

CALCULATION ON THE LIGHT SCATTERING FUNCTION OF HEXAGONAL ICE CRYSTALS

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ABSTRACT

Hexagonal ice crystal is a basic form of the particles in cirrus. The emphasis in this paper is on discussing the computational results of scattering phase matrices for infrared absorptive band by making use of the model developed by the authors. Comparisons have been made between infrared and visible light as well as between hexagonal columns and plates and equivalent spheroids and spheres. It is found that the effects of light absorption on phase function may cause the decreases of scattering energy in side direction and the directions near backscattering, and weaken halo and rainbow peaks which are induced by the refractions and internal reflections. There is a general agreement in the phase functions for ice crystals with different shapes. However, the scattering intensity in side direction for spheres and in backward and back-side directions for spheroids is weaker as compared with hexagons. It is believed that these features would, doubtless, have influences on the detection of cirrus by laser and radiation transfer in the atmosphere.

1. INTRODUCTION

The light scattering by ice crystals is not only an inevitably involved subject concerning the light transfer in the atmosphere and the radiation balance in earth-atmospheric system, but also a very interesting problem in the development of atmospheric remote sensing technique. The cold cloud in nature usually contains ice crystals. Particularly, the cirrus is composed of a large number of such particles, which are of several to hundreds of micrometers in size and of various shapes. The most basic and common crystals are hexagonal, either in columnar or in plated shapes. It is known that the Maxwell electromagnetic equations and the corresponding boundary conditions can only be adopted to solve scattering questions on the particles with very regular shapes, such as spheres and cylinders. For those with complex shapes as hexagons solving the equations would get into many troubles. In recent years, there have been some theoretical computations and experimental studies concerning the light scattering of nonspherical ice crystals^[1-3]. However, they are confined to deal with the problems of non-polarized light or the two components of a linear polarized light^[4,5]. In their previous paper^[6] the authors, based on the principles of geometrical and physical optics, made use of the ray tracking and coordinate transformation methods and derived a series of mathematical expressions for the calculation of light scattering function of hexagonal ice crystals involving the properties of polarization. However, we show in that paper only two

From Van de Hulst's and the authors' papers^[6,7] and Fig. 1, the electrical fields of scattered and incident light may be expressed as matrices:

$$\begin{bmatrix} E_i \\ E_r \end{bmatrix}_{Z'OP} = \begin{bmatrix} A_2 & A_3 \\ A_4 & A_1 \end{bmatrix} \begin{bmatrix} E_{X'O} \\ E_{Y'O} \end{bmatrix}_{Z'OX'} \quad (1)$$

where A_1, A_2, A_3, A_4 are four matrix elements which, in general, are all complexes and contain the information of electrical field intensity and the phase changes of scattering processes. $Z'OX'$ is the incident plane, the electrical fields of the incident light $E_{X'O}$ and $E_{Y'O}$ are two respective components along axes OX' and OY' (OY' not shown in Fig. 1); OZ' is the propagating direction of incident light, $Z'OP$ the scattering plane and OP the propagating direction of scattering light, E_i and E_r are the scattering components in the plane $Z'OP$ and perpendicular to this plane respectively. Using principles of geometric optics and ray tracking, the matrix expression for scattering field transformation is written as

$$\mathbf{A} = \mathbf{A}' + \mathbf{A}'' = \begin{bmatrix} A'_2 & A'_3 \\ A'_4 & A'_1 \end{bmatrix} + \begin{bmatrix} A''_2 & A''_3 \\ A''_4 & A''_1 \end{bmatrix}, \quad (2)$$

where $\mathbf{A} = \begin{bmatrix} A_2 & A_3 \\ A_4 & A_1 \end{bmatrix}$, \mathbf{A}' is the diffraction contribution and \mathbf{A}'' is the refraction and reflection contribution. The specific expressions for the matrix elements and \mathbf{A}' and \mathbf{A}'' can be found from Ref. [6]. The matrix expression for the Stokes' parameters of scattered light is then

$$\begin{pmatrix} I \\ Q \\ U \\ V \end{pmatrix} = \mathbf{F}(\theta, \phi) \begin{pmatrix} I_0 \\ Q_0 \\ U_0 \\ V_0 \end{pmatrix},$$

where I_0, Q_0, U_0, V_0 are the Stokes' parameters of incident light. The Stokes transformation matrix $\mathbf{F}(\theta, \phi)$ of scattered light has the following form:

