# THE STATISTICAL STRUCTURE OF LORENZ STRANGE ATTRACTORS

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ABSTRACT

The statistical characteristic quantities and marginal probability distribution of the Lorenz strange attractors were computed numerically. The results indicate that after a sufficiently long time the statistical characteristic quantities and marginal probability distribution tend to stable states, and the motion on the strange attractor is ergodic.

## Ⅰ. INTRODUCTION

In 1963, Lorenz used a simple three-mode system of nonlinear ordinary differential equations, truncating from the mode equations of two-dimension Rayleigh-Bénard convection problem, to discuss the problem of predictability. It is as follows[1,2]

$$\begin{cases} \frac{dx_1}{dt} = -\frac{3}{2}Px_1 + \frac{2}{3}aPx_2, \\ \frac{dx_2}{dt} = ax_1x_3 - \frac{3}{2}x_2 + aRx_1, \\ \frac{dx_3}{dt} = -\frac{1}{2}ax_1x_2 - 4x_3, \end{cases} \tag{1}$$

where $x_1$ stands for the intensity of the convection, $x_2$ and $x_3$ stand for the deviation of temperature, $a$ is a constant, $P$ the Prandtl number and $R$ the Rayleigh number. $P$, $R$ and $a$ are all the parameters of the system. Lorenz discussed the stability of Eq. (1) for different values of the parameters. He showed that under unstable condition, the dynamical system (1) would eventually tend to a chaotic motion in a specific region in phase space $X$ ($x_1$, $x_2$, $x_3$). Even if the difference between two initial values is very small, that is $|\delta X_0| \ll 1$, the distance between the two moving points in phase space $X$ would become very large as $t$ increases. In the twenty years followed, the continuing concern over this problem has resulted in many studies of various aspects of the problem.

Eq. (1) is deterministic, so that if the initial condition is given accurately, solution should be deterministic, that is, at any time $t$, the solution has a definite value $X$ ($t$, $X_0$). How do we understand the chaotic properties of the strange attractors in this case? D. Ruelle guessed that the motion on the Lorenz strange attractors is ergodic[3]. I. Shimada and T. Nagashima computed the Lyapunov exponent and indicated that except some point sets of zero Lebesgue measure the motion on the strange attractors is ergodic[4]. M. Lucke[5] and E. Knobloch[6], starting from the hypothesis of ergodic, derived the relation of some statistical quantities of Lorenz system with parameter $R$. Their results are in better agreement with numerical computation, indicating the reasonability of the ergodic hypothesis indirectly. In this paper, by solving Eq. (1) numerically, we attempt to directly indicate that the motion on the strange attractors follows an invariant and stable probability distribution and is ergodic.

## II. MOMENT PHASE SPACE

When an initial value $X_0$ is given, one would get a definite trajectory by solving Eq. (1). When $R$ is in a range of some values, solutions $X(t, X_0)$ would become very sensitive to $X_0$. In order to study the statistical structure of Lorenz strange attractors, we let $X_0$ be indeterministic but obey a deterministic probability distribution $P(X_0, t_0)$. For $t>t_0$, the probability distribution density of solution $X(t)$ is $P(X, t)$[7]. We can define the moments according to $P(X, t)$

$$M_{k,l,m}(t)=\langle k,l,m\rangle=\int x_1^k x_2^l x_3^m P(x_1,x_2,x_3,t)dx_1dx_2dx_3. \tag{2}$$

$P(X, t)$ satisfies the following probability conservation equation[8]

$$\frac{\partial P}{\partial t}+\frac{\partial}{\partial x_i}\left(\frac{dx_i}{dt}P\right)=0.$$

Thus it is easy to derive the evolution equations of moments. They are

$$\begin{aligned}\frac{d}{dt}\langle k,l,m\rangle=&\left(-\frac{3}{2}Pk-\frac{3}{2}l-4m\right)\langle k,l,m\rangle\\&+\frac{2}{3}aPk\langle k-1,l+1,m\rangle+aRl\langle k+1,l-1,m\rangle\\&+al\langle k+1,l-1,m+1\rangle-\frac{1}{2}am\langle k+1,l+1,m-1\rangle,\\&k,l,m=0,1,2,\cdots,n\cdots.\end{aligned} \tag{3}$$

