\eta\vec{V}_1)+(\frac{\epsilon^2}{\delta}-\epsilon^2)\nabla\vec{V}_1=0, \tag{14}$$

$$\epsilon^2\vec{V}_{2t}+\epsilon(\vec{V}_2\cdot\nabla)\vec{V}_2+\vec{k}\times\vec{V}_2=-\nabla p_2 \tag{15}$$

$$\epsilon^2\eta_t+\epsilon\nabla(\eta\vec{V}_2)+\epsilon\nabla(y\vec{V}_2)+\epsilon\nabla\vec{V}_2=0, \tag{16}$$

and Eq.(5) becomes

$$p_1=\pi,\ p_2=\epsilon\pi+\eta-y. \tag{17}$$

From Eq. (13) and the geostrophic balance $\vec{V}_1=\vec{k}\times\nabla p_1$, we have

$$\vec{V}_1=\vec{k}\times\nabla p_1-\epsilon^2\nabla p_{1t}-\epsilon^2J(p_1,\nabla p_1), \tag{18a}$$

$$\nabla\vec{V}_1=\nabla(\vec{k}\times\nabla p_1)-\epsilon^2\nabla^2p_{1t}-\epsilon J(p_1,\nabla^2p_1), \tag{18b}$$

where $J(f,g)=f_xg_y-f_yg_x$ is the horizontal Jacobian operator and ∇^2 is the two dimensional Laplacian. Substituting Eq.(18) into Eq.(13), from Eq.(17) we have

$$(\eta+\frac{\epsilon^2}{\delta}\nabla^2\pi)_t+J(\pi,\frac{\epsilon^2}{\delta}\nabla^2\pi+\eta)=0. \tag{19}$$

Similarly, from Eq. (15) and the geostrophic balance $\vec{V}_2=\vec{k}\times\nabla p_2$, we have

$$\vec{V}_2=\vec{k}\times\nabla p_2-\epsilon^2\nabla p_{2t}-\epsilon J(p_2,\nabla p_2). \tag{20}$$

Substituting Eq.(20) into Eq.(16), and eliminating the high terms of ϵ (since $\epsilon\ll 1$), we can obtain

$$\eta_t-J(\eta-y,\pi+\nabla^2(\eta-y)+(\eta-y)\nabla^2(\eta-y)+\frac{1}{2}\nabla(\eta-y)\cdot\nabla(\eta-y))=0. \tag{21}$$

Denoting the upper depth $h=1+\eta$, considering the case of $\delta=\epsilon^2$ and eliminating the high terms (ϵ^2 items) in Eqs.(19) and (21), we can obtain the following equations

$$(\nabla^2\pi+h)_t+J(\pi,\nabla^2\pi+h)=0, \tag{22a}$$

$$h_t-J(h-y,\pi+(h-y)\nabla^2(h-y)+\frac{1}{2}\nabla(h-y)\cdot\nabla(h-y))=0. \tag{22b}$$

Eq.(22) is the two-layer frontal geostrophic model for fronts on a sloping continental shelf. For this model, we can obtain the following conservations by using the boundary conditions:

$$\frac{d}{dt}\iint F(\frac{\epsilon^2}{\delta}\nabla^2\pi+h)dxdy=0, \tag{23a}$$

$$\frac{d}{dt}\iint G(h-y)dxdy=0, \tag{23b}$$

where F and G are two arbitrary integrable functions.

3. The nonlinear stability of fronts in the ocean

Define $\tilde{h}= h- y$ in Eqs.(22) and (23), we can obtain

$$\tilde{h}_t + J(\pi+ \tilde{h}\nabla^2\tilde{h}+ \frac{1}{2}\nabla\tilde{h}\cdot\nabla\tilde{h}, \tilde{h})= 0, \tag{22a)'}$$

$$(\nabla^2\pi+ \tilde{h}+ y)_t + J(\pi, \nabla^2\pi+ \tilde{h}+ y)= 0, \tag{22b)'}$$

and we have the conservations as follows:

$$\frac{d}{dt}\iint F(\nabla^2\pi+ \tilde{h}+ y)dxdy= 0, \tag{23a)'}$$

$$\frac{d}{dt}\iint G(\tilde{h})dxdy= 0. \tag{23b)'}$$

For the governing equation (22)', the horizontal area is a zonal period channel

$$D= \{-1\leqslant x\leqslant 1, -1\leqslant y\leqslant 1\}. \tag{24}$$

The horizontal boundary conditions are those of no normal flow and conservation of circulation at each level:

$$\pi_x = 0, \quad \frac{d}{dt}\{\int_{-1}^{1}\pi_y\, dx|_{y=-1,1}\}= 0, \quad y= -1,1, \tag{25a}$$

$$\tilde{h}_x = 0, \quad \frac{d}{dt}\{\int_{-1}^{1}\tilde{h}_y\, dx|_{y=-1,1}\}= 0, \quad y= -1,1. \tag{25b}$$

In this case, Eq. (23)' holds.