$$\mathbf{F} = \begin{pmatrix} \frac{1}{2}(M_2 + M_3 + M_4 + M_1) & \frac{1}{2}(M_2 - M_3 + M_4 - M_1) & S_{23} + S_{41} & D_{23} + D_{41} \\ \frac{1}{2}(M_2 + M_3 - M_4 - M_1) & \frac{1}{2}(M_2 - M_3 - M_4 + M_1) & S_{23} - S_{41} & D_{23} - D_{41} \\ S_{24} + S_{31} & S_{24} - S_{31} & S_{21} + S_{34} & D_{21} - D_{34} \\ D_{24} + D_{13} & D_{24} - D_{13} & D_{12} + D_{43} & S_{21} - S_{34} \end{pmatrix},$$

where $M_k = A_k A_k^* = |A_k|^2$,

$$S_{kl} = S_{lk} = \frac{1}{2}(A_l A_k^* + A_k A_l^*),$$

$$-D_{kl} = D_{lk} = \frac{1}{2}(A_l A_k^* - A_k A_l^*), \quad l, k = 1, 2, 3, 4.$$

Finally, the scattering phase function matrix can be obtained by normalization of \mathbf{F}

$$\mathbf{P} = \mathbf{C} \mathbf{F}, \quad (3)$$

where the coefficient $\mathbf{C} = \frac{4\pi}{\sigma_s}$, σ_s is the scattering cross-section of particle. In general, the scattering phase function matrix contains 16 elements.

Supposing that the ice crystals can randomly take any orientation in space, especially when a strong turbulence exists in clouds, \mathbf{P} is only a function of scattering angle, and only eight matrix elements out of 16 are not zero and six elements are independent of one another, that is,

$$\mathbf{P}(\theta) = \begin{pmatrix} P_{11} & P_{12} & 0 & 0 \\ P_{12} & P_{22} & 0 & 0 \\ 0 & 0 & P_{33} & -P_{43} \\ 0 & 0 & P_{43} & P_{44} \end{pmatrix}.$$

In order to describe the direction of incident light with respect to ice crystal, it is necessary to construct two coordinate systems with their origins being set at the centre of crystal (see Fig. 1). The system $OXYZ$ is fixed with respect to the crystal, and the axes OX , OZ are perpendicular to certain side and the top of the crystal respectively. The other system $OX'Y'Z'$ is used to describe the incident light, and the axes OX' , OY' and OZ' are along the two components of incident electrical field and its propagation direction respectively (OY and OY' are not shown in Fig. 1, but their directions can easily be imagined). Therefore, the orientation of ice crystal with respect to incident light is completely determined by 9 direction cosines between 6 axes of the two coordinate systems. However, of the 9 direction cosines, only 3 are mutually independent due to the fact that there are 6 mathematical relations among them. Therefore the determination of the orientation of ice crystal would be greatly simplified, that is, only three variables would be necessary for determining its orientation. In Fig. 1, we use ψ_1 , ψ_2 and η as the three variables, where ψ_1 represents the angle that describes the crystal rotation round its symmetrical axis, ψ_2 and η respectively express the azimuthal and polar angles of the symmetrical axis in the coordinate system $OX'Y'Z'$. Supposing that the crystal takes random orientation in space, the scattered light should be the average of all crystal contributions from various orientations, i. e.

$$\mathbf{P}(\theta, \phi, \eta, \psi_2) = \frac{1}{2\pi} \int_0^{2\pi} \mathbf{P}(\theta, \phi, \eta, \psi_2, \psi_1) d\psi_1, \quad (4)$$

and

$$\mathbf{P}(\theta) = \frac{1}{4\pi} \int_0^{2\pi} \int_0^\pi \mathbf{P}(\theta, 0, \eta, \psi_2) \sin \eta d\eta d\psi_2. \quad (5)$$

In fact, these three variables ψ_1 , ψ_2 and η should be related to the three afore-mentioned direction cosines. Their relations are easily found to be

$$\cos \psi_1 = \cos \alpha_{31} / \sin \alpha_{33},$$

$$\cos \psi_2 = \cos \alpha_{13} / \sin \alpha_{33},$$

$$\cos \eta = \cos \alpha_{33},$$

where α_{13} , α_{31} and α_{33} denote the angles between OZ and OX' , OX and OZ' , and OZ and OZ' respectively. Numerically integrating Eqs. (4) and (5) can obtain the scattering phase function. For infrared bands, the refraction indexes should be taken as the complex with a non-vanishing imaginary part.

