GENERAL FORMS OF DYNAMIC EQUATIONS FOR ATMOSPHERE IN NUMERICAL MODELS WITH TOPOGRAPHY

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ABSTRACT

The atmospheric dynamic equations have been transformed from the z-coordinate system into a generalized vertical coordinate system by using a so-called DDD transformation method. Then the generalized system is assumed being pressure, sigma or incorporated pressure-sigma coordinate system and corresponding equations are obtained with the second-order accuracy. It is pointed out that the usual equations are only of the first-order accuracy when their space-differential terms are approximated by central finite differences. Therefore the usual forms of the equations may result in quite large errors on steep slopes of mountains included in a model.

I. INTRODUCTION

Studies of topographic effects by using numerical models have been made more and more in recent years. The most evident advantage of using numerical models is easily to do controllable experiments. It is possible for people to test some physical factors and their effects for any times based on the properties of the objects in consideration and on observational data. What is more is that people can even do "null experiments" with the aid of numerical models to enhance artificially some factors in order to give prominence to them. Hunt has systematically discussed the importances of numerical models as experimental tools[1].

A series of problems will be encountered, however, when topography, especially large and steep mountain, is involved in the model. The appropriate treatment of those problems is the key to success of the numerical models with topography. In the early models with topography, computation instabilities often took place because of the unsuitable treatment of mountains involved. As a result, the mumerical predictions by the models with topography were usually of worse quality than those by the models without topography. However, after continuous efforts over more than twenty years some problems of more importance in the models with topography are being solved perfectly for the time being. In order to introduce topography in numerical prediction models there would be quite a few difficulties if the conventional pressure coordinate is used[2]. The most common selection is the so-called topographical coordinate system with the ground surface being a coordinate surface. In the studies of large-scale and synoptic-scale weather systems, for instance, we can choose sigma (σ) or incorporated p-σ coordinates[3-5]. On the selections of vertical coordinates, Kasahara has a detailed discussion in his paper[6]. However, whatever coordinate system is selected, the original equation set in z-coordinate has to be transformed into that in the new coordinate, which is called coordinate transformation. The transformed equation set should be equivalent to the original one. It means that the transformation must be reversible. In other words, the original equations will be obtained by simply replacing the selected vertical coordinate with z-coordinate. Generally speaking, the reversibility is strictly satisfied when transformation is made by using differential regulations. In grid numerical models, however, we use finite difference equations instead of differential ones to make time-integrations, and thus the completely equivalent equations in various differential forms obtain somewhat different computable results when they are finite-differenced for calculation. This is because that we have to use the grid values of some physical quantity A and its vertical variation in the topographic coordinate system in order to calculate horizontal derivatives of A in the z-coordinate, where the calculation of the pressure gradient force term serves as an example. In order to reduce the errors in transformation we should utilize as much as possible the transformation formulas with higher accuracy and less truncation error to transform the horizontal derivative terms despite the identity of the formulas in differential meanings.

Two schemes can be used to transform the differential equations in the z-coordinate into those in the topographical coordinate. The first can be called differential-transformation scheme in which the derivative regulations for implicit functions are directly used. This scheme is well-known to us and we call it classical scheme in this paper. The second can be called difference-transformation scheme, in which the differential equations in z-coordinate are finite-differenced with certain accuracy in certain horizontal coordinate system to the corresponding difference equations which are then transformed to the topographic coordinate system (difference process) and become the new differential equations by taking limit (differential process). For calculation the new differential equations need to be finite-differenced again (difference process). Therefore this scheme can also be designated as the DDD scheme. It is easy to verify that the equations of the same forms as those in the classical scheme will be obtained in the DDD scheme if the difference formulas of the first order accuracy are used in transformation. This means that the classical scheme is the commonly used one and strict in mathematical sense though, it reduces the accuracy of the equation set in z-coordinate to the first order due to the transformation. As a result, it is very difficult to improve the computational accuracy of the new equations.

From the above discussions we know that the higher accuracy of difference scheme we use, the more accurate new equation set is yielded in the topographic coordinate. However, the difference scheme with too high order will complicate the forms of equations. It should be determined, according to the properties of space and time variations of physical quantities, to use what kind of finite-difference schemes. For example, the local time derivative term requires only the transformation formula with the first order accuracy, but the horizontal derivative terms require the formulas with the second order accuracy, the pressure gradient force term even requires that with the fourth order accuracy.

The first, the second and the fourth order finite difference formulas in the A-type horizontal coordinate system are listed below[7].

