Spatial and Time Structure of a Gravity Wave in Horizontal Atmosphere of Heterogeneous Stratification

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ABSTRACT

By the use of the WKBJ method combined with the characteristic line method, the asymptotic solution of a gravity wave envelope in the atmosphere of horizontal heterogeneous stratification and time-varying stratification is obtained. The solution shows that not only the variation of amplitude of the gravity wave but also the variation of wavelength and the width of the envelope are affected by the horizontal heterogeneity. As the wave envelope moves from a region of strong stratification to a weak one, the horizontal wavelength will become shorter, the width of the envelope will narrow and its amplitude will increase. The variation of stratification with time cannot lead to the variation of wavelength and envelope width, but the amplitude of the wave envelope will increase while the amplitude of the wave decreases in time.

Key words: gravity wave envelope, WKBJ method, characteristic line method

1. Introduction

Observations show that gravity waves play an important role and are taken as an important trigger and maintenance mechanism in the process of the development of storm weather. Based on this point, it is of great significance to study the development of gravity waves so as to understand the generating mechanism of storms and to make predictions efficiently. It is also an important problem of meso-scale dynamics to seek the amplifying or developing condition of gravity wave amplitude. Many previous studies show that the increment of amplitude of gravity waves may result from the instability of shear stream, terrain forcing, or diabatic heating. In addition to the points mentioned above, it is noted that the development of gravity waves is also subject to the heterogeneity of circulation. At the beginning of the 1980s, the WKBJ method was first applied to discuss the development of disturbances in a heterogeneous medium by Zeng and Lu (1980) and Zeng (1983); then the effects of heterogeneous streams and stratification to gravity waves (or inertia-gravity waves) as well as Rossby waves were studied by other researchers (Cao, 1980; Liu and Liu, 1987; He 1989; Wu, 1990; Zhong and Zhang, 1992; Wang and Zhou, 1994; Wang and Zhang, 1992a, b; Wang, 1997; Zhuo and Sun, 1990). But most of the works were merely limited to discussing the developing condition of the waves qualitatively based on the generalized wave action function, leaving the spatial structure of the wave envelope in a heterogeneous medium and the quantitative character of its time evolution untouched. In this paper, the WKBJ method combined with the characteristic line method is applied to obtain the analytic expression of a gravity wave envelope in a heterogeneous stratified atmosphere. Then the law governing the evolution of the envelope in such an atmosphere and the function of time-varying stratification are further discussed.

2. Basic equations

Assume that the varying scale of potential temperature in the large-scale background field is one order greater than the characteristic scale of the disturbed quantity. Then stratification stability, N^2, is a slow-varying function of time and space (x, t). In order to emphasis the essence of our study, a two-dimensional perturbation equation set of a static background field was considered in terms of the above condition and the Boussinesq approximation, namely

$$\begin{cases} \dfrac{\partial u}{\partial t} = -\dfrac{\partial}{\partial x}\left(\dfrac{p}{\rho_s}\right) \\ \lambda\dfrac{\partial w}{\partial t} = b - \dfrac{\partial}{\partial z}\left(\dfrac{p}{\rho_s}\right) \\ \dfrac{\partial b}{\partial t} + N^2 w = 0 \\ \dfrac{\partial u}{\partial x} + \dfrac{\partial w}{\partial z} = 0\,, \end{cases} \tag{1}$$

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where $\lambda = 0, 1$ is for the static and non-static approximation respectively, $b = g\theta/\theta_s$, θ is the deviation of potential temperature from the ground state, and ρ_s and θ_s are density and potential temperature in the ground state. If p/ρ_s is eliminated from (1), it can be shown that

$$\left(\lambda\frac{\partial^2}{\partial t^2} + N^2\right)\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2}{\partial t^2}\frac{\partial^2 w}{\partial z^2} + 2\frac{\partial N^2}{\partial x}\frac{\partial w}{\partial x} + \frac{\partial^2 N^2}{\partial x^2}w = 0\,. \quad (2)$$

The upper and lower boundary conditions take the following expressions

$$w\,\big|_{z=0} = 0\,, \qquad w\,\big|_{z=H} = 0\,. \quad (3)$$

Therefore, the initial and boundary value problem can be solved once the initial condition is provided. The solution that satisfies the boundary condition (3) can be supposed as

$$w = \bar{w}(x,t)\sin\left(\frac{n\pi z}{H}\right), \qquad n = 1, 2, \ldots \quad (4)$$