In thit paper, some main statitical characteristic quantities, expected value $\bar{X}(t)$, standard deviation $\sigma_{ij}(t)$ and marginal probadility distribution density $f(x_i, t)$ are computed. They are

$$\begin{cases}\bar{x}_i(t)=\int x_iP(x_1,x_2,x_3,t)dx_1dx_2dx_3,\\ \sigma_{ij}(t)=\int(x_i-\bar{x}_i)(x_j-\bar{x}_j)P(x_1,x_2,x_3,t)dx_1dx_2dx_3,\\ f(x_i,t)=\int P(x_1,x_2,x_3,t)\dfrac{dV}{dx_i},\quad dV=dx_1dx_2dx_3.\end{cases} \tag{4}$$

In the computation the continuous functions $P(X_0, t_0)$ and $P(X, t)$ are replaced by the discrete functions $P(X_0, t_0)$ and $P(X, t)$ at $N$ points, so that (4) becomes

$$\begin{cases}\bar{x}_i\cong\dfrac{1}{N}\sum\limits_{j=1}^{N}x_i^{(j)}(t),\\ \sigma_{ij}\cong\dfrac{1}{N}\sum\limits_{k=1}^{N}(x_i^{(k)}-\bar{x}_i)(x_j^{(k)}-\bar{x}_j),\\ f(x_i,t)\cong\int\phi_N(x_1,x_2,x_3,t)\dfrac{dV}{dx_i},\end{cases} \tag{5}$$

where $\phi_N$ is the probability distribution density on $N$ points at time $t$. We will later discuss the effect of the dispersion approximation on the results. Besides, we also define the distance between the $j$th point and average-value point as

$$r^{(j)}=\sqrt{\sum_{i=1}^{3}(x_i^{(j)}-\bar{x}_i)^2}. \tag{6}$$

Then the mean distance of the $N$ points with respect to the average-value point is

$$\bar{r}(t)=\frac{1}{N}\sum_{j=1}^{N}r^{(j)} \tag{7}$$

and the standard deviation $\sigma_{rr}$ is

$$\sigma_{rr}(t)=\frac{1}{N}\sum_{j=1}^{N}(r^{(j)}-\bar{r})^2. \tag{8}$$

The mean value $\bar{X}(\bar{x}_1, \bar{x}_2, \bar{x}_3)$ can be regarded as a phase space, which is known as mean value phase space $\bar{X}$, and the deviation $\sigma_{ij}$ as deviation phase space $\Delta$. Because of $\sigma_{ij}=\sigma_{ji}$ $(i\neq j)$, $\Delta$ has only six components. $\bar{X}$ and $\Delta$ combine and construct a phase space, called moment phase space $M$, which contains nine components. Therefore the $N$ points with definite average value and deviation in the phase space $X$ correspond to a single point in moment phase space $M$.

## III. THE ATTRACTORS IN MOMENT PHASE SPACE $M$

Putting $a=\frac{1}{\sqrt{2}}$, $P=10$, $R=320$ and solving Eq. (1) numerically for the initial condition $X_0=(0, 1, 0)$ we obtain strange attractors shown in Fig. 1, which agrees with Lorenz'