$$\left(\frac{\partial A}{\partial s}\right)_z=\left(\frac{\partial A}{\partial s}\right)_\eta-\frac{\partial A}{\partial z}\left(\frac{\partial z}{\partial s}\right)_\eta \qquad \text{(first order)} \qquad (1)$$

$$\left(\frac{\partial A}{\partial s}\right)_z=\left(\frac{\partial A}{\partial s}\right)_\eta-\frac{\partial}{\partial s}\left[(z-z_0)\frac{\partial A}{\partial z}\right]_\eta \tag{2}$$

$$=\left(\frac{\partial A}{\partial s}\right)_\eta-\frac{\partial}{\partial s}\left(z\frac{\partial A}{\partial z}\right)_\eta+z\frac{\partial}{\partial s}\left(\frac{\partial A}{\partial z}\right)_\eta \quad \text{(second order)} \tag{3}$$

$$\left(\frac{\partial A}{\partial s}\right)_z=\frac{4}{3}\left\{\left(\frac{\partial A}{\partial s}\right)_\eta-\frac{\partial}{\partial s}\left[(z-z_0)\frac{\partial A}{\partial z}\right]_\eta\right\}-\frac{1}{3}\left\{\left(\frac{\delta A}{\delta s}\right)_\eta-\frac{\delta}{\delta s}\left[(z-z_0)\frac{\delta A}{\delta z}\right]_\eta\right\} \quad \text{(fourth order)} \tag{4}$$

where $\delta A/\delta s$ represents the central-difference with doubled grid distance, s is the argument such as the time t, the horizontal coordinates x and y, the subscripts z and η represent the z- and the new η-coordinates, respectively, A could be any variable or expression. z_0 is the height at the computation point, $\left(\frac{\partial A}{\partial z}\right)_\eta$ is the mean vertical variation of A between z_0 and z.

The nuclear problem in topographic coordinate system is how to calculate the pressure gradient force appropriately and accurately. Many meteorologists have done a lot of careful and effective studies on that problem[8-12]. So far there are three schemes for the calculation of the pressure gradient force (sometimes abbreviated to PGF hereafter), i. e. the special difference scheme such as Corby's scheme, the so-called hydrostatic subtraction scheme and the interpolation scheme. The Corby's scheme is equivalent to the second-order accuracy in fact, because Corby took into account the vertical distribution features of the atmospheric quantities such as pressure and temperature, although his transformation formula is based on the classical scheme. Yan and Qian[7] obtained completely the same scheme as Corby's by the DDD scheme shown in Eq. (2). The hydrostatic subtraction scheme tries to change the small residual of two large terms into that of two small terms, but it only subtracts the mean geopotential height of an isobaric surface, therefore the accuracy of the PGF calculation has not improved much. Its utilization in models is then decreasing.

As far as the interpolation scheme is concerned, it is a good scheme, though not very much in consistence with the transformation principle. In this scheme, the geopotential heights at isobaric or horizontal surfaces are gotten by interpolation from those at topographic surfaces and, therefore, it is quite accurate especially in the free atmosphere. The main problem is that sometimes the height values under the surface are required in order to calculate the PGF at the coordinate surface near the ground and large errors may result from extrapolation especially when diabatic heating effect is included in the model, and temperatures near the ground will then change drastically from space to space.

In this paper the emphasis is put on the general forms of the atmospheric dynamic equations and their accuracies in calculation. The last section will be devoted to the verification of some concepts with a computation based on a set of ideal data.

II. THE FORMS OF ATMOSPHERIC DYNAMIC EQUATIONS IN A GENERALIZED (η) COORDINATE SYSTEM

The dynamic equations of atmosphere in the z-coordinate system can be written in the following flux forms:

$$\frac{\partial \rho u}{\partial t}+\nabla\cdot\rho u\mathbf{V}+\frac{\partial \rho uw}{\partial z}+\frac{\partial p}{\partial x}=\rho F_x+\rho fv, \tag{5}$$

$$\frac{\partial \rho v}{\partial t}+\nabla\cdot\rho v\mathbf{V}+\frac{\partial \rho vw}{\partial z}+\frac{\partial p}{\partial y}=\rho F_y-\rho fu, \tag{6}$$

$$\frac{\partial p}{\partial z}=-\rho g\ , \tag{7}$$

$$\frac{\partial \rho}{\partial t}+\nabla\cdot\rho\mathbf{V}+\frac{\partial \rho w}{\partial z}=0, \tag{8}$$

$$\frac{\partial \rho T}{\partial t}+\nabla\cdot\rho T\mathbf{V}+\frac{\partial \rho Tw}{\partial z}=\frac{\omega+\varepsilon}{c_p}+\rho F_T, \tag{9}$$

$$\frac{\partial \rho q}{\partial t}+\nabla\cdot\rho q\mathbf{V}+\frac{\partial \rho qw}{\partial z}=\rho(E-C)+\rho F_q, \tag{10}$$

where the designations are conventional.

In order to transform Eqs. (5)—(10) to the new coordinate system, we use Eq. (2) or (3) to transform the horizontal derivative terms and Eq. (1) the local time derivative. At the mean time we integrate the equations with the vertica l coordinate z from z_2 to z_1 by using the next formulas:

$$\frac{\partial}{\partial t_z}\int_{z_2}^{z_1} A\mathrm{d}z=\frac{\partial}{\partial t_\eta}\left(A\frac{\partial z}{\partial \eta}\right)(\eta_1-\eta_2)-A_1\frac{\partial z_1}{\partial t_\eta}+A_2\frac{\partial z_2}{\partial t_\eta}, \tag{11}$$

$$\int_{z_2}^{z_1}\nabla_z A\mathrm{d}z=\nabla_\eta\left(A\frac{\partial z}{\partial \eta}\right)\cdot(\eta_1-\eta_2)-\nabla_\eta[(z_1-z_{10})A_1]+\nabla_\eta[(z_2-z_{20})A_2], \tag{12}$$

where A is any quantity, subscripts z and η mean that the derivatives are in the z- and the η-coordinate systems, respectively, ∇ represents the gradient or the divergence operator, subscripts 1 and 2 represent the values at the coordinate surfaces η_1 and η_2, and the 0 represents the value at the point in question. In Eq. (12) the A_1 and A_2 are the average values of A at the η_1 and η_2 surfaces between the 0 point and the adjacent point. The above designations will have the same meanings in the next content.