In order to satisfy its generality, from now on we merely consider one of its components that is chosen at random. If we take $l = n\pi/H$ and put formula (4) into formula (2), the horizontal structure equation can be obtained as

$$\frac{\partial^2}{\partial t^2}\left(\lambda\frac{\partial^2\bar{w}}{\partial x^2} - l^2\bar{w}\right) + N^2\frac{\partial^2 w}{\partial x^2} + 2\frac{\partial N^2}{\partial x}\frac{\partial\bar{w}}{\partial x} + \frac{\partial^2 N^2}{\partial x^2}w = 0\,, \quad (5)$$

and its initial condition can be written as

$$\bar{w}\,\big|_{t=0} = w_0(x)\,, \quad \frac{\partial\bar{w}}{\partial t}\Big|_{t=0} = w_1(x)\,, \quad (6)$$

where $w_0(x)$ represents the initial velocity disturbance of the gravity wave (Wang and Zhang, 1992b) and $w_1(x)$ stands for the effect of the initial buoyancy disturbance structure, both of which can cause a gravity wave disturbance of a different horizontal structure. This paper aims at discussing the effect of heterogeneous stratification on gravity waves, and for the purpose of simplifying the mathematics, we take a simple case,

$$\frac{\partial\bar{w}}{\partial t}\Big|_{t=0} = w_1(x) = 0\,.$$

This will not make any essential difference in the final result.

3. The development of wave envelope disturbance in the case of taking N^2 as a constant

In the case of taking N^2 as a constant, the solution of equation (5) is discussed below, all for the purpose of comparing it with the latter case of N^2 not equal to a constant. The initial value takes the following envelope disturbance

$$\bar{w}\,\big|_{t=0} = A_0\,\mathrm{e}^{-x^2/a^2}\,\mathrm{e}^{\mathrm{i}k_0x}\,, \quad \frac{\partial\bar{w}}{\partial t}\Big|_{t=0} = 0\,. \quad (7)$$

The above formula should be regarded as taking its real part (the same in the following cases). A_0, a, and k_0 are constants. The solution of such an initial value problem can be achieved through Fourier transformation,

$$\bar{w}(x,t) = \frac{1}{2}\int_{-\infty}^{\infty}\frac{aA_0}{2\sqrt{\pi}}\,\mathrm{e}^{-a^2(k-k_0)^2/4}[\mathrm{e}^{\mathrm{i}(kx+\omega t)} + \mathrm{e}^{\mathrm{i}(kx-\omega t)}]dk\,, \quad (8)$$

where k and ω satisfy the following dispersion relations

$$\omega^2 = k^2N^2/(\lambda k^2 + l^2)\,, \quad (9a)$$

$$k^2 = \omega^2 l^2/(N^2 - \lambda\omega^2)\,. \quad (9b)$$

On condition that the static equilibrium approximation ($\lambda = 0$) is satisfied, $\omega^2 = k^2N^2/l^2$ and the structure function of the gravity wave envelope can be obtained through integrating (8), namely

$$\bar{w}(x,t) = \frac{A_0}{2}\{\exp[-(x + Nt/l)^2/a^2 + \mathrm{i}k_0(x + Nt/l)] + \exp[-(x - Nt/l)^2/a^2 + \mathrm{i}k_0(x - Nt/l)]\}\,. \quad (10)$$

From (10), it is clear that when N^2 is treated as a constant, the gravity wave is non-dispersive under the approximation of static equilibrium and the wave envelope breaks into two parts that move toward their own side with a speed equal to N/l. Once $N^2 > 0$, the amplitude of the gravity wave envelope will decrease in time according to an exponential pattern. It is shown that the amplification of the wave amplitude does not result from homogeneous stratification, but is activated by stratification of the convective instability or wind shear of the basic field. Under the approximation of non-static equilibrium, the gravity wave envelope is dispersive in the horizontal direction; hence, the approximation solution of (8) can be obtained through constant phase method, which is omitted here.

4. The development of the wave envelope in the case of $N^2 = N^2(x,t)$

Formula (5) is a partial differential equation with a variable coefficient under the circumstance of heterogeneous stratification of $N^2 \neq$ constant. Generally, it is hard to obtain its analytical solution. Therefore, the WKBJ method is applied to discuss the development of the envelope on the condition of a slow varying stratification. In this way, it is easy to achieve the dispersive relation, the amplitude equation, as well as the wave action conservative equation. We then have

$$\sigma^2 = k^2N^2/(\lambda k^2 + l^2), \tag{11}$$

$$\frac{D_g}{DT}(K^2|w_{00}|^2) + (K^2|w_{00}|^2)\frac{\partial C_g}{\partial X} = -\frac{|w_{00}|^2k^2}{2\sigma^2}\frac{\partial N^2}{\partial T} - \frac{|w_{00}|^2k}{\sigma}\frac{\partial N^2}{\partial X}, \tag{12}$$

$$\frac{D_gE}{DT} + E\frac{\partial C_g}{\partial X} = 0, \tag{13}$$

where $X = \varepsilon t$ and $T = \varepsilon t$ are slow-varying coordinates, ε is a small parameter, $K^2 = \lambda k^2 + l^2$, and

$$E = \sigma\frac{K^2|w_{00}|^2}{k^2}$$

is the generalized wave action of the gravity wave. Equation (11) is similar to Eq. (9) in expression, but σ is a function of X, T in (11). In terms of (11), envelope velocity can be obtained as

$$C_g = \frac{\partial\sigma}{\partial k} = \frac{k}{\sigma}\frac{N^2 - \lambda\sigma^2}{\lambda k^2 + l^2} = \frac{\pm Nl^2}{(\lambda k^2 + l^2)^{3/2}}. \tag{14}$$