Supposing that $(\pi, \tilde{h})= (\pi_0, \tilde{h}_0)$ is a steady solution to the frontal geostrophic equation (22)', a disturbance $(\delta\tilde{h}, \delta\pi)$ to this steady basic state is defined as follows:

$$\tilde{h}(x, y, t)= \tilde{h}_0(x, y)+ \delta\tilde{h}(x, y, t), \quad \pi(x, y, t)= \pi_0(x, y)+ \delta\pi(x, y, t), \tag{26}$$

and satisfies the boundary conditions as follows:

$$\delta\pi_x = 0, \quad \int_{-1}^{1}\delta\pi_y\, dx|_{y=-1,1}= 0, \quad y= -1,1, \tag{27a}$$

$$\delta\tilde{h}_x = 0, \quad \int_{-1}^{1}\delta\tilde{h}_y\, dx|_{y=-1,1}= 0. \quad y= -1,1. \tag{27b}$$

Thus, we have

$$J(\pi_0+ \tilde{h}_0\nabla^2\tilde{h}_0+ \frac{1}{2}\nabla\tilde{h}_0\cdot\nabla\tilde{h}_0, \tilde{h}_0)= 0, \tag{28a}$$

$$J(\pi_0, \nabla^2\pi_0+ \tilde{h}_0+ y)= 0. \tag{28b}$$

A sufficient condition for Eq.(28) is

$$\pi_0+ \tilde{h}_0\nabla^2\tilde{h}_0+ \frac{1}{2}\nabla\tilde{h}_0\cdot\nabla\tilde{h}_0= F_1(\tilde{h}_0), \tag{29a}$$

$$\pi_0= F_2(\nabla^2\pi_0+ \tilde{h}_0+ y), \tag{29b}$$

where F_1 and F_2 are two arbitrary differentiable functions.

By the conservations in Eq.(23)′ and the boundary conditions (25) and (27), we have the following conservations

$$H(\bar{q})=\frac{1}{2}\iint_D (|\nabla\pi|^2+\tilde{h}|\nabla\tilde{h}|^2+\tilde{h}^2\nabla^2\tilde{h})dxdy, \tag{30}$$

$$C(\bar{q})=\iint_D \Phi_1(\tilde{h})dxdy+\iint_D \Phi_2(\nabla\pi+\tilde{h}+y)dxdy, \tag{31}$$

where $\bar{q}=(q_1,q_2)=(\tilde{h},\nabla^2\pi+\tilde{h}+y)$, Φ_1 and Φ_2 are arbitrary first order continuously differentiable functions.

According to the study of Arnol'd (1965, 1969), suppose that there are positive constants $C_i>0$, $i=1,\cdots,4$, such that

$$C_1<F'_1(\xi)<C_2, \tag{32a}$$

$$C_3<-F'_2(\xi)<C_4, \tag{32b}$$

where F_1 and F_2 are defined in Eq.(29).

Consider the following invariant functional

$$L(\bar{q})\equiv H(\bar{q}_0+\delta\bar{q})-H(\bar{q}_0)+C(\bar{q}_0+\delta\bar{q})-C(\bar{q}_0), \tag{33}$$

where H and C are the invariant functionals defined in Eqs.(30) and (31). And Eq.(33) can be written as

$$\begin{aligned}L(\bar{q})=&\frac{1}{2}\iint_D\{\nabla\delta\pi\cdot\nabla\delta\pi-\tilde{h}\nabla\delta\tilde{h}\cdot\nabla\delta\tilde{h}+\nabla^2h(\delta\tilde{h})^2\}dxdy\\&-\iint_D\{\int_{q_{10}}^{q_{10}+\delta q_1}F_1(\xi)d\xi-F_1(q_{10})\delta q_1\}dxdy,\\&+\iint_D\{\int_{q_{20}}^{q_{20}+\delta q_2}F_2(\xi)d\xi-F_2(q_{20})\delta q_2\}dxdy,\end{aligned} \tag{34}$$

where $\delta\bar{q}=(\delta q_1,\delta q_2)=(\delta\tilde{h},\nabla^2\delta\pi+\delta\tilde{h})$ is a finite-amplitude disturbance, $\bar{q}=\bar{q}_0+\delta\bar{q}$.

By the assumption in Eq.(32), we have

$$\frac{C_1(\delta q_1)^2}{2}<\int_{q_{10}}^{q_{10}+\delta q}F_1(\xi)d\xi-F_1(\xi)\delta q<\frac{C_2(\delta q_1)^2}{2}, \tag{35a}$$

$$\frac{C_3(\delta q_2)^2}{2}<\int_{q_{20}}^{q_{20}+\delta q}F_2(\xi)d\xi-F_2(\xi)\delta q<\frac{C_4(\delta q_2)^2}{2}. \tag{35b}$$

Substituting Eq.(35) into Eq.(34), we can obtain the following inequality:

$$\begin{aligned}&\frac{1}{2}\iint_D\{\nabla\delta\pi\cdot\nabla\delta\pi-\tilde{h}\nabla\delta\tilde{h}\cdot\nabla\delta\tilde{h}+\nabla^2\tilde{h}(\delta\tilde{h})^2-C_2(\delta q_1)^2-C_4(\delta q_2)^2\}dxdy\\&\qquad<L(\bar{q})<\\&\frac{1}{2}\iint_D\{\nabla\delta\pi\cdot\nabla\delta\pi-\tilde{h}\nabla\delta\tilde{h}\cdot\nabla\delta\tilde{h}+\nabla^2\tilde{h}(\delta\tilde{h})^2-C_1(\delta q_1)^2-C_3(\delta q_2)^2\}dxdy.\end{aligned} \tag{36}$$