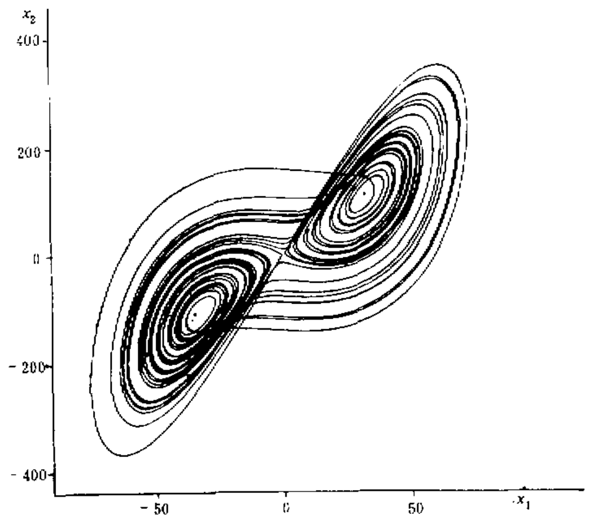
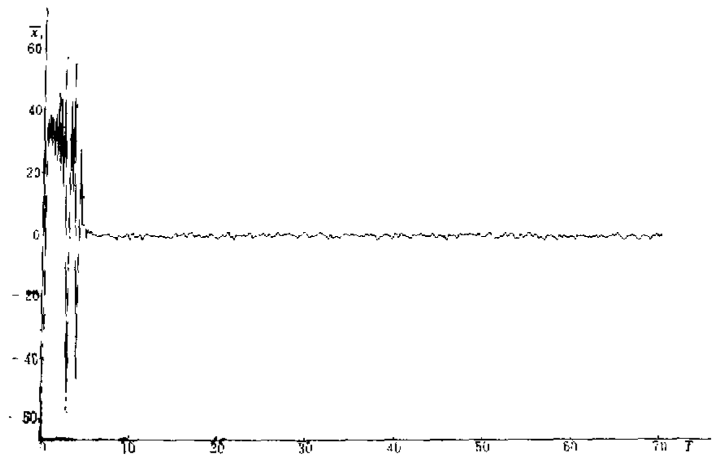


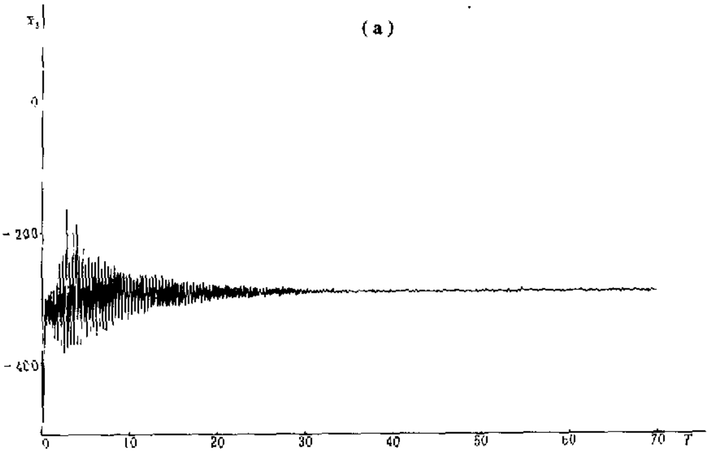

Fig. 1. Lorenz strange attractor with $a=\frac{1}{\sqrt{2}}$, $P=10$, $R=320$.

results. In the computation, 4000 points which satisfy normal distribution with expected values (50, 100, 100) and deviation $\sigma_{ii}=0.01$, $\sigma_{ij}=0.0$, were generated randomly, and Eq. (1) was integrated for each of these $N$ initial pointse with a time step length of $\tau=0.01$. Fig. 2 (a—d) is the time evolution curves of $\bar{x}_1(t)$, $\bar{x}_3(t)$, $\sigma_{11}(t)$, and $\sigma_{13}(t)$. The evolution of $\bar{x}_2$ $(t)$ is similar to that of $\bar{x}_1(t)$, $\sigma_{12}, \sigma_{33}, \sigma_{22}, \bar{r}$ and $\sigma_{rr}$ are similar to $\sigma_{11}$, and $\sigma_{13}$ to $\sigma_{23}$.

