A Comparative Analysis of Computational Stability for Linear and Non–Linear Evolution Equations

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ABSTRACT

For several difference schemes of linear and non–linear evolution equations, taking the one–dimensional linear and non–linear advection equations as examples, a comparative analysis for computational stability is carried out and the relationship between non–linear computational stability, the construction of difference schemes, and the form of initial values is discussed. It is proved through comparative analysis and numerical experiment that the computational stability of the difference schemes of the non–linear evolution equation are absolutely different from that of the linear evolution equation.

Key words: evolution equation, difference scheme, computational stability, initial value

1. Introduction

Climate numerical simulations, numerical forecasts, and ocean current numerical simulations can all be summed up as the numerical computation of evolution equations. So it is very important to assure the long–time computational stability of difference schemes. For linear evolution equations, Von Neumann and Richtmyer (1950) first used Fourier analysis to give a stability criteria. Later, Hirt (1968) also put forward a kind of method to analyze this problem, namely, the heuristic analysis method. Up until now, the computational stability of linear evolution equations is basically solved. For the computational stability of non–linear evolution equations, however, so far the general judgement method is still unobtainable. Zeng (1978), Ji (1981a, b), Zeng and Ji (1981), Wang and Ji (1990, 1994), Ji and Wang (1991), Wang et al. (1995), and Ji et al. (1998) systematically studied the non–linear computational instability for the difference schemes of non–linear evolution equations and inquired into the reasons for non–linear computational instability. In this paper, taking the one–dimensional linear and non–linear advection equations as examples, a comparative analysis of the computational stability is carried out. Furthermore, discussion is made on the relationship between non–linear computational stability, the construction of difference schemes, and the form of initial values.

2. Equations and difference schemes

Let us consider the one–dimensional linear advection equation,

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$$\frac{\partial u}{\partial t}+U\frac{\partial u}{\partial x}=0\ , \qquad U>0,\quad a\leqslant x\leqslant b\ ,\quad 0\leqslant t\leqslant T\ , \tag{1}$$

$$u(x,0)=\varphi(x)\ , \tag{2}$$

and the one-dimensional non-linear advection equation,

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}=0\ , \qquad a\leqslant x\leqslant b\ ,\quad 0\leqslant t\leqslant T\ , \tag{3}$$

$$u(x,0)=\varphi(x)\ . \tag{4}$$

These equations can be analyzed numerically, using the following difference schemes,

Scheme 1 (FTBS scheme)

$$\frac{u_j^{n+1}-u_j^n}{\Delta t}+\frac{U}{\Delta x}(u_j^n-u_{j-1}^n)=0\ , \tag{5}$$

Scheme 2 (Lax-Wendroff scheme)

$$\frac{u_j^{n+1}-u_j^n}{\Delta t}+\frac{U}{2\Delta x}(u_{j+1}^n-u_{j-1}^n)-\frac{U^2\Delta t}{2\Delta x^2}(u_{j+1}^n-2u_j^n+u_{j-1}^n)=0 \tag{6}$$

for the linear equation, and

Scheme 3 (FTBS scheme)

$$\frac{u_j^{n+1}-u_j^n}{\Delta t}+\frac{u_j^n}{\Delta x}(u_j^n-u_{j-1}^n)=0\ , \tag{7}$$

Scheme 4 (Lax-Wendroff scheme)

$$\frac{u_j^{n+1}-u_j^n}{\Delta t}+\frac{u_j^n}{2\Delta x}(u_{j+1}^n-u_{j-1}^n)-\frac{(u_j^n)^2\Delta t}{2\Delta x^2}(u_{j+1}^n-2u_j^n+u_{j-1}^n)=0 \tag{8}$$

for the non-linear equation.

