THE SECOND ORDER APPROXIMATION TO THE NONLINEAR WAVE IN BAROTROPIC ATMOSPHERE

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ABSTRACT

A kind of technique of computer extension of perturbation series is presented and used in seeking for the second-order approximation to a large-scale travelling wave solution of the barotropic primitive equations. Numerical experiments show that the second-order approximation keeps major characters of the travelling wave solution and is indeed more exact than the zero-order and the first order approximations.

I. PERTURBATION METHOD AND SECOND ORDER PROBLEMS

Consider the barotropic primitive equations in dimensionless form on β-plane

$$\begin{cases} \varepsilon\dfrac{\partial u}{\partial t}-\Omega_a v=-\dfrac{\partial}{\partial x}\left(\phi+\varepsilon\dfrac{u^2+v^2}{2}\right), \\ \varepsilon\dfrac{\partial v}{\partial t}+\Omega_a u=-\dfrac{\partial}{\partial y}\left(\phi+\varepsilon\dfrac{u^2+v^2}{2}\right), \\ \varepsilon\dfrac{\partial \phi}{\partial t}+\varepsilon\left(u\dfrac{\partial \phi}{\partial x}+v\dfrac{\partial \phi}{\partial y}\right)=-\mu^2(1+\mu^{-2}\varepsilon\phi)\left(\dfrac{\partial u}{\partial x}+\dfrac{\partial v}{\partial y}\right), \\ \Omega_a\equiv f+\varepsilon\left(\dfrac{\partial v}{\partial x}-\dfrac{\partial u}{\partial y}\right), \end{cases} \tag{1}$$

where ε is the Rossby number, a small parameter for large-scale flow, and μ is another dimensionless parameter characterizing the compressibility of the flow. Our goal is to seek for some travelling wave solutions of Eq. (1) in a channel $0\leqslant y\leqslant\pi$ with rigid walls at $y=0$, π.

Assuming the solution to be expanded into the series of ε, and using the perturbation method, we can convert Eq. (1) into a sequence of problems as follows[1]:

Zero-order equation

$$\begin{cases} \Delta X_0=0\,, \\ \left.\dfrac{\partial X_0}{\partial \eta}\right|_{\eta=0,\pi}=0, \end{cases} \tag{2}$$

$$\begin{cases} \Delta A_0=0\,, \\ \left.\dfrac{\partial A_0}{\partial \eta}\right|_{\eta=0,\pi}=-\left.\dfrac{\partial X_0}{\partial \xi}\right|_{\eta=0,\pi}, \end{cases} \tag{3}$$

$$\begin{cases} -c_0\dfrac{\partial}{\partial\xi}(\Delta-\mu^{-2})\psi_0+J(\psi_0,(\Delta-\mu^{-2})\psi_0+\beta\eta)=0 , \\ \dfrac{\partial\psi}{\partial\xi}\Big|_{\eta=0,\pi}=0; \end{cases} \tag{4}$$

First order equation

$$\begin{cases} \Delta\chi_1=c_0\dfrac{\partial}{\partial\xi}\Delta\psi_0-J(\psi_0,\beta\eta+\Delta\psi_0) , \\ \dfrac{\partial\chi_1}{\partial\eta}\Big|_{\eta=0,\pi}=0, \end{cases} \tag{5}$$

$$\begin{cases} \Delta A_1=(\beta\eta+\Delta\psi_0)\Delta\psi_0+\nabla\psi_0\cdot\nabla(\beta\eta+\Delta\psi_0)-\Delta\left(\dfrac{u_0^2+v_0^2}{2}\right), \\ \dfrac{\partial A_1}{\partial\eta}\Big|_{\eta=0,\pi}=-\left[\dfrac{\partial\chi_1}{\partial\xi}-\beta\eta u_0\right]_{\eta=0,\pi} , \end{cases} \tag{6}$$

$$\begin{cases} \mathscr{L}(\psi_1)=-\dfrac{\partial}{\partial\xi}[c_1(\Delta-\mu^{-2})\psi_0-c_0\mu^{-2}A_1]+J(\psi_0,-\mu^{-2}A_1) \\ \qquad +(\beta\eta+(\Delta-\mu^{-2})\psi_0)\Delta\chi_1+\nabla\chi_1\cdot\nabla(\beta\eta+(\Delta-\mu^{-2})\psi_0) , \\ \dfrac{\partial\psi_1}{\partial\xi}\Big|_{\eta=0,\pi}=0; \end{cases} \tag{7}$$