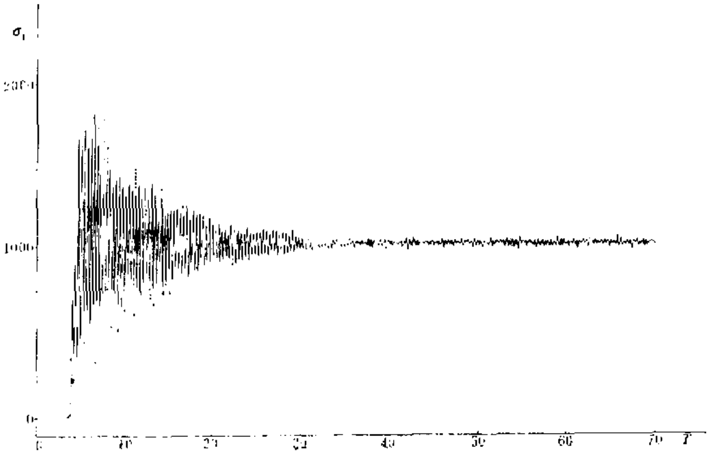
It can be seen from the figures that in the earlier stage ($T\leqslant30$), $\bar{x}_1(t)$, $\bar{x}_3(t)$, $\sigma_{13}(t)$, $\sigma_{11}(t)$ all have a large-amplitude variation, but in the later time stage all of the statistical characteristic quantities tend to specific values. Though there are small undulations near the specific values, it wil be seen that these undulations are completely random and they are caused by the dispersion of $P$ $(X_0, t_0)$ and $P$ $(x, t)$, not the property of the motion discribed by Eq. (1). Thus after a long time, it seems that the trajectory in the $M$ phase space tends to a definite point. Moreover, one can see that the initial deviation is 0.01 and the final is about 1000, that is to say the final amplitude of deviation increases more than about 300 times that of the initial deviation.



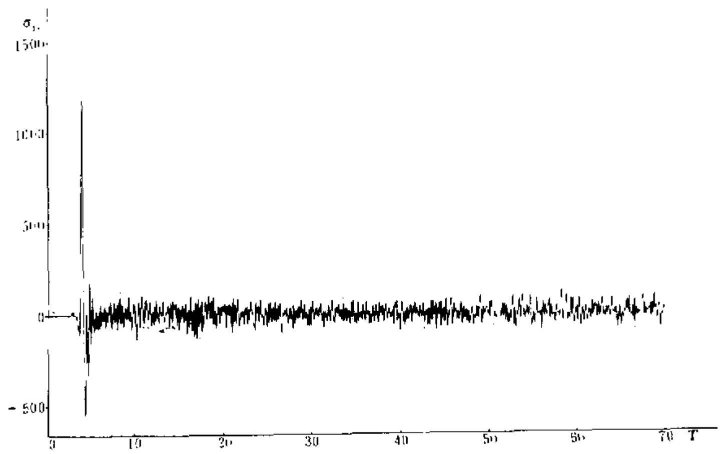

(a)




(b)




(c)




(d)

Fig. 2. Evolution of some components of phase space $M$.
(a) $\bar{x}_1(t)$; (b) $\bar{x}_3(t)$; (c) $\sigma_{11}(t)$; (d) $\sigma_{13}(t)$.

In order to study the property of the undulation of the statistical characteristic quantities near their definite value, we divided the evolution process into two parts with a boundary $T=30$ and studied the power spectrum of every component in space $M$ respectively. Fig. 3(a—d) gives the power spectrum of $\bar{x}_1$ and $\sigma_{11}$. The spectrum of $\bar{x}_2$, $\sigma_{13}$ and $\sigma_{23}$ is similar to that of $\bar{x}_1$, and the others to $\sigma_{11}$. It can be seen from the figures that in the first stage, statistical characteristic quantities are periodically decay oscillations. The period is $T_0=1/f_0=1/2.77=0.36$, which equals approximately to the average time needed for circling around one of the two states of Eq. (1). However in the later time stage, though the periodic motion with the primary frequency has not been vanished completely, the power spectrum of the