3. Comparative analysis for the computational stability of the difference schemes

First, we carry out an heuristic analysis for schemes 1 and 2, taking scheme 2 as an example. By means of a Taylor expansion for (6), we obtain

$$\frac{u_j^{n+1}-u_j^n}{\Delta t}=\frac{\partial u_j^n}{\partial t}+\frac{1}{2}\frac{\partial^2 u_j^n}{\partial t^2}\Delta t+\frac{1}{6}\frac{\partial^3 u_j^n}{\partial t^3}\Delta t^2+\frac{1}{24}\frac{\partial^4 u}{\partial t^4}\Delta t^3+O(\Delta t^4)\ , \tag{9}$$

$$\frac{U}{2\Delta x}(u_{j+1}^n-u_{j-1}^n)=U\left(\frac{\partial u_j^n}{\partial x}+\frac{1}{6}\frac{\partial^3 u_j^n}{\partial x^3}\Delta x^2\right)+O(\Delta x^4)\ , \tag{10}$$

$$\frac{U^2\Delta t}{2\Delta x^2}(u_{j+1}^n-2u_j^n+u_{j-1}^n)=\frac{U^2}{2}\frac{\partial^2 u_j^n}{\partial x^2}\Delta t+\frac{U^2\Delta x^2}{24}\frac{\partial^4 u}{\partial x^4}\Delta t+O(\Delta x^4)\ . \tag{11}$$

Substituting (9), (10), and (11) into (6), omitting superscripts and subscripts, we have

$$\frac{\partial u}{\partial t}+U\frac{\partial u}{\partial x}=-\frac{1}{2}\Delta t\left(\frac{\partial^2 u}{\partial t^2}-U^2\frac{\partial^2 u}{\partial x^2}\right)-\frac{1}{6}\left(\Delta t^2\frac{\partial^3 u}{\partial t^3}+U\Delta x^2\frac{\partial^3 u}{\partial x^3}\right)$$
$$-\frac{1}{24}\Delta t\left(\Delta t^2\frac{\partial^4 u}{\partial t^4}-U^2\Delta x^2\frac{\partial^4 u}{\partial x^4}\right)+O(\Delta t^4,\Delta x^4)\ . \tag{12}$$

The terms on the rhs of (12) become

$$-\frac{1}{2}\Delta t\left(\frac{\partial^2 u}{\partial t^2}-U^2\frac{\partial^2 u}{\partial x^2}\right)=\frac{1}{6}U^2\Delta t(U^2\Delta t^2-\Delta x^2)\frac{\partial^4 u}{\partial x^4}+O(\Delta t^4,\Delta t^2\Delta x^2)\ , \tag{13}$$

$$-\frac{1}{6}\left(\Delta t^2\frac{\partial^3 u}{\partial t^3}+U\Delta x^2\frac{\partial^3 u}{\partial x^3}\right)=\frac{1}{6}U(U^2\Delta t^2-\Delta x^2)\frac{\partial^3 u}{\partial x^3}+O(\Delta t^4,\Delta t^2\Delta x^2)\ , \tag{14}$$

$$-\frac{1}{24}\Delta t\left(\Delta t^2\frac{\partial^4 u}{\partial t^4}-U^2\Delta x^2\frac{\partial^4 u}{\partial x^4}\right)=\frac{1}{24}U^2\Delta t(U^2\Delta t^2-\Delta x^2)\frac{\partial^4 u}{\partial x^4}+O(\Delta t^4,\Delta t^2\Delta x^2)\ . \tag{15}$$

Substituting (13), (14), and (15) into (12), we obtain the modified partial differential equation of scheme 2,

$$\frac{\partial u}{\partial t}+U\frac{\partial u}{\partial x}=\frac{1}{6}U(U^2\Delta t^2-\Delta x^2)\frac{\partial^3 u}{\partial x^3}+\frac{1}{8}U^2\Delta t(U^2\Delta t^2-\Delta x^2)\frac{\partial^4 u}{\partial x^4}$$
$$+O(\Delta t^4,\Delta t^2\Delta x^2)\ . \tag{16}$$

From (16), we know the second–order dissipative coefficient is 0 and the fourth–order disspative coefficient is

$$\mu_{2r}=\frac{1}{8}U^2\Delta t(U^2\Delta t^2-\Delta x^2)\ ,\qquad r=2\ . \tag{17}$$

Similarly, we can obtain the modified partial differential equation of scheme 1,

$$\frac{\partial u}{\partial t}+U\frac{\partial u}{\partial x}=-\frac{1}{2}U(U\Delta t-\Delta x)\frac{\partial^2 u}{\partial x^2}-\frac{1}{6}U(U\Delta t-\Delta x)(2U\Delta t-\Delta x)\frac{\partial^3 u}{\partial x^3}$$
$$+O(\Delta t^3,\Delta t\Delta x^2)\ . \tag{18}$$

The second–order dissipative coefficient is

$$\mu_{2r}=-\frac{1}{2}U(U\Delta t-\Delta x)\ ,\qquad r=1\ . \tag{19}$$

The ample and necessary condition of computational stability of schemes 1 and 2 is (Warming and Hyett 1974)

$$(-1)^{r-1}\mu_{2r}>0\ . \tag{20}$$

Hence, we have the following theorem.