Here, the wave parameter equations are

$$\begin{cases} \dfrac{D_g\sigma}{DT} = \left(\dfrac{\partial\sigma}{\partial T}\right)_{k,X} = \dfrac{k^2}{2\sigma(\lambda k^2 + l^2)}\dfrac{\partial N^2}{\partial T}, \\ \dfrac{D_gk}{DT} = -\left(\dfrac{\partial\sigma}{\partial X}\right)_{K,T} = -\dfrac{K^2}{2\sigma(\lambda k^2 + l^2)}\dfrac{\partial N^2}{\partial X}, \\ \dfrac{D_g\phi}{DT} = C_gk - \sigma, \end{cases} \tag{15}$$

where φ is the phase angle and a function of (x,t), namely, a function of (X,T). Here

$$\frac{D_g}{DT} = \frac{\partial}{\partial T} + C_g\frac{\partial}{\partial X}.$$

Equations (12) and (13) are the general results of much research and it is obvious that the energy of the gravity wave is conservative when stratification is homogeneous and constant. Only on the condition of a heterogeneous and non-constant stratification can the gravity wave develop. Based on these points, the relation between the development of a gravity wave and N^2 has been thoroughly discussed. Here, we will not repeat it again. However, the wave action equation and wave energy equation can merely discuss the relation between the development of the gravity wave and heterogeneous stratification qualitatively, and cannot give the structure of the envelope as well as its time evolution. So, the characteristic line method is applied to seek an analytical solution in order to give a quantitative analysis of such a relation. In outline, the characteristic line method is as follows. Given $x_0 \in (-\infty, +\infty)$ and an initial value, an ordinary equation which is obtained by transforming (13) and (15) into equations in physical coordinates can be solved to achieve the variation of ϕ, σ, k, E on the line passing through the point x_0. By varying the value of $x_0 \in (-\infty, +\infty)$, every characteristic line can obtain a series of values for ϕ, σ, k, E; thus the final solution can be achieved (Whitham, 1974).

First, according to the equations: $X = \varepsilon t$, $T = \varepsilon t$, (13) and (15) can be transformed into equations in physical coordinates and their characteristic pattern can be given as

$$\begin{cases} \dfrac{d\sigma}{dt} = \dfrac{k^2}{2\sigma(\lambda k^2 + l^2)}\dfrac{\partial N^2}{\partial t}, \\ \dfrac{dk}{dt} = -\dfrac{k^2}{2\sigma(\lambda k^2 + l^2)}\dfrac{\partial N^2}{\partial x}, \\ \dfrac{d\phi}{dt} = C_gk - \sigma, \\ \dfrac{dE}{dt} = -E\dfrac{\partial C_g}{\partial x}, \end{cases} \tag{16}$$

where $C_g = dx/dt$, and its expression is given in (14). The two kinds of envelope velocity of $C_g > 0$ and $C_g < 0$ are determined by the sign of σ in (14): $C_g > 0$ corresponds to negative σ while $C_g < 0$ corresponds to positive σ. In the same way, two characteristic lines correspond to the two possible signs of σ. At first, we discuss the case of $C_g > 0$. The characteristic line determined by $dx/dt = C_g$ can be written as

$$x = x(x_0, t), \tag{17}$$

where $x_0 = x(x_0, 0)$ is the x coordinate of the characteristic line at $t = 0$. Assume the initial disturbance can be written as

$$\hat{w}\,|_{t=0} = A_0(x)\exp[i\phi_0(x)], \quad \frac{\partial\hat{w}}{\partial t}\Big|_{t=0} = 0, \tag{18}$$

where $\phi_0(x)$ satisfies

$$\frac{d\phi_0}{dx} = k_0(x),$$

and $A_0(x)$, $k_0(x)$ are slow-varying functions of x. Therefore, there exists an approximation, namely,

$$k\,|_{t=0} = k_0(x),$$

$$\sigma\,|_{t=0} = \sigma_0(x) = \pm\sqrt{N^2k_0^2/(\lambda k_0^2 + l^2)}, \tag{19a}$$

$$\phi\,|_{t=0} = \phi_0(x),$$

$$E\,|_{t=0} = \sigma_0\frac{\lambda k_0^2 + l^2}{k_0^2}|A_0(x)|^2 = E_0(x). \tag{19b}$$