By the method used in [13] and the hydrostatic equation we can get the dynamic differential equations of atmosphere in the η-coordinate at last:

$$\mathscr{L}(u)=\frac{\partial}{\partial x_\eta}\left(z\frac{\partial p}{\partial \eta}\right)-\frac{\partial}{\partial \eta}\left(z\frac{\partial p}{\partial x_\eta}\right)+\rho\frac{\partial z}{\partial \eta}(F_x+fv)+R(u)\ , \tag{13}$$

$$\mathscr{L}(v)=\frac{\partial}{\partial y_\eta}\left(z\frac{\partial p}{\partial \eta}\right)-\frac{\partial}{\partial \eta}\left(z\frac{\partial p}{\partial y_\eta}\right)+\rho\frac{\partial z}{\partial \eta}(F_y-fu)+R(v)\ , \tag{14}$$

$$\frac{\partial p}{\partial \eta}=-\rho g\frac{\partial z}{\partial \eta}, \tag{15}$$

$$\mathscr{L}(1)=R(1), \tag{16}$$

$$\mathscr{L}(T)=\rho\frac{\partial z}{\partial \eta}\left(\frac{\omega+\varepsilon}{\rho c_p}+F_T\right)+R(T), \tag{17}$$

$$\mathscr{L}(q)=\rho\frac{\partial z}{\partial\eta}(E-C+F_q)+R(q), \tag{18}$$

where

$$\mathscr{L}(A)=\frac{\partial}{\partial t_\eta}\left(\rho\frac{\partial z}{\partial\eta}A\right)+\nabla_\eta\cdot\left(\rho\frac{\partial z}{\partial\eta}A\mathbf{V}\right)+\frac{\partial}{\partial\eta}\left(\rho\frac{\partial z}{\partial\eta}A\dot{\eta}\right) \tag{19}$$

is a differential operator, A could be any quantity or unity, $\dot{\eta}$ is the vertical velocity in the η-coordinate system.

$R(A)$ is a correction term with the following form:

$$R(A)=\frac{\partial}{\partial\eta}\{\nabla_\eta\cdot[(z-z_0)(\rho A-\rho_0A_0)\mathbf{V}]\} \tag{20}$$

$$=\frac{\partial}{\partial\eta}\{\nabla_\eta\cdot(z\rho A\mathbf{V})-z\nabla_\eta\cdot\rho A\mathbf{V}-\rho A\nabla_\eta\cdot z\mathbf{V}+\rho zA\nabla_\eta\cdot\mathbf{V}\}. \tag{20'}$$

The pressure-gradient force (PGF) terms in Eqs. (13) and (14) have changed their forms because of integrating by parts. The unchanged form obtained after transformation is

$$\mathrm{PGF}/g=-\nabla\left(\rho\frac{\partial z}{\partial\eta}RT\right)+\frac{\partial}{\partial\eta}\nabla_\eta[p(z-z_0)]=-\nabla\left\{\left(\rho\frac{\partial z}{\partial\eta}RT\right)\right.$$

$$\left.-\frac{\partial}{\partial\eta}[p(z-z_0)]\right\}. \tag{21}$$

The Eqs. (13)—(18) are called the atmospheric dynamic equations with the difference-differential consistence. All derivative terms except the local-time term are of the second-order accuracy. The $R(A)$ term vanishes if the first-order formula Eq. (1) is used for transformation. It is also easy to prove that the $R(A)$ term will tend to zero in the differential case and the equations here will completely be the same as those in the classical scheme. However, the term $R(A)$ will not be zero certainly in the difference form. Therefore the accuracy of the equation set here may be different from that of the classical scheme. Besides, Eqs. (13)—(18) will take the original forms of Eqs. (5)—(10) if we set $\eta=z$.

The Eqs. (13)—(18) have universal properties because η is not connected with any assumptions. Especially if Eq. (15) is replaced by the nonhydrostatic form, then it can be used to study the small-scale phenomena in the atmosphere. In this paper, however, we will only discuss the hydrostatic motions. From Eq. (15) we have

$$\rho\frac{\partial z}{\partial\eta}=-\frac{1}{g}\frac{\partial p}{\partial\eta}. \tag{22}$$

Substitute it into Eqs. (13)—(19), a set of equations without the explicit density is obtained:

$$\mathfrak{L}(u)=\frac{\partial}{\partial x_\eta}\left(z\frac{\partial p}{\partial\eta}\right)-\frac{\partial}{\partial\eta}\left(z\frac{\partial p}{\partial x_\eta}\right)-\frac{1}{g}\frac{\partial p}{\partial\eta}(F_x+fv)+R(u), \tag{23}$$

$$\mathfrak{L}(v)=\frac{\partial}{\partial y_\eta}\left(z\frac{\partial p}{\partial\eta}\right)-\frac{\partial}{\partial\eta}\left(z\frac{\partial p}{\partial y_\eta}\right)-\frac{1}{g}\frac{\partial p}{\partial\eta}(F_y-fu)+R(v), \tag{24}$$

$$\mathfrak{L}(T)=-\frac{1}{g}\frac{\partial p}{\partial\eta}\left[\frac{RT}{p}(\omega+\varepsilon)+F_T\right]+R(T), \tag{25}$$

$$\mathscr{L}(q)=-\frac{1}{g}\frac{\partial p}{\partial\eta}(E-C+F_q)+R(q)\,, \tag{26}$$

with Eqs. (15) and (16) being unchanged.