Theorem 1. For scheme 1 (FTBS) and scheme 2 (Lax– Wendroff) of the one–dimensional linear advection equation, the ample and necessary condition of computational stability is

$$U\frac{\Delta t}{\Delta x}\leqslant 1\ . \tag{21}$$

Second, we carry out an heuristic analysis for schemes 3 and 4, taking scheme 4 as an example. By means of a Taylor expansion for (8), we obtain

$$\frac{u_j^{n+1}-u_j^n}{\Delta t}=\frac{\partial u_j^n}{\partial t}+\frac{1}{2}\frac{\partial^2 u_j^n}{\partial t^2}\Delta t+\frac{1}{6}\frac{\partial^3 u_j^n}{\partial t^3}\Delta t^2+O(\Delta t^3)\ , \tag{22}$$

$$\frac{u_j^n}{2\Delta x}(u_{j+1}^n-u_{j-1}^n)=u_j^n\left(\frac{\partial u_j^n}{\partial x}+\frac{1}{6}\frac{\partial^3 u_j^n}{\partial x^3}\Delta x^2\right)+O(\Delta x^3)\ , \tag{23}$$

$$\frac{(u_j^n)^2 \Delta t}{2\Delta x^2}(u_{j+1}^n - 2u_j^n + u_{j-1}^n) = \frac{(u_j^n)^2}{2}\frac{\partial^2 u_j^n}{\partial x^2}\Delta t + O(\Delta x^2) \ . \tag{24}$$

Substituting (22), (23), and (24) into (8), omitting superscripts and subscripts, we obtain

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = -\frac{1}{2}\Delta t\left(\frac{\partial^2 u}{\partial t^2} - u^2\frac{\partial^2 u}{\partial x^2}\right) - \frac{1}{6}\left(\Delta t^2\frac{\partial^3 u}{\partial t^3} + u\Delta x^2\frac{\partial^3 u}{\partial x^3}\right) + O(\Delta t^3, \Delta x^2) \ . \tag{25}$$

From (25), we can see

$$\frac{\partial u}{\partial t} = -u\frac{\partial u}{\partial x} + O(\Delta t, \Delta x^2) \ . \tag{26}$$

Same as in (13), (14), we have

$$\frac{\partial^2 u}{\partial t^2} = 2u\left(\frac{\partial u}{\partial x}\right)^2 + u^2\frac{\partial^2 u}{\partial x^2} + O(\Delta t, \Delta x^2) \ , \tag{27}$$

$$\frac{\partial^3 u}{\partial t^3} = -6u\left(\frac{\partial u}{\partial x}\right)^3 - 9u^2\frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial x^2} - u^3\frac{\partial^3 u}{\partial x^3} + O(\Delta t, \Delta x^2) \ . \tag{28}$$

Substituting (27) and (28) into (25), we obtain the modified partial differential equation of scheme 4,

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = -\Delta t u\left(\frac{\partial u}{\partial x}\right)^2 + \Delta t^2 u\left(\frac{\partial u}{\partial x}\right)^3 + \frac{3}{2}\Delta t^2 u^2\frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial x^2}$$
$$+ \frac{1}{6}(\Delta t^2 u^3 - \Delta x^2 u)\frac{\partial^3 u}{\partial x^3} + O(\Delta t^2, \Delta t\Delta x^2) \ . \tag{29}$$