4.1 *The solution in the case of $N^2 = N^2(x)$*

When $N^2 = N^2(x)$, then $\partial N^2/\partial t = 0$ and (16) can be rewritten as

$$
\begin{cases}
\dfrac{\mathrm{d}\sigma}{\mathrm{d}t}=0\,,\\[4pt]
\dfrac{\mathrm{d}k}{\mathrm{d}t}=-\dfrac{k^2}{2\sigma(\lambda k^2+l^2)}\dfrac{\mathrm{d}N^2}{\mathrm{d}x}\,,\\[4pt]
\dfrac{\mathrm{d}\phi}{\mathrm{d}t}=C_{\mathrm{g}}k-\sigma\,,\\[4pt]
\dfrac{\mathrm{d}E}{\mathrm{d}t}=-E\dfrac{\partial C_{\mathrm{g}}}{\partial x}\,.
\end{cases}
\tag{20}
$$

Consider the initial condition of (19) and integrate (20) from 0 to t along the characteristic line passing through x_0. We thus achieve

$$
\begin{cases}
\sigma\,\big|_{x_0}=\sigma_0(x_0)\,,\\[4pt]
k^2\,\big|_{x_0}=k_0(x_0)^2\Big[\dfrac{N^2(x_0)-\lambda\sigma_0^2(x_0)}{N(x)-\lambda\sigma_0^2(x_0)}\Big]\,,\\[4pt]
\phi\,\big|_{x_0}=\phi_0(x_0)+\displaystyle\int_0^t C_{\mathrm{g}}k\mathrm{d}t-\sigma_0(x_0)t\,.
\end{cases}
\tag{21}
$$

Notice that for a given characteristic line, x_0 is fixed and $x=x(x_0,t)=x_1(t)$. Thus x is merely determined by t. Equation (21) shows that k^2 on the characteristic line through x_0 is merely a function of x. Now further consider that N^2 is also only a function of x. Then the last equation in (20) can be rewritten as

$$\frac{\mathrm{d}E}{\mathrm{d}t}=-E\frac{\mathrm{d}C_{\mathrm{g}}}{\mathrm{d}x}\,. \tag{22a}$$

In the same way, integrating (22a) can lead to

$$E\,\big|_{x_0}=E_0(x_0)C_{\mathrm{g}}(x_0)/C_{\mathrm{g}}(x)\,. \tag{22b}$$

By the use of the generalized wave action

$$E=\sigma\frac{K^2|w_{00}|^2}{k^2}\,,$$

we can get

$$w_{00}\,\big|_{x_{00}}=\Big[\frac{N^2(x_0)-\lambda\sigma^2(x_0)}{N^2(x)-\lambda\sigma^2(x_0)}\Big]^{3/4}A_0(x_0)\,. \tag{23}$$

In the respect that $x_0=x_0(x,t)$ is an inverse function defined by (17), the analytic solutions of the gravity wave can be obtained from (21) and (23), namely,

$$
\begin{cases}
\sigma=\sigma(x_0)=\sigma_0(x_0)\,,\\[4pt]
k(x,t)=k_0(x_0)\Big[\dfrac{N^2(x_0)-\lambda\sigma_0^2(x_0)}{N^2(x)-\lambda\sigma_0^2(x_0)}\Big]^{1/2}\,,\\[4pt]
\phi(x,t)=\phi_0(x_0)+\displaystyle\int_{x_0}^{x}k\mathrm{d}x-\sigma_0(x_0)t\,,\\[4pt]
w_{00}(x,t)=\Big[\dfrac{N^2(x_0)-\lambda\sigma_0^2(x_0)}{N^2(x)-\lambda\sigma_0^2(x_0)}\Big]^{3/4}A_0(x_0)\,.
\end{cases}
\tag{24}
$$

Let the asymptotic solution of (5) adopt the following expression

$$
\begin{aligned}
\bar{w}(x,t)&=\hat{w}(x,t)\exp[\mathrm{i}\phi(x,t)]\approx w_{00}(x,t)\exp[\mathrm{i}\phi(x,t)]\\
&=\frac{1}{2}\hat{w}_1(x,t)\exp[\mathrm{i}\phi_1(x,t)]\\
&\quad+\frac{1}{2}\hat{w}_2(x,t)\exp[\mathrm{i}\phi_2(x,t)]\,.
\end{aligned}
\tag{25}
$$

Then for $C_{\mathrm{g}}>0$, $\hat{w}_1$ and ϕ_1 in the asymptotic solution of (5) are

$$\hat{w}_1(x,t)=\Big[\frac{N^2(x_{0+})-\lambda\sigma_0^2(x_{0+})}{N^2(x)-\lambda\sigma_0^2(x_{0+})}\Big]^{3/4}A_0(x_{0+})\,, \tag{26}$$

$$\phi_1(x,t)=\phi_0(x_{0+})+\int_{x_{0+}}^{x}C_{\mathrm{g}}k\mathrm{d}x+\sigma_0(x_{0+})t\,, \tag{27}$$

$$k(x,t)=k_0(x_{0+})\Big[\frac{N^2(x_{0+})-\lambda\sigma_0^2(x_{0+})}{N^2(x)-\lambda\sigma_0^2(x_{0+})}\Big]^{1/2}\,,$$