Second order equation

$$\begin{cases} \Delta\chi_2=\mu^{-2}\Big[c_0\dfrac{\partial\psi_1}{\partial\xi}+c_0\dfrac{\partial A_1}{\partial\xi}+c_1\dfrac{\partial\psi_0}{\partial\xi}+J(A_1,\psi_0) \\ \qquad -\nabla\chi_1\cdot\nabla\psi_0-\psi_0\Delta\chi_1\Big]\equiv H_2, \\ \dfrac{\partial\chi_2}{\partial\eta}\Big|_{\eta=0,\pi}=0, \end{cases} \tag{8}$$

$$\begin{cases} \Delta A_2=c_0\dfrac{\partial}{\partial\xi}\Delta\chi_1+(\beta\eta+\Delta\psi_0)\Delta\psi_1+\Delta\psi_1\Delta\psi_0+\nabla\psi_1\cdot\nabla(\beta\eta+\Delta\psi_0) \\ \qquad +\nabla\psi_0\cdot\nabla(\Delta\psi_1)-J(\chi_1,\beta\eta+\Delta\psi_0)-\Delta[\nabla\psi_0\cdot\nabla\psi_1+J(\psi_0,\chi_1)]\equiv Q_2, \\ \dfrac{\partial A_2}{\partial\eta}\Big|_{\eta=0,\pi}=-\left[\dfrac{\partial\chi_2}{\partial\xi}+\beta\eta\left(\dfrac{\partial\chi_1}{\partial\xi}-\dfrac{\partial\psi_1}{\partial\eta}\right)\right]_{\eta=0,\pi} , \end{cases} \tag{9}$$

$$\begin{cases} \mathscr{L}(\psi_2)=-c_2\dfrac{\partial}{\partial\xi}(\Delta-\mu^{-2})\psi_0+c_0\mu^{-2}\dfrac{\partial A_2}{\partial\xi}-\mu^{-2}J(\psi_0,A_2)+\nabla\chi_2 \\ \qquad \cdot\nabla(\beta\eta+(\Delta-\mu^{-2})\psi_0)+(\beta\eta+(\Delta-\mu^{-2})\psi_0)\Delta\chi_2 \\ \qquad -c_1\dfrac{\partial}{\partial\xi}(\Delta\psi_1-\mu^{-2}\phi_1)+J(\Delta\psi_1,\Delta\psi_1-\mu^{-2}\phi_1) \\ \qquad +\nabla\chi_1\cdot\nabla(\Delta\psi_1-\mu^{-2}\phi_1)+(\Delta\psi_1-\mu^{-2}\phi_1)\Delta\chi_1\equiv G_2, \\ \dfrac{\partial\psi_2}{\partial\xi}\Big|_{\eta=0,\pi}=0; \end{cases} \tag{10}$$

where ψ, χ,A and c represent stream function, velocity potential, ageostrophic geopotential and phase speed respectively, and

$$\mathscr{L}(F) \equiv c_0 \frac{\partial}{\partial \xi}(\Delta - \mu^{-2})F - J(\psi_0, \Delta F) - J(F, \beta\eta + \Delta\psi_0). \tag{11}$$

In Ref. [1], Zeng has already given the answers to the first two problems as follows:

$$\begin{cases} \chi_0 = A_0 = 0, \\ \psi_0 = -U\eta + {}_0\psi \sin m\xi \cdot \sin n\eta, \\ c_0 = U + \dfrac{\beta + \mu^{-2}U}{\kappa_{11} - \mu^{-2}}, \end{cases} \tag{12}$$

where ${}_0\psi$, m, n and U are some free parameters and $\kappa_{11} \equiv -(m^2 + n^2)$;

$$\begin{cases} \chi_1 = c_0\mu^{-2}/\kappa_{11} m \cdot {}_0\psi(c_1 s_1) + {}_1X_1^+(c_1c_0)e^{m\eta} + {}_1X_1^-(c_1c_0)e^{-m\eta}, \\ A_1 = -\dfrac{1}{2}\beta U\eta^2 - \beta/\kappa_{11} n \cdot {}_0\psi(s_1c_1) + \dfrac{1}{4}n^2 \cdot {}_0\psi^2(c_2c_0) \\ \qquad + \dfrac{1}{4}m^2 \cdot {}_0\psi^2(c_0c_2) + \beta \cdot {}_0\psi(s_1s_1)\eta + {}_1X_1^+(s_1c_0)e^{m\eta} \\ \qquad - {}_1X_1^-(s_1c_0)e^{-m\eta}, \\ \psi_1 = -{}_1U_1\eta - {}_1U_2\eta^2 + ({}_1X_1^+ - {}_1X_1^-)(s_1c_1) + {}_1\Psi_{20}^{cc}(c_2c_0) + {}_1\Psi_{02}^{cc}(c_0c_2) \\ \qquad + {}_1B_0(s_1c_1)\eta + {}_1B_1(s_1s_1)\eta + {}_1B_2(s_1c_1)\eta^2 + {}_1d_1(c_2c_0)e^{l\eta} \\ \qquad + {}_1d_2(c_2c_0)e^{-l\eta} + {}_1X_1^+(s_1c_0)e^{m\eta} + {}_1X_1^-(s_1c_0)e^{-m\eta}, \\ c_1 = \dfrac{1}{\kappa_{11} - \mu^{-2}}\left[\kappa_{11} \cdot {}_1U_1 - \pi\mu^{-2}(c_0 - U)\left(\kappa_{11}U + \dfrac{1}{2}\beta\right)\right], \end{cases} \tag{13}$$