III. RESULTS AND DISCUSSIONS

Some observational data from natural clouds show that ice crystals formed in cirrus under a temperature lower than -20°C are often hexagonal. Specifically, they are hexagonal

columns where temperature is below -25°C and plates between -20° and -25°C . In general, they are the mixture of the both forms^[4]. The scale of crystals in cirrus often reaches several hundreds of micrometers and the ratio of their diameters to lengths is about 1—1/5. In this calculation, we choose the ratio to be 2/5, the diameter $120\mu\text{m}$ and length $300\mu\text{m}$ for columns and diameter $75\mu\text{m}$ and length $30\mu\text{m}$ for plates. The wavelengths used in the computations include $0.55\mu\text{m}$, $0.6328\mu\text{m}$ for visible band and $1.3\mu\text{m}$, $3.8\mu\text{m}$, $10.6\mu\text{m}$ for infrared band.

The results of computation at the non-absorptive visible wavelengths of 0.55 and $0.6328\mu\text{m}$ have been discussed in Ref. [6] in detail. The computed scattering matrixes in the absorptive infrared band at wavelengths of 1.3 , 3.8 and $10.6\mu\text{m}$ are shown in Figs. 2—7. They are the curves of the six elements as a function of the scattering angle. Comparing with Fig. 9 in Ref. [6], we see that the curve for $\lambda=1.3\mu\text{m}$ is very similar to that for $\lambda=0.55\mu\text{m}$. Since there is no much difference in the real part of refractive index for ice between these two wavelengths, and the imaginary part for $\lambda=1.3\mu\text{m}$ is very small ($m_i=1.2\times 10^{-5}$), this consistence is therefore reasonable. For example, in Fig. 2 the curve for $\lambda=1.3\mu\text{m}$ has two evident halo peaks at scattering angles of 21° and 43° , respectively, and they are different in one order. Near the scattering angles of 126° and 156° , there appear a minimum and a wider maximum respectively. The maximum is similar to a "rainbow peak" appearing in spherical water drop scattering and this is due to a large number of backward rays after multiple internal reflections. It is also evident that the scattering phase functions both in the forward (0°) and backward (180°) directions have considerable energy.

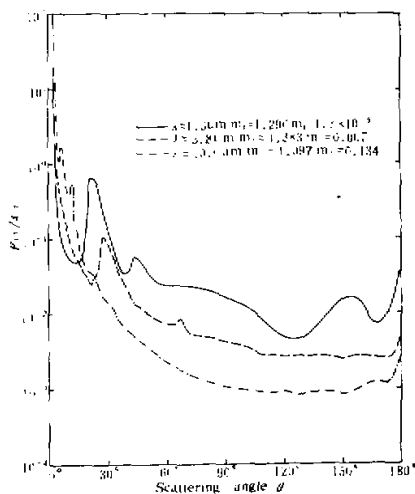


Fig. 2. Element $P_{11}/4\pi$ as a function of scattering angle θ at different wavelengths. Ice crystal is hexagonal column with diameter/length = $120\mu\text{m}/300\mu\text{m}$.

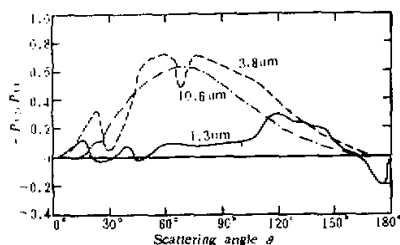


Fig. 3. As in Fig. 2, except for the element $-P_{12}/P_{11}$.

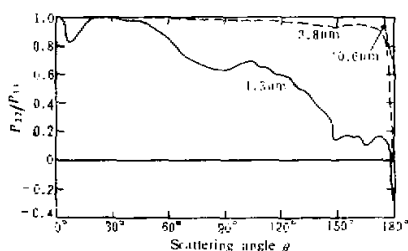


Fig. 4. As in Fig. 2, except for the element P_{22}/P_{11} .

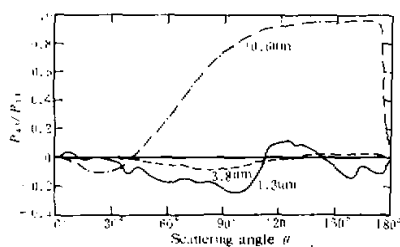


Fig. 5. As in Fig. 2, except for the element P_{43}/P_{11} .

