THE APPLICATION OF DELTA FUNCTION TO THE ALBEDO OF CLOUDS

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ABSTRACT

It is verified that there is δ-phase function characteristic in both of TS and SW simplified models, and on the basis of TS model, a more accurate model calculating the albedo and transmissivity of cloud layers is derived.

I. INTRODUCTION

Zdunkowski and Crandall[1] (1971) indicated that in the long-wave radiation, clouds can be treated as purely absorbing media, whose scattering effects may be approximately ignored. In the short-wave radiation, however, the multiple scattering effects must be considered and as a result, must complicate the radiation calculation. Radiation Commission[2] (1977) recommended two independent simplified models to compute the albedo of the scattering atmosphere. It will be proved that they are both associated with δ-function, and a more accurate method will be derived on the basis of δ-function and one of those models.

II. FORMULA FOR COMPUTING ALBEDO AND TRANSMISSIVITY

For the standard radiation problem, in which there is no ground reflection, the procedures for calculating the albedo and transmissivity in short-wave radiation are described as[2]

$$\mu' \frac{dI(\tau,\mu',\mu)}{d\tau} = -I(\tau,\mu',\mu) + \frac{\omega_0}{2}\int_{-1}^{1} I(\tau,\mu'',\mu)P(\mu',\mu'')d\mu'' + \frac{F_0}{4\pi}\omega_0 P(\mu',\mu)\exp(-\tau/\mu) \qquad (1)$$

$$F(\tau,\mu) = 2\pi\int_{-1}^{1} I(\tau,\mu',\mu)\mu' d\mu', \quad F\downarrow(\tau^*,\mu)=0, \quad F\uparrow(0,\mu)=0, \qquad (2)$$

$$a(\tau^*,\mu) = F\uparrow(\tau^*,\mu)/F_0\mu, \quad t(\tau^*,\mu) = F\downarrow(0,\mu)/F_0\mu + \exp(-\tau^*/\mu), \qquad (3)$$

where I is the diffuse radiance, F_0 the incident solar flux, ω_0 the single scattering albedo, τ the upward increasing optical depth, τ^* the τ of the total scattering atmosphere, μ' and μ are the cosine of emergent and the solar zenith angles, respectively, $a(\tau^*,\mu)$ and $t(\tau^*,\mu)$ the plane albedo and transmissivity of the total atmosphere illuminated by the sun in the direction of μ, and P the phase function.

By the use of the reciprocity principle proposed by Chandrasekhar[3] (1960), Stamnes and Swanson[4] (1981) offered a simple method to compute $a(\tau^*,\mu)$ and $t(\tau^*,\mu)$ defined in Eq. (3) for standard radiation problem, which can be summarized as

$$\mu\frac{d\tilde{I}(\tau,\mu)}{d\tau}=-\tilde{I}(\tau,\mu)+\frac{\omega_0}{2}\int_{-1}^{1}\tilde{I}(\tau,\mu')P(\mu',\mu)d\mu', \tag{4}$$

$$\tilde{I}\uparrow(0,\mu)=0,\quad \tilde{I}\downarrow(\tau^*,\mu)=I_{in}=\frac{F_0}{\pi}, \tag{5}$$

$$a(\tau^*,\mu)=\tilde{I}\uparrow(\tau^*,\mu)/I_{in},\quad t(\tau^*,\mu)=\tilde{I}\downarrow(0,\mu)/I_{in}. \tag{6}$$

Eqs. (4)—(6) relate the angular distribution of emergent intensity $\tilde{I}\uparrow(\tau^*,\mu)$ and the transmitted intensity $\tilde{I}\downarrow(\tau^*,\mu)$ due to isotropic incident illumination to the plane albedo and transmissivity resulting from a parallel sun beam. As a result, the nonlinear differential equation (1) is reduced to a linear one and, in addition, we need not calculate the flux integral (2). Next, we will use Eqs. (1)—(3) or (4)—(6) in conjunction with δ function to derive simplified formula to compute $a(\tau^*,\mu)$ and $t(\tau^*,\mu)$.

III. INTRODUCTION OF δ FUNCTION

Seeing that the average diameter of cloud drops (more than 4 μm) are far larger than the short-wavelength, the scattering of sun light by clouds falls under the scattering problems of "large drop", in which usually many terms involved in the phase function are needed by Mie theory. However, it is easily seen from the pattern of phase function for a "large drop" that this kind of phase function may be approximated by δ function. Assuming

$$P(\mu,\mu')=A\delta(\mu-\mu')+B, \tag{7}$$

taking the polar axis in the incident direction, and according to the normalized condition and the definition of the asymmetry factor of the phase function, we have

$$\frac{1}{2}\int_{-1}^{1}P(1,\mu')d\mu'=1, \tag{8}$$

$$\frac{1}{2}\int_{-1}^{1}P(1,\mu')\mu'd\mu'=g, \tag{9}$$

where g is asymmetry factor. Putting (7) into (8) and (9) yields $A=2g$, $B=1-g$, and therefore

$$P(\mu,\mu')=2g\delta(\mu-\mu')+1-g. \tag{10}$$

The troubles from the multiple scattering lie in treating the integral in Eq. (4). Therefore, for simplicity, we use $\tilde{I}(\tau,\mu)$ and $\tilde{I}(\tau,-\mu)$ to stand for the average intensity for μ from 0 to 1 and μ from 0 to -1, respectively, and as a result, the integral in (4) becomes

$$\int_{-1}^{1}\tilde{I}(\tau,\mu')P(\mu',\mu)d\mu'=2g\tilde{I}(\tau,\mu)+(1-g)[\tilde{I}(\tau,\mu)+\tilde{I}(\tau,-\mu)]. \tag{11}$$