where

$$\begin{cases} (c_ic_j) \equiv \cos im\xi \cdot \cos jn\eta, \quad (c_is_j) \equiv \cos im\xi \cdot \sin n\eta, \\ (s_ic_j) \equiv \sin im\xi \cdot \cos jn\eta, \quad (s_is_j) \equiv \sin im\xi \cdot \sin jn\eta. \end{cases} \tag{14}$$

All coefficients in these formulas are completely determined by the zero-order solutions except for one free parameter ${}_1U_1$.

It is clear that the zero-order approximation is just Rossby wave and the first order approximation is its nonlinear version. In [2], we have investigated and tested both of them by using a numerical experiment.

If ε is not too small, we shall have to solve the second order Eqs. (8)—(10) to get the further corrections of the nonlinear wave solutions. Now, the problem are:

1. How to calculate the right terms of Eqs. (8)—(10), H_2, Q_2 and G_2.
2. How to solve the Eqs. (8)—(10), especially the potential vorticity Eq. (10).

We shall give the answers to these questions in the following two sections and some results of numerical experiments in the last section.

II. A COMPUTER METHOD TO COMPLETE CERTAIN CONTINUOUS OPERATIONS OF ELEMENTARY FUNCTIONS

In order to obtain H_2, Q_2 and G_2, we have to manage numerous elementary operations concerning trigonometric, exponential and power functions. For example, one of the terms in G, $J(\psi_1, \Delta\psi_1 - \mu^{-2}\phi_1)$, can be expanded into more than two hundred terms and each of them has a general form as

$$P_{i,j,k,r}(\xi,\eta)=\left\{\begin{matrix}\cos im\xi\\ \sin im\xi\end{matrix}\right\}\cdot\left\{\begin{matrix}\cos jn\eta\\ \sin jn\eta\end{matrix}\right\}\cdot\eta^{k}\cdot e^{r\eta}. \tag{15}$$

Noticing that the sum, difference, product and derivatives of any two functions in (15) still have the same form but different subscripts, we take some of the functions in (15) as base functions in our case and represent them by specific four-dimensional subscript variables. Table 1 gives a correspondence between the four-dimensional subscripts and base functions.

Table 1 A Correspondence between the Four-Dimensional Subscripts i, j, k, r and $P_{i',j',k',r'}(\xi,\eta)$

	0	1	2	3	4	5	6	7	8	9	10
i	1	$\cos m\xi$	$\cos 2m\xi$	$\cos 3m\xi$	$\cos 4m\xi$	$\sin m\xi$	$\sin 2m\xi$	$\sin 3m\xi$	$\sin 4m\xi$	—	—
j	1	$\cos n\eta$	$\cos 2\eta$	$\cos 3\eta$	$\cos 4n\eta$	$\sin n\eta$	$\sin 2n\eta$	$\sin 3n\eta$	$\sin 4n\eta$	—	—
k	1	η	η^2	η^3	η^4	—	—	—	—	—	—
r	1	$e^{m\eta}$	$e^{2m\eta}$	$e^{3m\eta}$	$e^{-m\eta}$	$e^{-2m\eta}$	$e^{-3m\eta}$	$e^{l\eta}$	$e^{2l\eta}$	$e^{-l\eta}$	$e^{-2l\eta}$

Thus, four-dimensional arrays

$$\{X[i,j,k,r], 0\leqslant i,j\leqslant 8, 0\leqslant k\leqslant 4, 0\leqslant r\leqslant 10\}$$

can be used to represent all the functions in the process of calculation. For example, $\psi_0=-U\eta+{}_0\psi(s_is_i)$ can be represented by $\{\Psi_0[i,j,k,r]\}$, where $\Psi_0[0,0,1,0]=-U$, $\Psi_0[5,5,0,0]={}_0\psi$ and the other subscript variables vanish. Correspondingly, certain continuous operations of $P_{i,j,k,r}(\xi,\eta)$ can be converted into some arithmetic and logical operations of $Z(i,j,k,r)$ by using the following definitions.