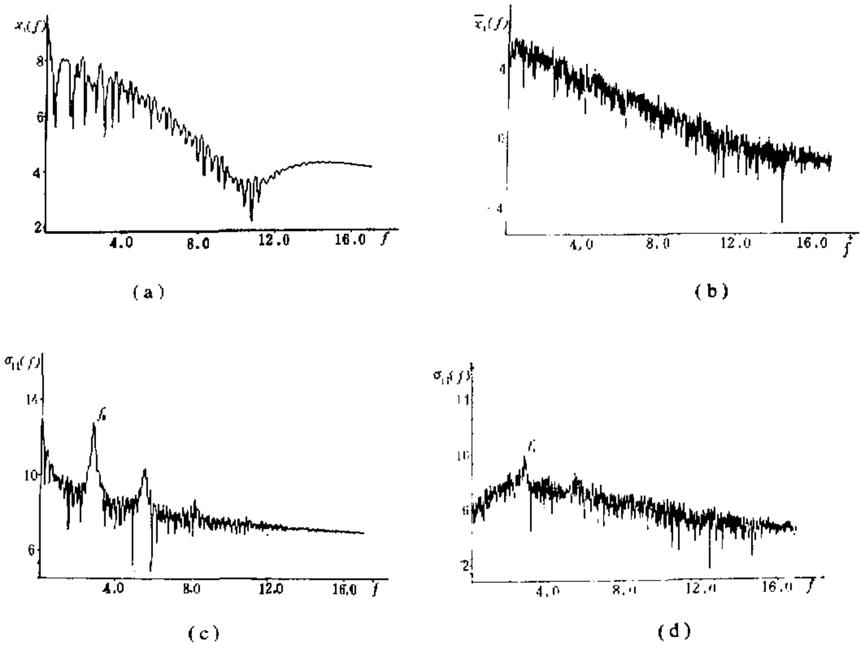



Fig. 3. Power spectrum of $\bar{x}_1(f)$ and $\sigma_{11}(f)$ in different time stages.
(a) $\bar{x}_1(f)$: $0<T<41$; (b) $\bar{x}_1(f)$; $29<T<70$; (c) $\sigma_{11}(f)$; $0<T<41$; (d) $\sigma_{11}(f)$; $29<T<70$.

statistical quantities of the system is close to white noise spectrum on the whole. This indicates that in the phase space $M$ every component takes random undulate motion near a steady value. For any component $m$ in phase space $M$, we take the average value to describe the steady value and use $\delta$ to describe the extent of the undulation. They are computed by the following equations

$$\begin{cases} \bar{m}=\lim\limits_{T_2\to\infty}\dfrac{1}{T_2-T_1}\displaystyle\int_{T_1}^{T_2} m(t)dt\approx\dfrac{1}{n_2-n_1}\sum_{i=n_1}^{n_2} m(i\tau), \\ \delta=\left[\lim\limits_{T_2\to\infty}\dfrac{1}{T_2-T_1}\displaystyle\int_{T}^{T_2}(m(t)-\bar{m})^2dt\right]^{1/2}=\left[\dfrac{1}{n_2-n_1}\sum_{i=n_1}^{n_2}(m(i\tau)-\bar{m})^2\right]^{1/2}. \end{cases} \tag{9}$$

The results are listed in Table 1.

Table 1 Steady Values and Undulation Amplitudes of Statistical Quantities with Normal Initial Distribution and $n_1=3000$, $n_2=7000$

| | $\bar{x}_1$ | $\bar{x}_2$ | $\bar{x}_3$ | $\sigma_{11}$ | $\sigma_{12}$ | $\sigma_{13}$ | $\sigma_{22}$ | $\sigma_{23}$ | $\sigma_{33}$ | $\bar{r}$ | $\sigma_{rr}$ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $\bar{m}$ | −0.01 | −0.04 | −288.94 | 1022.40 | 3250.03 | 0.37 | 14918.07 | 0.59 | 6163.06 | 132.69 | 4500.26 |
| $\delta$ | 0.51 | 1.92 | 1.34 | 17.63 | 68.92 | 44.79 | 317.86 | 139.56 | 122.41 | 1.16 | 99.03 |

**Table 2** Steady Values and Undulation Amplitudes of Statistical Quantities with Uniform Initial Distribution and $n_1=3000$, $n_2=7000$

| | $\bar{x}_1$ | $\bar{x}_2$ | $\bar{x}_3$ | $\sigma_{11}$ | $\sigma_{12}$ | $\sigma_{13}$ | $\sigma_{22}$ | $\sigma_{23}$ | $\sigma_{33}$ | $\bar{r}$ | $\sigma_{rr}$ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $\bar{m}$ | −0.01 | −0.03 | −288.94 | 1022.42 | 3250.21 | 0.39 | 14919.59 | 1.02 | 6164.07 | 132.69 | 4500.39 |
| $\delta$ | 0.53 | 2.01 | 1.67 | 22.55 | 86.79 | 47.38 | 380.52 | 145.50 | 131.91 | 1.48 | 107.68 |