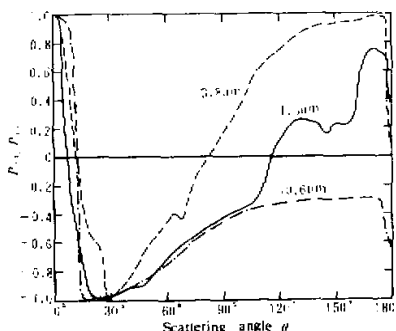


Fig. 6. As in Fig. 2, except for the element P_{32}/P_{11} .

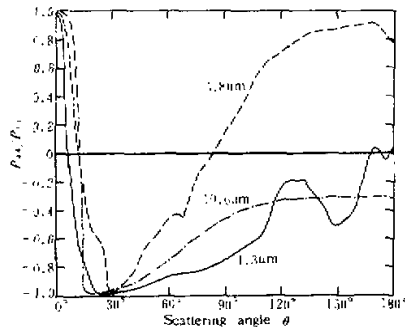


Fig. 7. As in Fig. 2, except for the element P_{44}/P_{11} .

All the above-mentioned features are more or less the same for the curve of $\lambda=3.8\mu\text{m}$, but the halo peaks move rightward to the scattering angles 28° and 66° . This can easily be explained by the principle of halo formation. As we know, the scattering angle of halo peaks θ_0 follows the relation

$$\sin\left[\frac{1}{2}(\theta_0 + A)\right] = m_r \sin\left(\frac{1}{2}A\right), \quad (6)$$

where A is the "prismatic top angle" between the surface of crystal on which incident ray hits and the surface from which the first refracted ray goes out, m_r is the real part of ice refractive index which is a function of wavelength. Therefore, the position of halo peak is related to wavelength. According to Eq. (6), the peaks with scattering angles of 28° and 66° should be generated by the "prismatic top angles" 60° and 90° . Since our calculation results from the random orientation of ice crystal in space, physically considering, there are always some crystal orientations in space which may make Eq. (6) be satisfied for such top angles. Of course, the average of three dimensional orientations of crystals would affect the intensity of halos.

Another feature of the curve for $\lambda=3.8\mu\text{m}$ is that its two halo peaks are much smaller than that for $\lambda=1.3\mu\text{m}$. This is due to the absorption effect when rays go through the crystals. The scattering maximum in near-back direction around 156° vanishes due to the

absorption, and the whole scattering energy is smaller than that for $\lambda=1.3 \mu\text{m}$.

The effect of absorption on the scattering phase function for $\lambda=10.6 \mu\text{m}$ is more evident. It is seen that the curve for $P_{11}/4\pi$ is generally in one order less than that for $\lambda=1.3 \mu\text{m}$ even in the forward (0°) or backward (180°) directions. Since refractive index is $m_r=1.097$ at wavelength of $10.6 \mu\text{m}$ the three top angles of ice crystal 60° , 90° and 120° will form three halo peaks according to Eq. (6). Their corresponding scattering angles are 7° , 12° and 24° respectively. But we can only see two small and narrow peaks in Fig. 2 and the third one (24°) is too small to be seen. This may be induced by the less probability of the favourable orientation of crystal in space due to three dimensional average and the large absorption at this wavelength.

Fig. 3 shows the element $-P_{12}/P_{11}$ as a function of scattering angles for three wavelengths. Physically, they represent the degree of polarization of scattered light when the incident lights are non-polarized natural lights. Since the Stokes parameters of scattered light in this case can be written as

$$\begin{pmatrix} I \\ Q \\ U \\ V \end{pmatrix} = \begin{pmatrix} P_{11} & P_{12} & 0 & 0 \\ P_{12} & P_{22} & 0 & 0 \\ 0 & 0 & P_{33} & -P_{43} \\ 0 & 0 & P_{43} & P_{44} \end{pmatrix}.$$

Therefore

$$\frac{P_{12}}{P_{11}} = \frac{Q}{I} = \frac{I_t - I_r}{I},$$

where I_t and I_r denote the components of scattering intensities in the scattering plane and in its perpendicular direction respectively. Obviously, value of $-P_{12}/P_{11}$ represents the degree of linear polarization of scattered light. For the curve with $\lambda=1.3 \mu\text{m}$ in Fig. 3, the values are generally positive at most of scattering angles except for two halo peaks and the directions around backscattering. The maximum degree of polarization for $\lambda=1.3 \mu\text{m}$ is about 30% (near 120°). At wavelengths with more absorption ($\lambda=3.8, 10.6 \mu\text{m}$), the $-P_{12}/P_{11}$ are quite different. Their maximum degrees of the polarizations reach about 60–70%, their positions shift rightward to scattering angles of 60° and 70° , and all the values are positive. It is noted that some minimum polarizations still exist at the positions of halos, even though the values are positive.