$$\sigma=\sigma(x_{0+})=\sigma_0(x_{0+})\,, \tag{28}$$

where $x_{0+}=x_{0+}(x,t)$ is determined by the characteristic line of $C_{\mathrm{g}}>0$. While $C_{\mathrm{g}}<0$, the same operation can be applied to obtain

$$\hat{w}_2(x,t)=\Big[\frac{N^2(x_{0-})-\lambda\sigma_0^2(x_{0-})}{N^2(x)-\lambda\sigma_0^2(x_{0-})}\Big]^{3/4}A_0(x_{0-})\,, \tag{29}$$

$$\phi_2(x,t)=\phi_0(x_{0-})+\int_{x_{0-}}^{x}C_{\mathrm{g}}k\mathrm{d}x+\sigma_0(x_{0-})t\,, \tag{30}$$

$$k(x,t)=k_0(x_{0-})\Big[\frac{N^2(x_{0-})-\lambda\sigma_0^2(x_{0-})}{N^2(x)-\lambda\sigma_0^2(x_{0-})}\Big]^{1/2}\,,$$

$$\sigma=\sigma(x_{0-})=\sigma_0(x_{0-})\,, \tag{31}$$

where $x_{0-}=x_{0-}(x,t)$ results from the characteristic line of $C_{\mathrm{g}}<0$.

If N^2 is regarded as a constant and the static equilibrium approximation is adopted, the solution that results from (25) is the same as that from (10) — the solution achieved by Fourier expansion. If we take the energy density of the envelope as

$$\varepsilon=\frac{(\lambda k^2+l^2)|w_{00}|^2}{k^2}$$

and take all the above conditions into it, it is easy to get

$$\varepsilon=\frac{l^2}{k_0^2}|A_0|^2\Big(\frac{N^2(x_0)}{N^2(x)}\Big)^{1/2}$$

on condition of the static equilibrium approximation. This shows that the spatial variation of stratification can lead to the variation of energy density of the envelope and consequently affect the variation of its vertical velocity. With a decrease in stratification, the energy density of the envelope will increase and the vertical velocity will be reinforced to form the storm weather. Otherwise, the density will decrease and then the storm weather will have difficulty forming.

In the condition of static equilibrium ($\lambda=0$), take $N^2=N_0^2\sec h^2x/a_0$, $A_0(x)=A_1\,\mathrm{e}^{-(x-x_{00})^2/a^2}$, and

$\phi_0(x) = k_0 x$. Here N_0^2, A_1, a_0, a, k_0 are constants, then

$$\sigma^2 = \frac{N_0^2 k^2}{l^2} \sec h^2\left(\frac{x}{a_0}\right).$$

Therefore the two characteristic lines and their inverse function can be achieved from

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \pm\frac{N_0}{l} \sec h\left(\frac{x}{a_0}\right)$$

as

$$\begin{cases} x = a_0 \ln\left[\mathrm{sh}\,\frac{x_{0+}}{a_0} + \frac{N_0 t}{a_0 l} + \sqrt{\left(\mathrm{sh}\,\frac{x_{0+}}{a_0} + \frac{N_0 t}{a_0 l}\right) + 1}\right], \\ x = a_0 \ln\left[\mathrm{sh}\,\frac{x_{0-}}{a_0} + \frac{N_0 t}{a_0 l} - \sqrt{\left(\mathrm{sh}\,\frac{x_{0-}}{a_0} + \frac{N_0 t}{a_0 l}\right) + 1}\right], \\ x_{0+} = a_0 \ln\left[\mathrm{sh}\,\frac{x}{a_0} + \frac{N_0 t}{a_0 l} + \sqrt{\left(\mathrm{sh}\,\frac{x}{a_0} + \frac{N_0 t}{a_0 l}\right) + 1}\right], \\ x_{0-} = a_0 \ln\left[\mathrm{sh}\,\frac{x}{a_0} + \frac{N_0 t}{a_0 l} - \sqrt{\left(\mathrm{sh}\,\frac{x}{a_0} + \frac{N_0 t}{a_0 l}\right) + 1}\right]. \end{cases} \tag{32}$$

Take (32) into (26)-(31), and we get

$$\bar{w}(x,t) = \frac{A_1}{2}\left[\frac{\mathrm{ch}\,(x/a_0)}{\mathrm{ch}\,(x_{0+}/a_0)}\right]^{3/2} \exp\left[-\frac{(x_{0+} - x_{00})^2}{a^2} + \mathrm{i}k_0 x_{0+}\right] + \frac{A_1}{2}\left[\frac{\mathrm{ch}\,(x/a_0)}{\mathrm{ch}\,(x_{0-}/a_0)}\right]^{3/2} \exp\left[-\frac{(x_{0-} - x_{00})^2}{a^2} + \mathrm{i}k_0 x_{0-}\right]. \tag{33}$$