The differential operator $\mathscr{L}(A)$ has the following form:

$$\mathscr{L}(A)=-\frac{1}{g}\left\{\frac{\partial}{\partial t_\eta}\left(A\frac{\partial p}{\partial\eta}\right)+\nabla_\eta\cdot\left(A\frac{\partial p}{\partial\eta}\mathbf{V}\right)+\frac{\partial}{\partial\eta}\left(A\frac{\partial p}{\partial\eta}\dot{\eta}\right)\right\}. \tag{27}$$

In Eqs. (17) and (25), $\omega=\frac{dp}{dt}$ can be written as

$$\omega=\frac{\partial p}{\partial t_\eta}+\nabla_\eta\cdot(p-p_0)\mathbf{V}+\dot{\eta}\frac{\partial p}{\partial\eta}\,. \tag{28}$$

By now we have gotten the complete equation set in the η-coordinate system. The diabatic heating term ε, the evaporation E and the condensation C, the eddy diffusion terms F_x, F_y, F_T and F_q, are not discussed here. In the next sections we will put our focus on the situations in the p- and the σ-coordinate systems, respectively.

III. THE EQUATIONS IN THE P-COORDINATE SYSTEM

In order to get the dynamic equations in the p-coordinate system, it is required to replace η with p. Thus, the equations in the previous section will change into

$$\mathscr{L}_p(u)=-\frac{\partial\phi}{\partial x_p}+fv+F_x+R_p(u)\,, \tag{29}$$

$$\mathscr{L}_p(v)=-\frac{\partial\phi}{\partial y_p}-fu+F_y+R_p(v)\,, \tag{30}$$

$$\mathscr{L}_p(T)=\frac{RT}{Pc_p}(\omega+\varepsilon)+F_T\,, \tag{31}$$

$$\mathscr{L}_p(q)=E-C+F_q+R_p(q)\,, \tag{32}$$

$$\mathscr{L}(1)=R_p(1)\,, \tag{33}$$

$$P\frac{\partial\phi}{\partial p}=-RT\,, \tag{34}$$

where the operator

$$\mathscr{L}_p(A)=\frac{\partial A}{\partial t_p}+\nabla_p\cdot(A\mathbf{V})+\frac{\partial}{\partial p}(A\omega)\,, \tag{35}$$

the correction term

$$R_p(A)=-\frac{\partial}{\partial p}\{\nabla_p\cdot[(\phi-\phi_0)(\rho A-\rho_0 A_0)\mathbf{V}]\}. \tag{36}$$

If the PGF term takes the form of Eq. (21), then

$$\mathrm{PGF}=-\nabla_p(RT)-\nabla_p\frac{\partial}{\partial p}(\phi-\phi_0)p\,, \tag{37}$$

where the second term on the right side can be transformed into

$$\nabla_p\left[\frac{\partial}{\partial p}(\phi-\phi_0)p\right]=\frac{\partial}{\partial p}[p\nabla_p\phi]=\nabla_p\phi+p\frac{\partial}{\partial p}\nabla_p\phi$$

$$=\nabla_p\phi+p\nabla_p\frac{\partial\phi}{\partial p}=\nabla_p\phi-p\nabla_p\left(\frac{RT}{p}\right)=\nabla_p\phi-\nabla_p(RT). \quad (38)$$

Hence we still get

$$\mathrm{PGF}=-\nabla_p\phi. \quad (39)$$

That is to say, the two methods of the PGF transformation have the same result in the p-coordinate system.

From Eqs. (29)—(34) we find that the only difference of the equation set in the DDD scheme from that in the classical scheme is the correction terms in Eqs. (29), (30), (32) and (33). In Eq. (31) there is no correction term because the quantity ρT is proportional to the vertical p-coordinate. In fact the state equation yields $\rho T=P/R$, the coordinate surface in the p-coordinate system coincides with that of ρT and the correction term $R_p(T)$ should be exact zero according to its definition Eq. (36).

Now we are going to discuss the characteristics of the correction terms. In order to save space, we take the continuity equation as example. It has the following form:

$$\nabla\cdot\mathbf{V}+\frac{\partial\omega}{\partial p}=-\frac{\partial}{\partial p}\{\nabla_p\cdot[(\phi-\phi_0)(\rho-\rho_0)\mathbf{V}]\}. \quad (40)$$

Integrating it with respect to pressure from p_1 to p_2, we get

$$\omega_2=\omega_1-\int_{p_1}^{p_2}\nabla\cdot\mathbf{V}dp-\{\nabla_p\cdot[(\phi-\phi_0)(\rho-\rho_0)\mathbf{V}]_2$$
$$-\nabla_p\cdot[(\phi-\phi_0)(\rho-\rho_0)\mathbf{V}]_1\}, \quad (41)$$

where subscripts 1 and 2 mean the values at p_1 and p_2 surfaces, respectively.