Definition 1. $Z=aX+bY$:

$$Z[i,j,k,r]=a\cdot X[i,j,k,r]+b\cdot Y[i,j,k,r].$$

Definition 2. $Y=\frac{\partial}{\partial\xi}X$:

$$Y[i,j,k,r]=\begin{cases} imX[i+4,j,k,r], & if\ i\leqslant 4,\\ -(i-4)mX[i-4,j,k,r], & if\ i>4.\end{cases}$$

Definition 3. $Y=\frac{\partial}{\partial\eta}X$:

$$Y1[i,j,k,r]=\begin{cases} jnX[i,j+4,k,r], & \text{if } j\leqslant 4,\\ -(j-4)nX[i,j-4,k,r], & \text{if } j>4;\end{cases}$$

$$Y2[i,j,k,r]=Y1[i,j,k,r]+(k+1)\cdot X[i,j,k+1,r],\ k\leqslant 3;$$

$$Y[i,j,k,r]=Y2[i,j,k,r]+e[r]\cdot X[i,j,k,r],$$

where $\{e[r], 0\leqslant r\leqslant 10\}$ is an one-dimensional array given in Table 2.

Table 2 One-Dimensional Array $\{e[r]\}$

r	0	1	2	3	4	5	6	7	8	9	10
$e[r]$	0	m	$2m$	$3m$	$-m$	$-2m$	$-3m$	l	$2l$	$-l$	$-2l$

Definition 4. $Z = X \times Y$:

Let $a = X[i, j, k, r] \cdot Y[i', j', k', r']$.

then $Z1[i_1, j_1, k_1, r_1] = a \cdot C1[i, i'] \cdot C1[j, j']$,

$Z2[i_1, j_2, k_1, r_1] = Z1[i_1, j_2, k_1, r_1] + a \cdot C1[i, i'] \cdot C2[j, j']$,

$Z3[i_2, j_1, k_1, r_1] = Z2[i_2, j_1, k_1, r_1] + a \cdot C2[i, i'] \cdot C1[j, j']$,

$Z[i_2, j_2, k_1, r_1] = Z3[i_2, j_2, k_1, r_1] + a \cdot C2[i, i'] \cdot C2[j, j']$,

where $k = k + k'$ and r, i_1, j_1, i_2, j_2 are given by two dimensional arrays $r1$ $[r, r']$ (Table 3), $i1$ $[i, i']$ (Table 4), $i2$ $[i, i']$ (Table 5).

Table 3 $r1$ $[r, r']$

r \ r'	0	1	2	3	4	5	6	7	8	9	10
0	0	1	2	3	4	5	6	7	8	9	10
1	1	2	3		0	4	5				
2	2	3			1	0	4				
3	3	3			2	1	0				
4	4	4	0	1	2	5	6				
5	5	4	0	1	6						
6	6	5	4	0							
7	7							8		0	
8	8										0
9	9							0		10	
10	10								0		

Table 4 $i1$ $[i, i']$ (Left) and $c1$ $[i, i']$ (Right)

i \ i'	0	1	2	3	4	5	6	7	8
0	0 1	1 1	2 1	3 1	4 1	5 1	6 1	7 1	8 1
1	1 1	0 1	1 1	2 1	0	6 1	5 1	6 1	0
2	2 1	1 1	0 1	0	0	5 −1	8 1	0	0
3	3 1	2 1	0	0	0	6 −1	0	0	0
4	4 1	0	0	0	0	0	0	0	0
5	5 1	6 1	5 −1	6 −1	0	0 1	1 1	2 1	0
6	6 1	5 1	8 1	0	0	1 1	0 1	0	0
7	7 1	6 1	0	0	0	2 1	0	0	0
8	8 1	0	0	0	0	0	0	0	0

Empties in Tables 3—5 can be filled with a large enough number which means that the operations of subscripts go beyond the bound in our case.

Table 5 $i2\ [i, i']$ (Left) and $C2\ [i, i']$ (Right)

$i \backslash i'$	0		1		2		3		4		5		6		7		8	
0	0	1	1	1	2	1	3	1	4	1	5	1	6	1	7	1	8	1
1	1	1	2	1	3	1	4	1		0	6	0	7	1	8	1		0
2	2	1	3	1	4	1		0		0	7	1	8	0		0		0
3	3	1	4	1		0		0		0	8	1		0		0		0
4	4	1		0		0		0		0		0		0		0		0
5	5	1	6	0	7	1	8	1		0	2	−1	3	−1	4	−1		0
6	6	1	7	1	8	0		0		0	3	−1	4	−1		0		0
7	7	1	8	1		0		0		0	4	−1		0		0		0
8	8	1		0		0		0		0		0		0		0		0

Some composition operations, e. g. $J(X\cdot Y)$, $\nabla X\cdot\nabla Y$, ΔX etc., can be defined by using the above basic definitions. All these can be written easily by computer programming language. Thus, we have given a kind of computer method to get H_2, Q_2 and G_2 instead of manual calculations.