In order to further investigate the causes of the undulation of the statistical quantities, we took $N_1=4000$ and $N_2=1000$ respectively for the same uniform initial distribution $P(X_0, t_0)$. The computed values of $\bar{m}$ and $\delta$ are listed in Table 3. It can be seen from the table that the undulation intensity $\delta$ of $N_1$ decreases by about one time compared with that of $N_2$. This indicates that undulation of statistical characteristic quantities near the steady values is caused by the approximation of dispersion of $P(X, t)$, so that it does not represent the feature of the motion.

**Table 3** Steady Values and Undulation iAmplitudes of Statstical Quantities for Different Dispersion Point Number with Uniform Initial Distribution ($n_1=3000$, $n_2=7000$)

| | $N$ | $\bar{x}_1$ | $\bar{x}_2$ | $\bar{x}_3$ | $\sigma_{11}$ | $\sigma_{12}$ | $\sigma_{13}$ | $\sigma_{22}$ | $\sigma_{23}$ | $\sigma_{33}$ | $\bar{r}$ | $\sigma_{rr}$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $\bar{m}$ | 1000 | 0.21 | 0.66 | −288.98 | 1021.72 | 3248.04 | −5.31 | 14905.14 | −22.95 | 6151.83 | 132.62 | 4491.06 |
| $\bar{m}$ | 4000 | −0.06 | −0.21 | −288.94 | 1022.51 | 3250.38 | 1.46 | 14918.43 | 7.29 | 6165.10 | 132.20 | 4499.70 |
| $\delta$ | 1000 | 1.06 | 4.04 | 2.58 | 34.35 | 135.75 | 95.71 | 618.64 | 292.31 | 290.95 | 2.24 | 199.79 |
| $\delta$ | 4000 | 0.48 | 1.86 | 1.52 | 20.47 | 79.67 | 46.24 | 354.04 | 144.19 | 132.53 | 1.35 | 108.99 |

Similarly, for the uniform initial distribution with the same parameters and the expected value (0, 1, 0), we performed the same computation. The results are listed in Table 2, which is in good agreement with that in Table 1.

The results and above analysis indicate that in moment phase space $M$ the final state of flow seems to be independent of initial condition and tends to a steady state when $t\to\infty$. Not only the moment $\bar{X}$ and $\sigma$ tend to steady state, but also the probability distribution density $P(X, t)$ does. The evolution of $f(X_2\ t)$ is shown in Fig. 4 (a—e).

It should be pointed out that $\bar{x}_1=\bar{x}_2=\sigma_{13}=\sigma_{23}=0$ due to symmetry, but because there must be some small computational errors, $\bar{x}_1$ and $\bar{x}_2$ are not exactly zero. According to Eq. (3), we know that $\sigma_{13}$ and $\sigma_{23}$ are dependent on $\bar{x}_1$ and $\bar{x}_2$ and have amplification effect. Thus deviation of $\sigma_{13}$ and $\sigma_{23}$ from zero are large.

## IV. THE ERGODIC PROBLEM

In order to investigate the ergodic nature of the motion on the strange attractors, the average $m_i^t$ and correlation moment $\sigma_{ij}^t$ for $t$ were computed for any components $x_i(t)$ in phase space $M$. They are

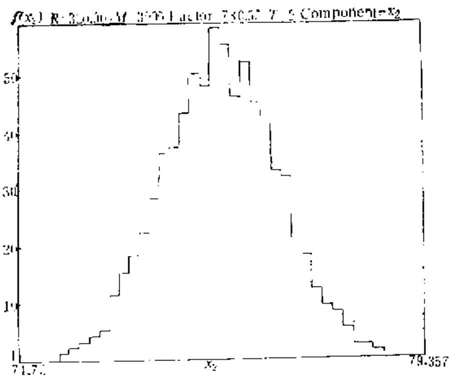
$$\begin{cases} m'_i = \lim\limits_{T_2\to\infty} \dfrac{1}{T_2-T_1}\displaystyle\int_{T_1}^{T_2} x_i(t)dt \cong \dfrac{1}{n_2-n_1}\sum_{j=n_1}^{n_2} x_i(j\tau), \\ \sigma'_{ij} = \lim\limits_{T_2\to\infty} \dfrac{1}{T_2-T_1}\displaystyle\int_{T_1}^{T_2} [x_i(t)-m'_i][x_j(t)-m'_j]dt \\ \qquad \cong \dfrac{1}{n_2-n_1}\displaystyle\sum_{k=n_1}^{n_2} [x_i(k\tau)-m'_i][x_j(k\tau)-m'_j]. \end{cases} \tag{10}$$