We can see from Eq. (41) that the form of the continuity equation here is quite different from that in the classical scheme. Two correction terms are in existence which have close relations to the slopes of the p_1 and the p_2 surfaces at the point in question and to the baroclinicity of the atmosphere. It is then able to infer that some errors in the ω computation will result from the classical form of the continuity equation when the weather systems develop very severely. We know that the vertical velocity is a key quantity in weather forecasting and, therefore, the errors in the ω field will influence the quality of the weather predictions.

Assuming that the slopes of the p_1 and the p_2 surfaces are independent of the horizontal coordinates, the density of the air is only the linear function of the height and there exists $\Delta\phi\Delta\rho<0$ in the atmosphere, i. e. the density decreases with the height, then we have the simplified form of Eq. (41) as follows

$$\omega_2=\omega_1-\int_{p_1}^{p_2}\nabla\cdot\mathbf{V}dp+\{[|\Delta\phi\Delta\rho|\nabla_p\cdot\mathbf{V}]_2-[|\Delta\phi\Delta\rho|V_p\cdot\mathbf{V}]_1\}. \quad (42)$$

With the further assumption of $|\Delta\phi\Delta\rho|_2=|\Delta\phi\Delta\rho|_1$, from the above equation we obtain

$$\omega_2=\omega_1-\int_{p_1}^{p_2}\nabla\cdot\mathbf{V}dp-|\Delta\phi\Delta\rho|_1\{\nabla_p\cdot\mathbf{V}_2-\nabla_p\cdot\mathbf{V}_1\}, \quad (43)$$

where the last term shows that the upward motion at the p_2 surface increases for the weather systems with the low-level convergence and the upper-level divergence, and vice versa. If the slope of the isobaric surface at the point in question is zero, the correction term then is zero too, it is the case that the p- and the z-coordinate systems there coincide.

Assuming $p_1=0$ and $p_2=p$ in Eq. (41) and $\omega_0=0$, $\rho=0$ at the upper boundary of the atmosphere, we obtain the vertical velocity at any isobaric surface p, that is

$$\omega=-\int_0^p \nabla\cdot\mathbf{V}\mathrm{d}p-\nabla_p\cdot[(\phi-\phi_0)(\rho-\rho_0)\mathbf{V}]. \tag{44}$$

IV. THE EQUATIONS IN THE σ-COORDINATE SYSTEM

In the σ-coordinate system we define

$$\sigma=\frac{p-p_c}{p_s-p_c}, \tag{45}$$

where p_c is the pressure value of a certain isobaric surface, p_s is the ground surface pressure. When $p_c=0$, we obtain the same definition of the σ-coordinate system as that proposed by Phillips[3]. From Eq. (45) we have

$$\frac{\partial p}{\partial\sigma}=p_s^*, \tag{46}$$

where $p_s^*=p_s-p_c$ is the pressure thickness of the σ-coordinate system.

Replacing η with σ defined in Eq. (45), we can directly obtain the equations in the σ-system from those in the η-system:

$$\mathscr{L}_\sigma(u)=-\frac{\partial}{\partial x_\sigma}(p_s^*\phi)+\frac{\partial}{\partial\sigma}\left(\sigma\phi\frac{\partial p_s^*}{\partial x_\sigma}\right)+p_s^*(F_x+fv)+R_\sigma(u), \tag{47}$$

$$\mathscr{L}_\sigma(v)=-\frac{\partial}{\partial y_\sigma}(p_s^*\phi)+\frac{\partial}{\partial\sigma}\left(\sigma\phi\frac{\partial p_s^*}{\partial y_\sigma}\right)+p_s^*(F_y-fu)+R_\sigma(v), \tag{48}$$

$$\mathscr{L}_\sigma(T)=p_s^*\left[\frac{RT}{pc_p}(\varepsilon+\omega)+F_T\right]+R_\sigma(T), \tag{49}$$

$$\mathscr{L}_\sigma(q)=p_s^*(E-C+F_q)+R_\sigma(q), \tag{50}$$

$$\mathscr{L}_\sigma(1)=R_\sigma(1), \tag{51}$$

$$\frac{\partial\phi}{\partial\sigma}=-\frac{RT}{\sigma+p_c/p_s^*}, \tag{52}$$

where the operator

$$\mathscr{L}_\sigma(A)=\frac{\partial}{\partial t_\sigma}(p_s^*A)+\nabla_\sigma\cdot(p_s^*A\mathbf{V})+\frac{\partial}{\partial\sigma}(p_s^*A\dot{\sigma}), \tag{53}$$

the correction term

$$R_\sigma(A)=-\frac{\partial}{\partial\sigma}[\nabla_\sigma\cdot(\phi-\phi_0)(\rho A-\rho_0A_0)\mathbf{V}]. \tag{54}$$

The other form of the PGF can be gotten from Eq. (2), that is

$$\begin{aligned}\mathrm{PGF}&=-\nabla_\sigma(p_s^*RT)-\frac{\partial}{\partial\sigma}\{\nabla_\sigma[(\sigma p_s^*+p_c)(\phi-\phi_0)]\}\\&=-\nabla_\sigma\{p_s^*RT+\frac{\partial}{\partial\sigma}[(\sigma p_s^*+p_c)(\phi-\phi_0)]\},\end{aligned} \tag{55}$$

which is somewhat different from that in Eqs. (47) and (48).