Equation (33) gives the analytical solution of the gravity wave envelope under the condition of heterogeneous stratification. It is clear that the structure of the envelope in the heterogeneous stratified atmosphere can be divided into three parts. The first part is the vacillation of the envelope disturbance expressed by $e^{ikx_{0+}}$ or $e^{ikx_{0-}}$, the second part is the envelope width expressed by $e^{-(x_{0+}-x_{00})^2/a^2}$ or $e^{-(x_{0-}-x_{00})^2/a^2}$, and the last part is the amplitude of the envelope expressed by $[N^2(x_{0+})/N^2(x)]^{3/4}$ or $[N^2(x_{0-})/N^2(x)]^{3/4}$. For the sake of discussing the properties of the solution, the characteristic line calculated from the first two equations of (32) is first shown in Fig. 1 under the conditions of taking N^2 as a constant or as a slow-varying function of (x, t). Here, $l = \pi/5 \times 10^3\ \mathrm{m}^{-1}$ and the concrete value of N is shown in Fig. 1. It is easy to see from the figure that the slopes of the characteristic lines are very small and have a scattered distribution in the region of big N^2 while the contrary case appears in the region of small N^2. In light of the distribution of the characteristic lines, the relation between the three parts of the envelope analytical solution and the heterogeneous stratification can be drawn out as follows.

(1) Owing to k_0 as a constant, x_{0+} and x_{0-} have a rapid variation and the wavelength is shorter in the region of densely gathered lines. Therefore, the wavelength of the envelope will shorten as it moves from a large stratification to a small one and it will lengthen in the reverse direction. This implies that the heterogeneity of stratification will change the scale of the envelope of the gravity wave during the course of its spread. In detail, the weaker the stratification instability is, the smaller the disturbed scale will be and the easier the meso- and micro-scale system will develop.

(2) It is clear that the envelope width will become narrow when the envelope moves from a region of large stratification to a small one. Otherwise, it will widen (see Fig. 1). According to this term, the maximum of amplitude will be concentrated on the two characteristic lines of $x_{0+} = x_{00}$ and $x_{0-} = x_{00}$.

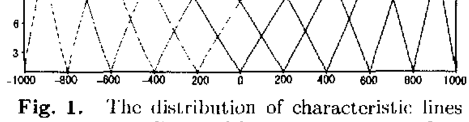

Fig. 1. The distribution of characteristic lines under the condition of heterogeneous stratification. The abscissa stands for x while the ordinate stands for t, and $N = 0.5 \times 10^{-2} \sec h[x/(5 \times 10^{-5})]\ \mathrm{s}^{-1}$.

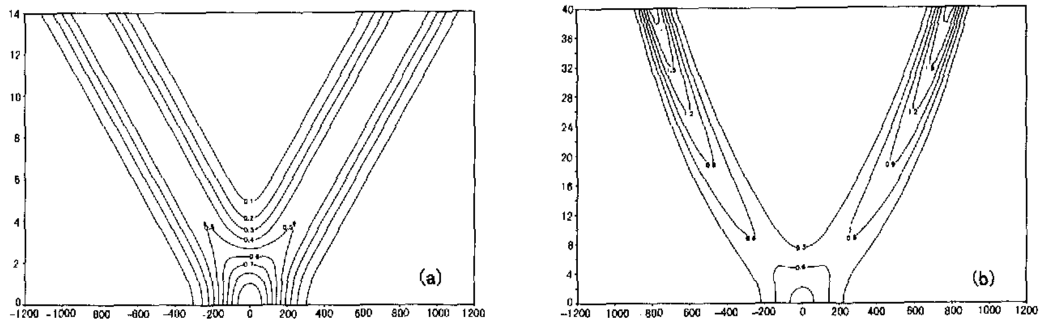

Fig. 2. Time variation of envelope amplitude ($a = 200$ km, $l = [\pi/(5\times10^3)]$ m^{-1}). (a) $N = 0.5\times10^{-2}$ s^{-1}, (b) $N = 0.5\times10^{-2}\sec h[x/(5\times10^5)]$ s^{-1}.

(3) The envelope amplitude will increase with a movement of the envelope from a region of large stratification to a small one and will decrease in the reverse dissection. According to this term, the maximum of amplitude will not occur at the two lines of $x_{0+} = x_{00}$ and $x_{0-} = x_{00}$. Furthermore, the enlarging of envelope amplitude means an increase of energy, which is of great advantage to the development of disturbances and is confirmed by many weather facts. In the real atmosphere, a meso-scale disturbance will often be stirred up as it moves from a strong stability of stratification to a weak one.