III. SOLUTIONS TO THE SECOND ORDER EQUATIONS

Taking $\varepsilon=0.3$, ${}_0\psi=0.35, m=n=1$ and $U={}_1U_1=0.5$, using the computer method described in the last section, we obtain the analytical expressions with numerical coefficients for $H_2(\xi,\eta)$, $Q_2(\xi,\eta)$ and $G_2(\xi,\eta)$. It is easy to get the inverse image of a function in (15) about Laplacian (This can be performed by a computer programme, too). Therefore, the solutions to the Eqs. (8) and (9) are given immediately as follows.

$$\begin{aligned}
X_2 = {} & \alpha_1(c_1s_1)+\alpha_2(c_1c_1)+\alpha_3(c_1s_1)\eta+\alpha_4(c_1c_1)\eta+\alpha_5(c_1c_1)\eta^2 \\
& +\alpha_6(s_2c_0)+\alpha_7(s_2c_2)+\alpha_8(c_1c_3)+\alpha_9(c_3c_1)+\alpha_{10}(s_2c_0)e^{l\eta} \\
& +\alpha_{11}(s_2c_0)e^{-l\eta}+\alpha_{12}(c_1c_0)e^{m\eta}+\alpha_{13}(c_1c_0)e^{-m\eta}+\alpha_{14}(s_2c_1)e^{2m\eta} \\
& +\alpha_{15}(s_2c_1)e^{-2m\eta}, \qquad (16)
\end{aligned}$$

$$\begin{aligned}
A_2 = {} & \beta_0+\beta_1\eta+\beta_2\eta^2+\beta_3\eta^3+\beta_4(s_1s_1)+\beta_5(s_1c_1)+\beta_6(s_1s_1)\eta \\
& +\beta_7(s_1c_1)\eta+\beta_8(s_1s_1)\eta^2+\beta_9(s_1c_1)\eta^2+\beta_{10}(s_1c_1)\eta^3 \\
& +\beta_{11}(c_2c_0)+\beta_{14}(c_0s_2)+\beta_{15}(c_2s_2)+\beta_{16}(c_2c_0)\eta \\
& +\beta_{17}(c_0c_2)\eta+\beta_{19}(c_0s_2)\eta+\beta_{21}(c_0s_2)\eta^2+\beta_{23}(s_3s_1) \\
& +\beta_{24}(s_1s_3)+\beta_{25}(c_2c_0)e^{l\eta}+\beta_{26}(c_2c_0)e^{-l\eta} \\
& +\beta_{27}(c_2c_0)\eta e^{l\eta}+\beta_{28}(c_2c_0)\eta e^{-l\eta}+\beta_{29}(s_1s_1)e^{l\eta} \\
& +\beta_{31}(s_1s_1)e^{-l\eta}+\beta_{31}(s_1c_1)e^{l\eta}+\beta_{32}(s_1c_1)e^{-l\eta}+\beta_{33}(s_3s_1)e^{l\eta} \\
& +\beta_{34}(s_3s_1)e^{-l\eta}+\beta_{35}(s_3c_1)e^{l\eta}+\beta_{36}(s_3c_1)e^{-l\eta}+\beta_{37}(s_1c_0)e^{m\eta}
\end{aligned}$$

$$+\beta_{38}(s_1c_0)e^{-m\eta}+\beta_{39}(c_2c_0)e^{2m\eta}+\beta_{40}(c_2c_0)e^{-2m\eta}$$
$$+\beta_{41}(s_3c_0)e^{3m\eta}+\beta_{42}(s_3c_0)e^{-3m\eta}. \quad (17)$$

The values of coefficients $\{\alpha\}$ and $\{\beta\}$ are omitted here. Figs. 1—4 show the first order approximations and the second-order corrections of divergence and ageostrophic vorticity given by X_1, X_2 and A_1, A_2.

Now, let us solve the Eq. (10). Assuming

$$\psi_2(\xi,\eta)=\sum_{j=0}^{\infty}({}_2\psi_j(\eta)\cos jm\xi+{}_2\bar{\Phi}_j(\eta)\sin jm\xi), \quad (18)$$

and substituting Eq. (18) into Eq. (8), we have

$$\begin{cases}\left[\dfrac{d^2}{d\eta^2}-(\kappa_{11}+j^2m^2)\right]{}_2\psi_j=L_j(\eta),\\ {}_2\psi_j(0)={}_2\psi_j(\pi)=0,\end{cases} \quad (19)_1$$

$$\begin{cases}\left[\dfrac{d^2}{d\eta^2}-(\kappa_{11}+j^2m^2)\right]{}_2\bar{\Phi}_j=\bar{L}_j(\eta),\\ {}_2\bar{\Phi}_j(0)={}_2\bar{\Phi}_j(\pi)=0.\end{cases} \quad (19)_2$$