The comparison of the equation set of (47)—(52) with that of the classical scheme indicates that the main difference between the two sets is also in that there are correction terms in the DDD scheme as in the p-coordinate system. Here we again discuss the correction term in the continuity equation (51), namely,

$$R_\sigma(1) = -\frac{\partial}{\partial\sigma}[\nabla_\sigma\cdot(\phi-\phi_0)(\rho-\rho_0)\mathbf{V}]. \tag{56}$$

By integrating Eq. (51) with respect to σ from 0 to 1, we get

$$\frac{\partial p_s^*}{\partial t_\sigma} = p_s^*\dot{\sigma}_0 - \nabla_\sigma\cdot\int_0^1 p_s^*\mathbf{V}d\sigma - [\nabla_\sigma\cdot(\phi-\phi_0)(\rho-\rho_0)\mathbf{V}]_{\sigma=0}^{\sigma=1}, \tag{57}$$

where the boundary condition at ground, i. e. $\dot{\sigma}=0$ at $\sigma=1$, has been used.

From Eq. (57) we see clearly that the correction term can change the time tendency of the ground surface pressure, which does not exist in the classical scheme. By the use of similar method as in the p-system, we have the conclusion that the correction term will increase the pressure at the windward slope and decrease that at the lee side. Such an effect is related to the wind direction and to the topography. Therefore, in a long-time integration, it will be important.

If we take p_c as an isobaric surface higher than mountains, and if, above p_c, we use the p-coordinate system and below p_c, the σ-coordinate system, then we get the so-called incorporated p-σ coordinate system and its dynamic equation set. Because the equations in the p- and the σ-coordinate systems are independent of each other, the only requirement for connecting them is to set up the conditions at the interface of the two systems. By the definition of ω, we have

$$\omega = \frac{\partial p}{\partial t_\sigma} + \nabla_\sigma\cdot[(p-p_0)\mathbf{V}] + \dot{\sigma}p_s^* = \sigma\left[\frac{\partial p_s^*}{\partial t_\sigma} + \nabla_\sigma\cdot(p_s^*-p_{s0}^*)\mathbf{V}\right] + \dot{\sigma}p_s^*. \tag{58}$$

At the surface with $\sigma=0$, $p=p_c$ and $\omega=\omega_c$, then

$$\omega_c = \dot{\sigma}_0 p_s^*, \tag{59}$$

where

$$\omega_c = -\int_0^{p_c}\nabla\cdot\mathbf{V}dp - \nabla\cdot[(\phi_c-\phi_{c0})(\rho_c-\rho_{c0})\mathbf{V}_c], \tag{60}$$

and the subscript c means the quantities at the $p=p_c$ surface.

Substituting Eq. (60) into (57), we obtain

$$\frac{\partial p_s^*}{\partial t_\sigma} = -\int_0^{p_c}\nabla\cdot\mathbf{V}dp - \nabla_\sigma\cdot\int_0^1 p_s^*\mathbf{V}d\sigma - \nabla_\sigma[(\phi_s-\phi_{s0})(\rho_s-\rho_{s0})\mathbf{V}_s]. \tag{61}$$

Therefore, the only difference of Eq. (61) from the tendency equation in the classical scheme is again the last correction term.

After calculation of $\frac{\partial p_s^*}{\partial t}$ by Eq. (61), we can integrate Eq. (51) from $\sigma=0$ to σ and get the vertical velocity $\dot{\sigma}$, that is

$$\dot{\sigma} = \dot{\sigma}_0 - \frac{1}{p_s^*}\left\{\sigma\frac{\partial p_s^*}{\partial t} + \nabla_\sigma\cdot\int_0^\sigma p_s^*\mathbf{V}d\sigma + \nabla_\sigma\cdot[(\phi-\phi_0)(\rho-\rho_0)\mathbf{V}]_{\sigma=0}^{\sigma=\sigma}\right\} \tag{62}$$

The above equations and procedures constitute, respectively, the basic equations and the solution method of the DDD scheme in the p-σ incorporated coordinate system.

V. VERIFICATION BY AN IDEAL DATA SET

In order to give the readers a quantitative understanding of the previous discussions, we design an ideal data set. The geopotential height and the pressure are related to each other by the following equation:

$$z(p)=z_5+\frac{1}{2}[T(p)+T_5]\cdot\ln\left(\frac{500}{p}\right)\cdot\frac{R}{g}, \tag{63}$$

where z_5 and T_5 are the height and the temperature at the 500 hPa level, respectively, which are determined by

$$T_5=273.0+(\varphi-\varphi_0)DT_1-\varphi^2/A_T-DT\cdot\cos[B_T(\lambda-\lambda_0)], \tag{64}$$

$$z_5=580.0+(\varphi-\varphi_0)Dz_1-\varphi^2/Az-Dz\cdot\cos[B_z(\lambda-\lambda_0)], \tag{65}$$

where φ and λ are the latitude and the longitude, respectively, then φ_0 and λ_0 are the reference latitude and longitude, and the others are all constants introduced as parameters. With different values of the parameters, the different temperature and pressure fields will be obtained.