Similarly, we can obtain the modified partial differential equation of scheme 3,

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = -\Delta t u\left(\frac{\partial u}{\partial x}\right)^2 + \Delta t^2 u\left(\frac{\partial u}{\partial x}\right)^3 + \frac{1}{2}(3\Delta t^2 u^2\frac{\partial u}{\partial x} - \Delta t u^2 + \Delta x u)\frac{\partial^2 u}{\partial x^2}$$
$$+ \frac{1}{6}(\Delta t^2 u^3 - \Delta x^2 u)\frac{\partial^3 u}{\partial x^3} + O(\Delta t^2, \Delta t\Delta x) \ . \tag{30}$$

Hence, the second–order dissipative coefficient of scheme 3 is

$$\mu_2 = \frac{1}{2}(3\Delta t^2 u^2\frac{\partial u}{\partial x} - \Delta t u^2 + \Delta x u) \ . \tag{31}$$

The second–order dissipative coefficient of scheme 4 is

$$\mu_2 = \frac{3}{2}\Delta t^2 u^2\frac{\partial u}{\partial x} \ . \tag{32}$$

Schemes 3 and 4 are stable only if the second–order dissipative coefficients are positive (Wu and Han 1988). Of course, they must be positive when $t = 0$ (Lin et al. 2000). Hence, we have the following theorems.

Theorem 2. For scheme 3 (FTBS) of the one–dimensional non–linear advection equation, the necessary condition of computational stability is

$$3\Delta t^2 u^2(x, 0)\frac{\partial u(x, 0)}{\partial x} - \Delta t u^2(x, 0) + \Delta x u(x, 0) > 0 \ . \tag{33}$$

Theorem 3. For scheme 4 (Lax–Wendroff) of the one–dimensional non–linear advection equation, the necessary condition of computational stability is

$$\frac{\partial u(x,0)}{\partial x} > 0 \ . \tag{34}$$

Through the above analysis, we have the following inferences.

Inference 1. The computational stability of the difference schemes of the one–dimensional linear advection equations is only concerned with the structure of the difference schemes. It is unconcerned with the form of the initial values.

Inference 2. The computational stability of the difference schemes of the one–dimensional non–linear advection equations is not only dependent on the structure of the difference schemes, but also on the form of the initial values and their partial derivatives.

4. Numerical examples

In order to verify the relationship between the computational stability of the difference schemes of the one–dimensional linear and non–linear advection equations, the structure of the schemes, and the form of the initial values, we perform the following numerical experiments. Four initial values are chosen,

(1) $u(x,0)=x$, (2) $u(x,0)=-x$, (3) $u(x,0)=1-e^{-x}$, (4) $u(x,0)=1-e^{x}$,
where $0\leqslant x\leqslant 1$, $0\leqslant t\leqslant 10$.

Numerically, we take $\Delta x=0.01$, $\Delta t=0.001$, and $U=1$. The results are shown in the Table 1.

Table 1. Computational results of numerical experiments

	Linear		Non–linear	
	Scheme 1	Scheme 2	Scheme 3	Scheme 4
initial value 1	stable	stable	stable	stable
initial value 2	stable	stable	unstable	unstable
initial value 3	stable	stable	stable	stable
initial value 4	stable	stable	unstable	unstable

From the results we can see that schemes 1 and 2 are stable for all initial values because Theorem 1 is satisfied. Schemes 3 and 4 are stable for initial values 1 and 3, owing to satisfying Theorems 2 and 3. They are unstable for initial values 2 and 4, however, since the stability conditions of Theorems 2 and 3 are not satisfied.

5. Conclusion and discussion

It is proved that the computational stability of the difference schemes of the non–linear evolution equation are absolutely different from that of the linear evolution equation through a comparative analysis and numerical experiment. For this reason, the analysis of computational stability must combine the construction of the difference scheme with the form of the initial value and its partial derivative. This is the main characteristic emphasized in this paper.

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线性与非线性发展方程计算稳定性的比较分析

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摘　要

针对线性与非线性发展方程的几种差分格式，以一维线性和非线性平流方程为例，对线性与非线性发展方程差分格式的计算稳定性进行了比较分析，揭示了差分格式结构和初值形式与计算稳定性的关系。理论分析和数值试验证明，线性与非线性发展方程差分格式的计算稳定性在本质上是完全不同的。

关键词: 发展方程，差分格式，计算稳定性，初值