For initial value $X_0=(0,1,0)$, the results are as in Table 4. Comparing Table 4 with Tables 1—3, we get

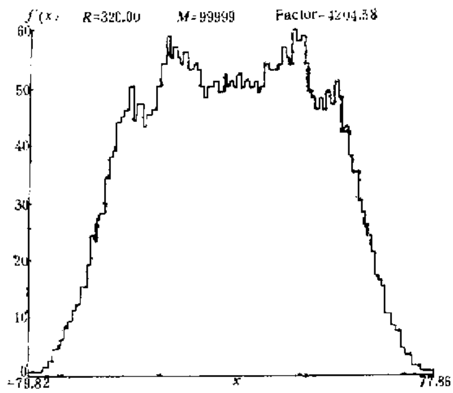
$$\bar{m}_i \cong m'_i, \qquad \sigma_{ij} \cong \sigma'_{ij} .$$

**Table 4** The Average $m'_i$ and Correlation Moment $\sigma'_{ij}$ for $t$ ($n_1=5000$, $n_2=105000$)

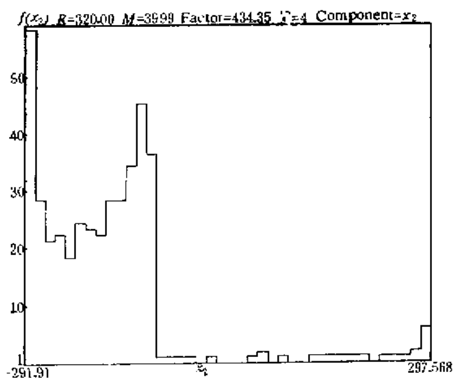
| $\bar{x}'_1$ | $\bar{x}'_2$ | $\bar{x}'_3$ | $\sigma'_{11}$ | $\sigma'_{12}$ | $\sigma'_{13}$ | $\sigma'_{22}$ | $\sigma'_{23}$ | $\sigma'_{33}$ | $\bar{r}'$ | $\sigma'_{rr}$ |
|---|---|---|---|---|---|---|---|---|---|---|
| 0.25 | 0.79 | −288.99 | 1022.82 | 3251.82 | −5.77 | 14900.94 | −29.87 | 6132.46 | 132.55 | 4486.52 |



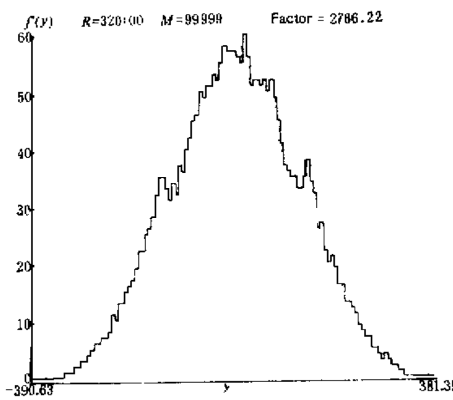

4 (a)




5 (a)




4 (b)