In fact, L_0, $\bar{L}_1$, L_2 and $\bar{L}_3$ satisfy the following equations,

$$\begin{cases}\bar{\mathscr{L}}_3(\bar{L}_3)=-\dfrac{2}{{}_0\psi mn}\bar{G}_4(\eta),\\ \bar{\mathscr{L}}_2(L_2)=-\dfrac{2}{{}_0\psi mn}G_3(\eta)+\dfrac{c_0-U}{{}_0\psi}\dfrac{6}{n}\bar{L}_3,\\ \bar{\mathscr{L}}_1(\bar{L}_1)=-\dfrac{2}{{}_0\psi mn}\bar{G}_2(\eta)-\dfrac{c_0-U}{{}_0\psi}\dfrac{4}{n}L_2-\mathscr{L}_3(\bar{L}_3),\\ \bar{\mathscr{L}}_0(L_0)=-\dfrac{1}{{}_0\psi mn}G_1(\eta)+\dfrac{c_0-U}{{}_0\psi}\dfrac{1}{n}\bar{L}_1-\dfrac{1}{2}\mathscr{L}_2(L_2),\end{cases} \quad (20)$$

where

$$\begin{cases}\mathscr{L}_j(\ )\equiv\left(\sin n\eta\dfrac{\partial}{n\partial\eta}+j\cos n\eta\right)(\ ),\\ \bar{\mathscr{L}}_j(\ )\equiv\left(\sin n\eta\dfrac{\partial}{n\partial\eta}-j\cos n\eta\right)(\ ),\end{cases} \quad (21)$$

and the other components of $\{L_j\}$, $\{\bar{L}_j\}$ vanish, because we have already obtained

$$G_2(\xi,\eta)=[-c_2m(\kappa_{11}-\mu^{-2})\cdot{}_0\psi\sin n\eta+G_1(\eta)]\cos m\xi$$
$$+\bar{G}_2(\eta)\sin 2m\xi+G_3(\eta)\cos 3m\xi+\bar{G}_4(\eta)\sin 4m\xi. \quad (22)$$

(The expressions for G_1, $\bar{G}_2$, G_3 and $\bar{G}_4$ can be seen in Appendex). Solving Eqs.(20) and (19) by a method combining analytical and numerical technique, we obtain

$$\psi_2(\xi,\eta)={}_2\psi_0(\eta)+{}_2\bar{\Phi}_1(\eta)\sin m\xi+{}_2\psi_2(\eta)\cos 2m\xi+{}_2\bar{\Phi}_3(\eta)\sin 3m\xi, \quad (23)$$

where ${}_2\psi_0(\eta)$, ${}_2\bar{\Phi}_1(\eta)$, ${}_2\psi_2(\eta)$ and ${}_2\bar{\Phi}_3(\eta)$ are shown in Fig. 3. Moreover, c, the second-order correction of phase speed, is determined as the eigenvalue of the Eq. (10). (There are two free parameters, D and E, in (23) , which come from the solving of Eq. (20). Here, we let $D=E=0.5$.)

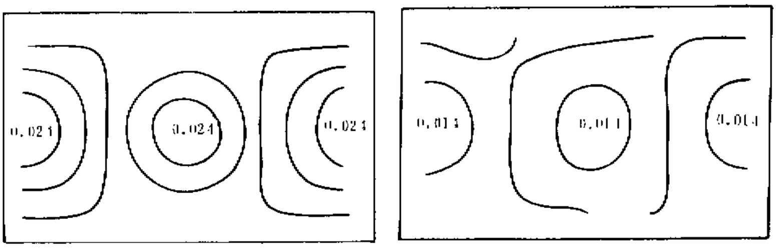

Fig. 1. The first order approximation $\varepsilon\Delta\chi_1 t^{*-1}$ (left) and the second order correction $\varepsilon\Delta\chi_2 t^{*-1}$ (right) of divergence (unit: $10^{-5}s^{-1}$).

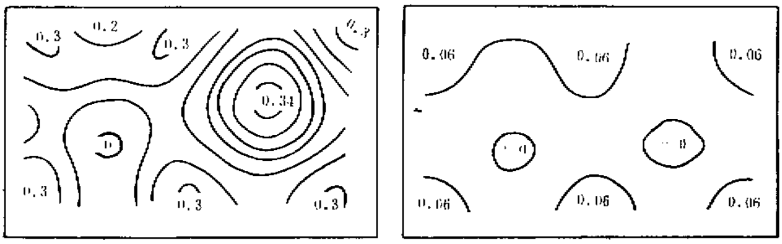

Fig. 2. The first order approximation $\varepsilon\Delta A_1 t^{*-1}$ (left) and the second order correction $\varepsilon^2\Delta A_2 t^{*-1}$ (right) of ageostrophic vorticity (unit: $10^{-5}s^{-1}$).