We assume that the temperature distribution in the vertical direction satisfies the following equation:

$$T(p)=T_5+\gamma\cdot\ln\left(\frac{p}{500}\right), \tag{66}$$

where γ is the constant with a value of 43.28 K above the 500 hPa level, while with a value of 50 K lower than that level. Then we can determine the temperature at any level from Eq. (66).

Having given the surface pressure distribution, we can obtain the surface height z_s from Eq. (63). The surface pressures are assumed to be

$$p_s=600-510A^2+960A \tag{67}$$

with

$$A=(\varphi-\varphi_0)^2/A_\varphi+(\lambda-\lambda_0)^2/A_\lambda, \tag{68}$$

where A_φ and A_λ are constants.

The computed p_s distribution is listed in Table 1a, where the pressure larger than 1000 hPa has been set equal to 1000 hPa. Table 1b is the z_s gradient distribution. It is seen that the largest value of the z_s gradient is 0.33%, which indicates that the slopes are not very steep. The maximum z_s value is about 4300 m.

Table 2a and 2b show the ideal distributions of the geopotential height and the temperature fields at the 500 hPa level, respectively. Between 70°E and 110°E there is wavy flow pattern and the atmosphere can be considered as a barotropic one. The same is true of other isobaric surfaces.

In order to calculate the correction term we need a numerical model. Thus we make use of the p–σ incorporated model as given in [13]. The geostrophic wind is computed first and then used for calculation of $R_p(u)$, $R_p(v)$ and $R_\sigma(v)$, $R_\sigma(T)$ terms in the p– and the σ–coordinate systems. Comparisons are made of these terms with the corresponding flux ones which lead to the correction terms. It is found that, in the p–system, both the magnitudes of $R_p(u)$ and $R_p(v)$ terms are about three orders smaller than those of corresponding flux terms. In the σ-system $R_\sigma(u)$, $R_\sigma(v)$ are at least one order smaller than the flux terms. Table 3a and 3b show the distributions of $\nabla_\sigma\cdot(p_s^*u\mathbf{V})$ and $R_\sigma(u)$, respectively, at the fourth model level. We can see that the magnitude of $\nabla_\sigma\cdot(p_s^*u\mathbf{V})$ has the order of 10^{-2} to 10^{-3}, while that of $R_\sigma(u)$ of 10^{-4}. The case for $R_\sigma(v)$ is the same. The distributions of $\nabla_\sigma(p_s^* T\mathbf{V})$ and $R_\sigma(T)$ are shown in Table 4a and 4b. It is seen that although generally $R_\sigma(T)$ is also smaller than $\nabla_\sigma(p_s^* T\mathbf{V})$, the differences are not over one order. For example, at some points to the north and the south of the z_s maximum, $R_\sigma(T)$ reaches 0.6—0.7 with the same magnitude as the $\nabla_\sigma(p_s^* T\mathbf{V})$ term.

Table 1a. Surface Pressure Distribution (in hPa)

N \ E	70°	75°	80°	85°	90°	95°	100°	105°	110°
50°	1000.0	1000.0	1000.0	993.3	978.6	993.3	1000.0	1000.0	1000.0
45°	1000.0	995.8	907.9	835.5	808.1	835.5	907.9	995.8	1000.0
40°	1000.0	914.6	788.4	693.0	658.0	693.0	788.4	914.6	1000.0
35°	1000.0	879.5	740.5	637.6	600.0	637.6	740.5	879.5	1000.0
30°	1000.0	914.6	788.4	693.0	658.0	693.0	788.4	914.6	1000.0
25°	1000.0	995.8	907.9	835.5	808.1	835.5	907.9	995.8	1000.0
20°	1000.0	1000.0	1000.0	993.3	978.6	993.3	1000.0	1000.0	1000.0

Table 1b. Z_s Gradient Distribution (in 10^{-2})

N \ E	70°	75°	80°	85°	90°	95°	100°	105°	110°
50°	0.13	0.13	0.28	0.31	0.31	0.31	0.27	0.13	0.13
45°	0.13	0.13	0.28	0.31	0.31	0.31	0.27	0.13	0.13
40°	0.27	0.27	0.33	0.28	0.23	0.28	0.32	0.26	0.26
35°	0.29	0.29	0.30	0.19	0.01	0.20	0.30	0.29	0.29
30°	0.24	0.24	0.30	0.26	0.22	0.26	0.29	0.24	0.24
25°	0.11	0.11	0.24	0.30	0.31	0.30	0.24	0.11	0.11
20°	0.11	0.11	0.24	0.30	0.31	0.30	0.24	0.11	0.11

Table 2a. Geopotential Heights at the 500 hPa (10 m)