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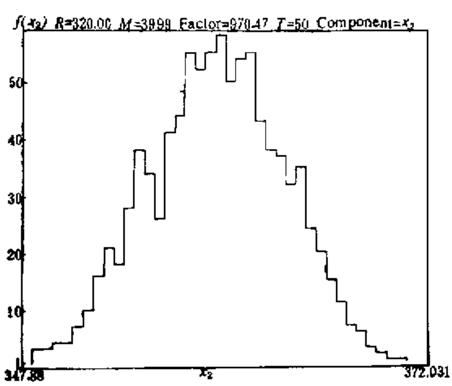
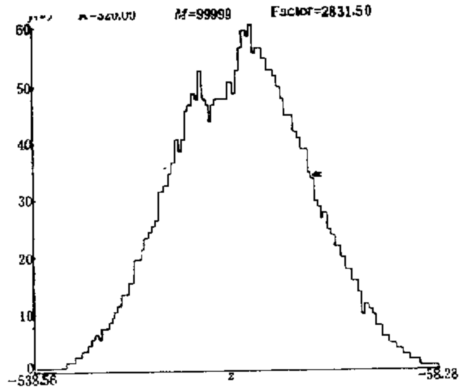



4 (c)

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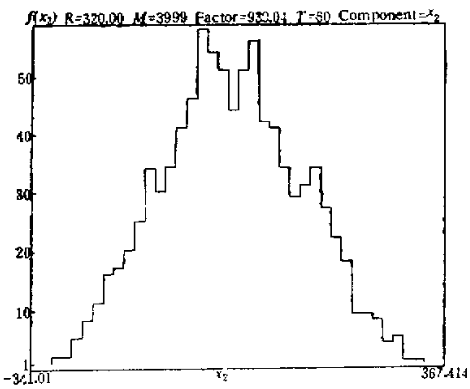
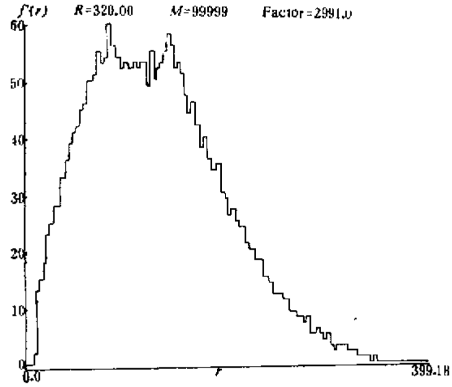
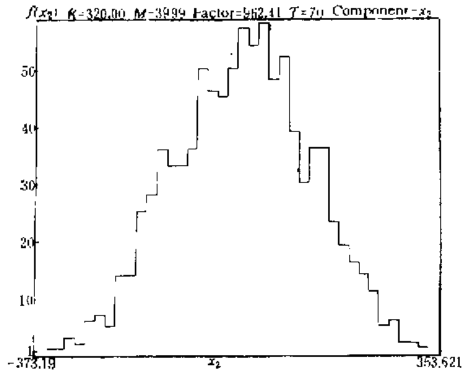



4 (d)




5(d)




4 (e)

Fig. 5. Marginal probability distribution density $f'$, $T_1=50$, $T_2=1050$, $M=(T_2-T_1)/\tau$, Factor is the amplifying factor of $f'$, initial value (0, 1, 0).
(a) $f'(x_1)$; (b) $f'(x_2)$; (c) $f'(x_3)$;
(d) $f'(r)$.

←Fig. 4. Time evolution of marginal probability distribution density $f(x_2, t)$, where $M=N-1$ and Factor is the amplifying factor of $f(x_2, t)$.
(a) $t=2$,; (b) $t=4$; (c) $t=50$;
(d) $t=60$; (e) $t=70$.

We have also carried out the statistical calculation of what $x(t)$ has passed in the intervals $T_2>t>T_1$ and obtained the marginal probability distribution density $f'(x_i)$ and $f'$ $(r)$ as shown in Fig. 5 (a—d). They are very similar to the distribution of points in phase space when $t$ is large enough. This indicates that the motion on the strange attractors is ergodic.

## V. CONCLUSIONS

According to the results of the numerical computation in this paper the following two conclusions are indicated:

1. Though Lorenz system (1) and its solution are deterministic, the motion on the strange attractors is ergodic and random on the whole.

2. The motion on the Lorenz attractors is chaotic, but its statistical characteristics are stable, and it has deterministic statistical structure and does not depend on initial conditions.

In this paper the numerical computations were carried out for only one group of parameters. In order to prove the above conclusions, more computations and strictly mathematical proof are needed.

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