In order to describe the structure and time variation of the envelope and its amplitude of gravity waves more vividly and intuitively, we also calculate the analytic solution of the envelope under the condition of the above stratification. The results are given in Figs. 2 and 3. Figure 2 shows the time variation of envelope amplitude and Fig. 3 gives the physical image of envelope disturbance

$$[N^2(x_{0+})/N^2(x)]^{3/4}\,e^{-(x_{0+}-x_{00})^2/a^2}\cos(k_0x_{0+}),$$

which moves along the positive direction of the x-axis. For the sake of comparison, the case of heterogeneous stratification is given in the figure. From Fig. 2, although the amplitude of the initial disturbance is the same, the time-varying tendency of envelope amplitude is different because of the different distribution of N^2 along the x-axis. While N^2 is a constant (Fig. 2a), the amplitude will decrease with time along the characteristic line. When N^2 is not a constant, the variation of amplitude is determined by the distribution of N^2 along the x-axis. Under the condition of the above heterogeneous stratification, the amplitude increases with time along the characteristic line. The figure of the envelope's movement shows that not only the envelope amplitude but also the width and wavelength are closely related to the distribution of the heterogeneous stratification, and that its structure and time-variation are different in different stratifications. For the example shown in Fig. 3b, as x increases, the envelope width becomes narrow and the wavelength becomes shorter while its amplitude increases. This case is favorable to the development of disturbances.

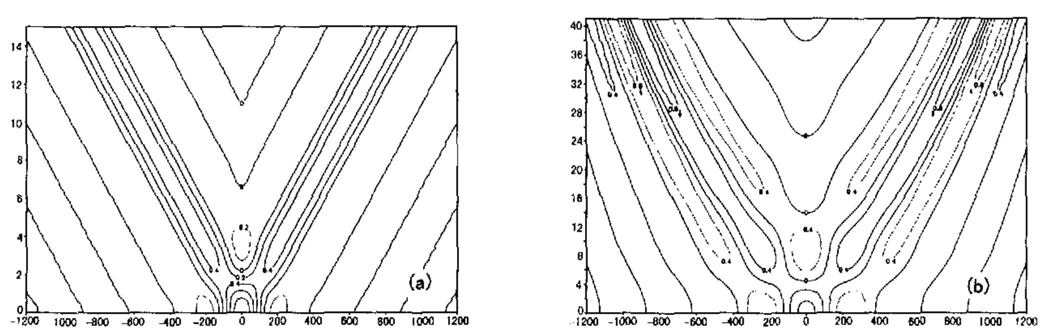

Fig. 3. The structure and time variation of the envelope moving along the x-axis ($a = 500$ km, $x_0 = 500$ km, $l = [\pi/(5\times10^3)]$ m^{-1}). (a) $N = 0.5\times10^{-2}$ s^{-1}, (b) $N = 0.5\times10^{-2}\sec h[8/(8\times10^5)]$ s^{-1}.

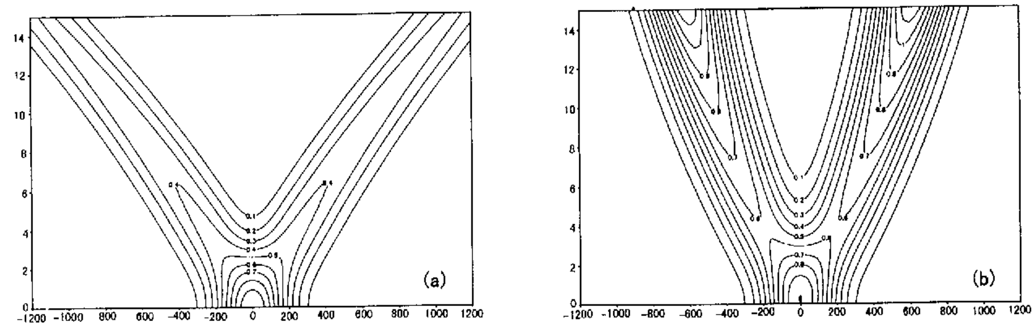

Fig. 4. Time variation of envelope amplitude when $N = 10^{-2}(1+t/T_c)$ (a) $T_c = 26$ hours, (b) $T_c = -26$ hours.

4.2 *The solution in the case of* $N^2 = N^2(t)$

This case is similar to that of $N^2 = N^2(x)$. When $N^2 = N^2(t)$, we obtain $\partial N/\partial x = 0$ and $\partial k/\partial t = 0$. The expression of C_g shows that C_g is merely a function of t along the characteristic line through x_0, therefore, it is also merely a function of x. Now (16) can be rewritten as

$$\begin{cases} \left.\dfrac{dk}{dt}\right|_{x_0} = 0\,, \\ \left.\dfrac{d\sigma}{dt}\right|_{x_0} = \dfrac{k^2}{2\sigma(\lambda k^2 + l^2)}\left.\dfrac{dN^2}{dt}\right|_{x_0}\,, \\ \left.\dfrac{d\phi}{dt}\right|_{x_0} = C_g K - \sigma\,, \\ \left.\dfrac{dE}{dt}\right|_{x_0} = -\dfrac{E}{C_g}\dfrac{dC_g}{dt} = \dfrac{Nl^2}{(\lambda k^2 + l^2)^{3/2}}\,. \end{cases} \tag{34}$$