Figs. 4–7 show the curves for P_{22}/P_{11} , P_{43}/P_{11} , P_{33}/P_{11} and P_{44}/P_{11} , respectively, which are associated with the physical meanings of depolarization, electric field orientation and ellipticity of polarization. For these curves, we shall not further discuss in detail.

Fig. 8 directly shows the linear depolarization at three wavelengths, which is defined as

$$\frac{I_r}{I_t} = \frac{P_{11} - P_{22}}{P_{11} + P_{22} + 2P_{43}},$$

and represents the perpendicular component of scattering light when the electric field of incident light is along the direction l . It can be seen that the less the absorption of scattering light in crystal at a wavelength, the larger the depolarization of scattering light is. Specifically for $\lambda=10.6 \mu\text{m}$, depolarizations appear only in the directions near backscattering (maximal value only 20%).

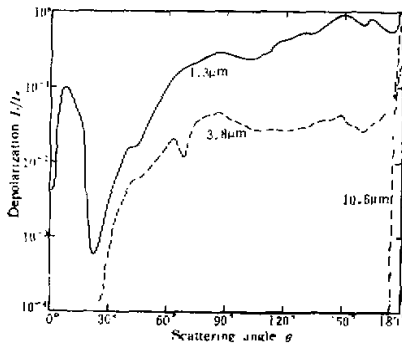


Fig. 8. As in Fig. 2, except for the ordinate being depolarization I_r/I_e .

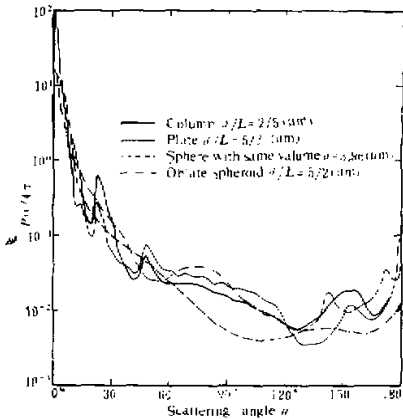


Fig. 9. Comparison of the scattering phase functions for hexagonal, spherical and spheroidal ice crystals. The wavelength is $0.7 \mu\text{m}$. d and L represent diameter and length of the crystals respectively.

The comparisons of the scattering phase functions of ice crystals between hexagonal columns and plates as well as between hexagonal and spherical and spheroidal crystals are shown in Fig. 9. The curve for the spheroid comes from the calculations in accordance with P. W. Barber's method^[9] and that for spherical crystal with same size is calculated by Mie theory. Strictly speaking, the geometric optical model is not appropriate to computing phase functions of hexagonal crystals with the size as shown in Fig. 9. For the purpose of comparison, however, we put them together with the others. All the results given in Fig. 9 seem to be physically reasonable. It is found that all the curves in the figure behave in an identical tendency. A careful analysis shows that the scattering intensity of hexagonal plate is greater than that of hexagonal column in the side direction of 50° – 120° ; while in the directions with the scattering angles larger than 120° , particularly in the vicinity of the 156° -scattering maximum, the hexagonal column scatters more energy. As for halos, the peak at 22° for columns is higher than for plates, but the halo at 46° behaves by contraries. This conclusion seems reasonable, because the ice crystals of different shapes have different probabilities of the orientations in space with respect to incident light which are favourable for the appearance of 22° or 46° halo.

The scattering phase functions of oblate spheroid and hexagonal plate also have identical tendencies. The differences lie only in that the scattering energy for oblate spheroid is smaller in the direction of 140° – 180° . In comparison with the curve of spherical crystal, it is found that its scattering energy is smaller than that of non-spherical crystals on the side between scattering angles of 60° and 120° . However, in the directions larger than 120° the mean curve for spherical crystal is in a consistent tendency with the non-spherical ones because of the effects of rainbow peaks.

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