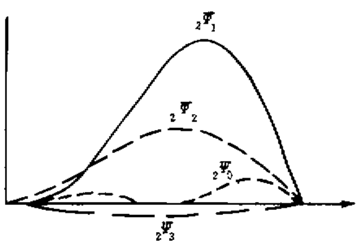

Fig. 3. Profiles of ${}_2\psi_0$, ${}_2\bar{\psi}_1$, ${}_2\psi_2$ and ${}_2\bar{\psi}_3$.

IV. NUMERICAL EXPERIMENTS

The first-order approximations and second order corrections of height field and wind field are calculated by using the following formula.

$$\begin{cases} \phi=\psi+A, \\ u=-\dfrac{\partial\psi}{\partial y}+\dfrac{\partial\chi}{\partial x}, \\ v=\dfrac{\partial\psi}{\partial x}+\dfrac{\partial\chi}{\partial y}. \end{cases} \tag{24}$$

The distributions of them are shown in Figs. 4—5.

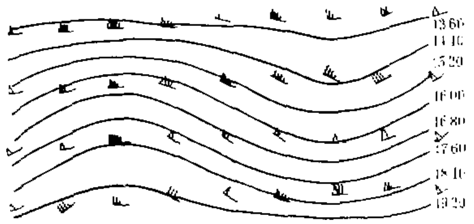

Fig. 4. The first-order approximation of height $(\phi_0+\varepsilon\phi_1)\Phi^*/g$ (unit: m) and wind $(\mathbf{V}_0+\varepsilon\mathbf{V}_1)U^*$ (unit: m s^{-1}).

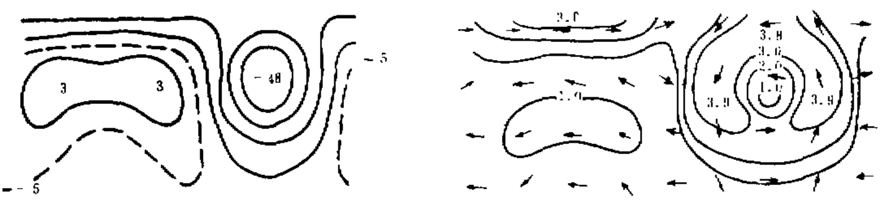

Fig. 5. The second order correction of height $\varepsilon^2\phi_2\Phi^*/g$ (left, unit: m) and wind $\varepsilon^2\mathbf{V}_2U^*$ (right, unit: m s^{-1}).

With the initial conditions given by the second order approximations, $F^{(0)}=F_0+\varepsilon F_1+\varepsilon^2F_2, F=\phi,u,v$, a numerical time integration for the barotropic primitive equation model is performed. The finite-difference scheme used here is the scheme B in Grammeltvedt's paper[3] and the grid size and the time interval are 196.35 km and 5 min, respectively.

Fig. 6 shows the height field of the numerical solution at day 19 (the dashed line represents that of initial conditions). The solid line and the dashed line almost coincide with each other, which means that the second order approximation keeps the non-dispersive behavior of the travelling wave solution very well. By day 19, the wave has moved a distance about four times the wavelength and the average phase speed is about 15.3 m s^{-1}. Moreover, the theoretical phase speed of the second order approximation, $(c_0+\varepsilon c_1+\varepsilon^2c_2)U^*$, is about 15.4 m s^{-1}.

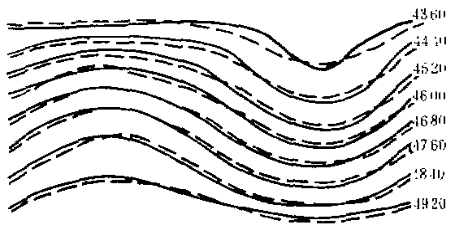

Fig. 6. The height of the numerical solution at day 19 (solid) and that of initial conditions (dashed).

Taking $\varepsilon=0, 3$, ${}_0\psi=0.7$ and $m=n=U={}_1U_1=D=E=1$, another set of numerical experiments is performed with three different initial conditions given by the zero-order (case 1), the first order (case 2) and the second order (case 3) approximations respectively. By using a finite-difference scheme, all the integrations have been carried out for more than 20 days.

Figs. 7—9 show the calculated heights of the free surface on the 10-th day for the three cases respectively, where the correpsonding theoretical solutions, i. e. ϕ_0, $\phi_0+\varepsilon\phi_1$ and $\phi_0+\varepsilon\phi_1+\varepsilon^2\phi_2$, are also given by the dashed lines. It can be seen in these figures that the time lapse in the phase is about one third, one tenth and one twentieth of a wavelength in the three cases.