On the other hand, by the boundary conditions (27a) and (27b), two Poincaré inequalities can be obtained:

$$\iint_D \nabla\delta\pi \cdot \nabla\delta\pi dxdy \leqslant \tilde{C}\iint_D (\nabla^2\delta\pi)^2 dxdy, \tag{37}$$

$$\iint_D \delta\tilde{h} \cdot \delta\tilde{h} dxdy \leqslant \frac{1}{\tilde{C}}\iint_D \nabla\delta\tilde{h} \cdot \nabla\delta\tilde{h} dxdy, \tag{38}$$

where $\tilde{C}$ and $\tilde{\tilde{C}}$ are positive constants.

Combining Eqs.(37) and (38) with Eq.(36), we can obtain an important inequality as follows:

$$L(\overline{q}) < \frac{1}{2}\iint_D (\tilde{C} - C_3)(\nabla^2\delta\pi + \delta\tilde{h})^2 - 2\tilde{C}(\nabla^2\delta\pi + \delta\tilde{h})\delta\tilde{h} + (\nabla^2\tilde{h}_0 - \tilde{\tilde{C}}(\min_{(x,y)\in D}\tilde{h}_0) - \tilde{C} - C_1)(\delta h)^2 dxdy. \tag{39}$$

Thus, by the definition of $\tilde{h} = h - y$ the nonlinear stability criterion for the fronts in a sloping bottom can be obtained as follows:

Criterion 1. The basic state (π_0, h_0) of the frontal geostrophic equation (22) satisfies Eqs.(29) and (32), and

$$C_3 > \tilde{C} \tag{40a}$$

$$-C_1 - \tilde{\tilde{C}}(\min_{(x,y)\in D}(h_0 - y)) + \nabla^2(h_0 - y) < \frac{\tilde{C}C_3}{\tilde{C} - C_3}, \tag{40b}$$

then the basic state is nonlinearly stable; i.e., for any $\in > 0$, there exists $\delta > 0$ such that, when

$$\|\overline{q}(x, y, 0) - \overline{q}_0(x, y)\| < \delta, \tag{41}$$

we have

$$\|\overline{q}(x, y, t) - \overline{q}_0(x, y)\| < \in, \tag{42}$$

where the mode $\|\delta\overline{q}\|$ is defined as

$$\|\delta\overline{q}\| = \|\overline{q} - \overline{q}_0\| = [-L(\overline{q})]^{1/2}. \tag{43}$$

Proof: Under the condition (40), we can prove that the invariant functional is negative definite by Eq.(39), i.e,

$$L(\overline{q}) \leqslant -\Lambda\iint_D |\nabla^2\delta\pi + \delta(h - y)|^2 + |\delta(h - y)|^2 dxdy,$$

where Λ is a positive constant. Thus, by the definition of the mode $\|\delta q\|$ in Eq.(43), we can prove that the basic state (π_0, h_0) is nonlinearly stable.

Swaters (1993) gave the nonlinear stability criterion using a simple two-layer model, the criterion is

$$\tilde{C}_3 > \tilde{C} \tag{44a}$$

$$-\tilde{C}_1 < \frac{\tilde{C}\tilde{C}_3}{\tilde{C} - \tilde{C}_3}. \tag{44b}$$

Comparing Eq.(40) to Eq.(44), we find that Eq.(40a) is the same as Eq.(44a); and Eq.(44b) is a special case (in the case of $\nabla^2(h_0 - y) - \bar{C}(\min_{\{(x,y)\in D\}}(h_0 - y)) = 0$) of Eq.(40b). Therefore, **Criterion 1** is better than the result of Swaters(1993). It is shown that the frontal geostrophic model established in this paper is more suitable for the study of fronts in the ocean.

4. Conclusion and discussion

Fronts are regions of large horizontal gradients of certain properties, usually including density. Fronts are important natural barriers in the ocean. While they hinder horizontal transfers of heat, momentum and other properties, they play a crucial role in enhancing vertical exchanges. They are also large reserviors of available potential energy, which are often released in the form of eddies and rings.

In this paper, the frontal geostrophic model, which was first developed by Cushman–Roisin (1986) and Cushman–Roisin et al. (1992), was extended to describe the dynamic system of density fronts on a sloping continent shelf. The new frontal geostrophic model is a two–layer one with a sloping bottom. The nonlinear stability of fronts in the coastal regions was discussed. And the nonlinear stability criteria of fronts in the coastal region were obtained. It is shown that our result is better than that of Swaters (1993).

It should be pointed out that we only focused on the theoretical analyses. It is our future aim how to apply the theoretical results to the actual oceanic problem. And the problem in the β–plane approximation is our another future aim.

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