After integrating (34) and considering initial conditions (19a) and (19b), we can obtain

$$\begin{cases} k(x,t) = k_0(x_0)\,, \\ \sigma(x,t) = \dfrac{k(x_0)N(t)}{[\lambda k_0^2(x_0) + l^2]^{1/2}}\,, \\ \phi(x,t) = \phi_0(x_0) + k_0(x_0)(x - x_0) - \displaystyle\int_0^t \sigma\, dt\,, \\ \hat{w}(x,t) = \dfrac{N(0)}{N(t)} A_0(x_0)\,, \end{cases} \tag{35}$$

$$w(x,t) = \frac{1}{2}\frac{N(0)}{N(t)}[A_0(x_{0+})\,e^{-i\phi_1} + A_0(x_{0-})\,e^{-i\phi_2}]\,, \tag{36}$$

where $x_0 = x_0(x,t)$ is determined by

$$x - x_0 = \frac{l^2}{[\lambda k_0^2(x_0) + l^2]^{3/2}} \int_0^t N(t)\,dt\,,$$

then

$$x_{0+} = x - \frac{l^2}{(\lambda k_0^2 + l^2)^{3/2}} \int_0^t N(t)\,dt\,, \tag{37a}$$

$$x_{0-} = x + \frac{l^2}{(\lambda k_0^2 + l^2)^{3/2}} \int_0^t N(t)\,dt\,, \tag{37b}$$

$$\phi_1 = \phi_0 + k_0(x - x_0) - \int_0^t \sigma\,dt\,, \tag{37c}$$

$$\phi_2 = \phi_0 + k_0(x - x_0) + \int_0^t \sigma\,dt\,. \tag{37d}$$

It is noted that k_0 does not change with the variation of space and time. If we take (20) into the expression of wave energy density and take the static equilibrium approximation, then

$$\varepsilon = \frac{l^2}{k_0^2}|A_0|^2 \frac{N^2(0)}{N^2(t)}$$

can be obtained. Thus it can be seen that the energy density of the envelope will increase and be in favor of the development of a disturbance, otherwise, the energy density will decrease. In the real atmosphere, the generation of storms often accompanies the transform of stratification from the stable state to the unstable one. As an example, take

$$N^2(t) = N_0^2(1 + t/T_c)^2\,,$$

where T_c is a constant and satisfies $\sigma \gg 1/T_c$ and the static equilibrium approximation is adopted. Then there are

$$x_{0+} = x - \frac{N_0}{l}\Big(t + \frac{t^2}{2T_c}\Big)\,, \tag{38a}$$

$$x_{0-} = x + \frac{N_0}{l}\Big(t + \frac{t^2}{2T_c}\Big)\,, \tag{38b}$$

$$\hat{w}(x,t) = \frac{1}{2}\frac{A_1}{1 + t/T_c}\Big[\exp\Big(-\frac{x_{0+}^2}{a^2} + ik_0x_{0+}\Big) + \exp\Big(-\frac{x_{0-}^2}{a^2} + ik_0x_{0-}\Big)\Big]\,. \tag{39}$$

The time variation of the atmosphere can be demonstrated directly through the calculation of the amplitude component in (39). When the stratification decreases with time, the amplitude of the envelope will increase (Fig. 4b); otherwise, the amplitude will decrease (Fig. 4a).

With regard to $C_g < 0$, we can treat it in the same way. Here, we will not repeat it.

5. Conclusions

The asymptotic solution of a gravity wave envelope is obtained in a horizontal heterogeneous stratified atmosphere and time-varying atmosphere through combining the WKBJ method with the characteristic line method. The structure and time variation of the gravity wave envelope in the heterogeneous stratified atmosphere are discussed, and furthermore, given a clearer image in the paper. The expression of the solution shows that the heterogeneous stratification has a great influence not only on the envelope amplitude, but also on the structure of the envelope. Namely, it affects the envelope width and wavelength. These results cannot be achieved by the wave action equation. The movement of the envelope from a region of great stratification to that of a weak one can shorten the wavelength and the envelope width, but its amplitude will grow. Although the time variation of stratification cannot lead to the variation of wavelength and envelope width, the amplitude will consequently change. In detail, the amplitude will grow as the stratification decreases with time.

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水平非均匀层结大气中重力波的时空结构

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摘　要

用WKBJ方法结合特征线法求得了重力波波包在水平非均匀层结和时变层结大气中演变的渐近解，结果表明层结水平非均匀性除引起重力波波幅的变化外，还引起波长和包络宽度的变化，当波包由层结大值区移向层结小值区时，水平波长变短，包络宽度变窄，同时振幅增加。层结随时间的变化不会引起波包波长和包络的宽度的变化，但层结随时间减小时，波包振幅增加。

关键词： 重力波波包，WKBJ方法，特征线法