N \ E	70°	75°	80°	85°	90°	95°	100°	105°	110°
50°	564.6	568.2	569.6	568.2	564.7	561.1	559.6	561.1	564.6
45°	569.4	573.0	574.6	573.3	569.8	566.2	564.6	565.9	569.4
40°	574.1	577.8	579.4	577.9	574.4	570.9	569.4	570.7	574.1
35°	578.7	582.5	584.0	582.5	579.0	575.4	574.0	575.4	578.7
30°	583.1	586.8	588.4	587.0	583.4	580.0	578.4	580.0	583.1
25°	587.4	591.0	592.5	591.3	587.8	584.2	582.6	583.9	587.4
20°	591.6	595.2	596.6	595.2	591.7	588.1	586.6	588.1	591.6

Table 2b. Temperature at the 500 hPa (°C)

N \ E	70°	75°	80°	85°	90°	95°	100°	105°	110°
50°	−3.8	−2.4	−1.8	−2.4	−3.9	−5.2	−5.8	−5.2	−3.8
45°	−2.5	−1.1	−0.7	−1.4	−2.8	−4.3	−4.7	−3.9	−2.5
40°	−1.2	−0.0	0.6	0.2	−1.3	−2.7	−3.4	−2.8	−1.2
35°	0.0	1.1	1.9	1.3	−0.3	−1.5	−2.1	−1.7	0.2
30°	1.2	2.4	3.0	2.6	1.1	−0.3	−1.0	−0.4	1.2
25°	2.3	3.7	4.1	3.4	2.0	0.5	0.1	0.9	2.3
20°	3.4	4.8	5.4	4.8	3.4	2.0	1.4	2.0	3.4

Table 3a. Distribution of $\nabla_\sigma \cdot (p_s^* \cdot u \cdot \mathbf{V})$ (10^{-3} hPa m s^{-2})

N \ E	75°	80°	85°	90°	95	100°	105°
45°	−2.0	0.6	1.7	2.2	2.7	−0.5	−3.7
40°	−3.6	−2.5	−2.0	0.6	5.2	2.5	−4.8
35°	−4.7	−5.5	−2.4	5.2	11.0	5.5	−3.8
30°	−4.4	−4.0	3.6	14.0	14.0	4.0	−4.9
25°	−4.3	0.0	7.2	13.0	9.1	−0.0	−6.3

Table 3b. Distribution of $R_\sigma(u)$ (10^{-3} hPa m s^{-2})

N \ E	75°	80°	85°	90°	95°	100°	105°
45°	−0.3	0.0	−0.4	−0.9	−0.6	−0.0	0.3
40°	−0.6	0.2	0.5	0.3	−0.2	−0.2	0.7
35°	−0.3	0.3	0.4	0.3	−0.2	−0.3	0.4
30°	0.2	0.2	−0.1	−0.4	−0.6	−0.2	0.4
25°	−0.6	0.1	0.1	−0.3	−0.4	−0.1	0.2

Table 4a. Distribution of $\nabla_\sigma \cdot (p_s^* \cdot T \cdot \mathbf{V})$ (10^{-1} hPa K s^{-1})

N \ E	75°	80°	85°	90°	95°	100°	105°
45°	−1.6	−1.8	−1.6	−0.8	1.2	1.7	−0.0
40°	−3.2	−3.4	−3.3	−1.7	1.7	3.4	1.6
35°	−4.3	−4.3	−2.4	0.9	3.8	4.3	2.3
30°	−4.1	−3.8	−0.0	4.2	5.4	3.8	1.4
25°	−2.9	−2.3	1.3	4.6	4.5	2.3	−0.8

Table 4b. Distribution of $R_\sigma(T)$ (10^{-1} hPa K s^{-1})

N \ E	75°	80°	85°	90°	95°	100°	105°
45°	0.1	0.0	0.3	0.7	0.5	−0.0	−0.2
40°	0.4	0.0	−0.6	−0.2	0.2	−0.0	−0.4
35°	0.5	0.0	−0.4	0.0	0.4	−0.0	−0.5
30°	0.4	0.0	−0.1	0.4	0.5	−0.0	−0.4
25°	0.2	0.0	−0.5	−0.6	−0.2	−0.0	−0.2

We have made other calculations with different distributions of temperature and geo–potential height fields. The basic results are similar to the above. However, for the atmosphere with stronger baroclinicity, $R_\sigma(T)$ is of more importance. We omit the results here for space saving.

From the above computation results and discussions we know that the classical form of the dynamic equation set in the p–system has enough accuracy, so it is not necessary to add the correction terms to the classical equations of u, v and q, unless the weather system develops very severely and the results in very large divergence. In the σ–system, however, since $R_\sigma(u)$, $R_\sigma(v)$ and $R_\sigma(T)$ are relatively important, we had better not omit them in the equations, especially, $R_\sigma(T)$ in the thermodynamic equation. In the verification test for the time being we only use geostrophic wind instead of real wind which can be very different from the former, therefore the importance of the correction terms may increase in the real atmosphere. In order to improve the accuracy of calculation and, as a result, *the quality of weather forecasting, it is necessary to include correction terms in the* σ–system, especially for long–time integrations.

The conclusions in this paper are derived from the A–type horizontal grid systems. However, they are universal, to some extent, for other types of grid systems as well. We will use the dynamic equations obtained in this paper to carry out more numerical experiments in order to discuss the effects of the correction terms on the real atmospheric processes.

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