Furthermore, obvious distortions in the shape of the numerical solution in the zero-order approximation are observed, while this distortion is very small in the numerical solutions with the first order and the second order approximations.

Fig. 10 shows the vibration of the total kinetic energy for the three numerical solutions. It is clear that this vibration is due to the fast (inertial-gravity) waves in the primitive equations. The decreasing of amplitudes of tast waves with the increasing of orders of approximations clearly shows the importance of the consistency of initial data. In fact, as far as the relation between wind field and pressure field is considered, the zero-order approximation is simply the geostrophic equilibrium, but the first order and the second order approximations satisfy the nonlinear balance equations to a certain extent and contain a proper amount of divergent winds.

From the above facts it is clear that for large-scale waves in the barotropic atmosphere governed by the primitive equations, the first-order and the second-order approximations are much better than the zero-order approximation, i. e. the nondivergent Rossby wave. As we have mentioned that the first-order and the second-order approximations are the nonlinear and non-dispersive revisions to the nondivergent Rossby wave, we can call them nonlinear Rossby waves.

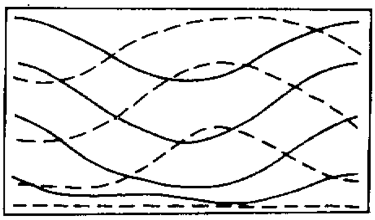

Fig. 7. The numerical (solid) and theoretical (dashed) heights of free surface with zero-order approximation on the 10-th day (case 1).

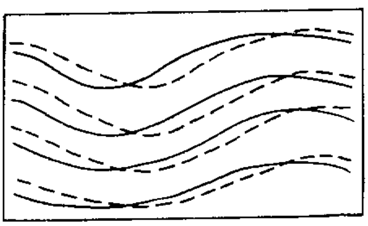

Fig. 8. As in Fig. 7 with the exception of the first-order approximation (case 2).

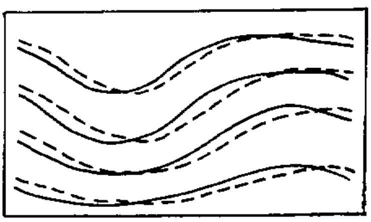

Fig. 9. As in Fig. 7 except the second-order approximation (case 3).

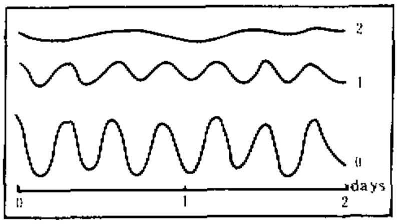

Fig. 10. The vibration of the total kinetic energy for various initial fields corresponding to the zero-order, the first-order and the second-order approximations respectively from below to above.

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APPENDIX

The expressions of G_1, G_2, G_3, G_4

$$
\begin{aligned}
G_1(\eta)=&-0.0004e^{\eta}+0.0050e^{-\eta}\\
&+\cos\eta[-0.0057-0.0001e^{-2\eta}+0.0008e^{-\sqrt{2}\eta}+0.0059\eta\\
&-0.0004\eta e^{-\sqrt{2}\eta}-0.0074\eta^2+0.0016\eta^3]\\
&+\cos3\eta[-0.0013-0.0004\eta]\\
&+\sin\eta[-0.0027+0.0001e^{-2\eta}-0.0008e^{-\sqrt{2}\eta}+0.0191\eta\\
&-0.0003\eta e^{-\sqrt{2}\eta}+0.0055\eta^2]\\
&+\sin3\eta[-0.0008+0.0012\eta-0.0004\eta^2],\\
G_2(\eta)=&[-0.0045+0.0001e^{-2\eta}-0.0006e^{-\sqrt{2}\eta}-0.0132\eta+0.0003\eta e^{-\sqrt{2}\eta}\\
&+0.0099\eta^2-0.0021\eta^3]\\
&+\cos\eta[0.0005e^{\eta}-0.0066e^{-\eta}]\\
&+\cos2\eta[0.0126-0.0001e^{-\sqrt{2}\eta}+0.0058\eta]\\
&+\sin\eta[-0.0005e^{\eta}-0.0066e^{-\eta}]\\
&+\sin2\eta[0.0046-0.0099\eta+0.0031\eta^2]\\
&+0.0006\sin4\eta,\\
G_3(\eta)=&\cos\eta[0.0019+0.0001e^{-2\eta}-0.0008e^{-\sqrt{2}\eta}+0.0004\eta e^{-\sqrt{2}\eta}]\\
&+\sin\eta[0.0007+0.0001e^{-2\eta}-0.0008e^{-\sqrt{2}\eta}+0.0003\eta e^{-\sqrt{2}\eta}]\\
&+0.0006\sin3\eta,\\
G_4(\eta)=&0.0001-0.0006\sin2\eta.
\end{aligned}
$$