Substituting (11) into (4) yields

$$\mu\frac{d\tilde{I}(\tau,\mu)}{d\tau}=-\tilde{I}(\tau,\mu)\left(1-\frac{1+g}{2}\omega_0\right)+\frac{1-g}{2}\omega_0\tilde{I}(\tau,-\mu). \tag{12}$$

If we designate $\tilde{I}_1(\tau,\mu)$ and $\tilde{I}_2(\tau,\mu)$ for upward and downward radiance, respectively, take μ to be absolute values, and take the upward increase of τ into account, Eq. (12) may be written in two:

$$\mu\frac{d\tilde{I}_1(\tau,\mu)}{d\tau}=-\tilde{I}_1(\tau,\mu)\left(1-\frac{1+g}{2}\omega_0\right)+\frac{1-g}{2}\omega_0\tilde{I}_2(\tau,\mu), \tag{13}$$

$$\mu\frac{d\tilde{I}_2(\tau,\mu)}{d\tau}=\tilde{I}_2(\tau,\mu)\left(1-\frac{1+g}{2}\omega_0\right)-\frac{1-g}{2}\omega_0\tilde{I}_1(\tau,\mu). \tag{14}$$

Eqs. (13) and (14) are the "Two Stream" radiation model, hereafter written as TS model for short. On the other hand, by Eddington's approximation, the radiance can be given by

$$I(\tau,\mu)=I_0(\tau)+I_1(\tau)\mu. \tag{15}$$

Obviously, the expression (15) cannot satisfy the boundary conditions (5), and therefore cannot be used for the aforementioned simplified procedure (Eqs. (4)—(6)). Putting (10) and (15) into (1), we have

$$\int_{-1}^{1} I(\tau,\mu'')P(\mu'',\mu')d\mu''=2I_0(\tau)+2gI_1(\tau)\mu', \tag{16}$$

$$\mu'\frac{d(I_0(\tau)+I_1(\tau)\mu')}{d\tau}=-I_0(\tau)-I_1(\tau)\mu'+\omega_0(I_0(\tau)+I_1(\tau)g\mu')$$
$$+\frac{\omega_0F_0}{4\pi}\exp(-\tau_0/\mu)(2g\delta(\mu-\mu')+1-g). \tag{17}$$

By integrating (17) and μ' times (17) over μ' from -1 to $+1$, we can respectively obtain

$$\frac{dI_1(\tau)}{d\tau}=-3(1-\omega_0)I_0(\tau)+\frac{3}{4\pi}\omega_0F_0\exp(-\tau/\mu), \tag{18}$$

$$\frac{dI_0(\tau)}{d\tau}=-(1-\omega_0g)I_1(\tau)+\frac{3}{4\pi}\omega_0g\mu F_0\exp(-\tau/\mu). \tag{19}$$

It is Eqs. (18) and (19) that are used by Shettle and Weinman[5] (1970) to get another model to compute $a(\tau^*,\mu)$ and $t(\tau^*,\mu)$, hereafter written as SW model.

The models, TS and SW, are two of the main simplified models recommended by Radiation Commission[2]. It is worth pointing out that we have derived the same models by different approaches from the original works, in both of which δ function was not used to approximate the phase function. Thus, it can be seen that the phase function expressed in (10) is physically meaningful.

IV. COMPARISON BETWEEN TS AND SW MODELS

In order to verify and improve SW and TS models, the calculations of $a(\tau^*,\mu)$ and $t(\tau^*,\mu)$ of cloud layers for conservative case ($\omega_0=1$) have been made.

First, for TS model, let $\omega_0=1$, and we can get the solutions to (13), (14) and (5) as

$$\tilde{I}_1(\tau,\mu)=I_{in}\frac{(1-g)\tau}{2\mu}\bigg/\left(H\frac{(1-g)\tau^*}{2\mu}\right), \tag{20}$$

$$\tilde{I}_2(\tau,\mu)=I_{in}\left(1+\frac{(1-g)\tau}{2\mu}\right)\bigg/\left(1+\frac{(1-g)\tau^*}{2\mu}\right), \tag{21}$$

and according to (6) we will obtain

$$a(\tau^*,\mu)=\frac{(1-g)\tau^*}{2\mu}\bigg/\left(1+\frac{(1-g)\tau^*}{2\mu}\right), \tag{22}$$

$$t(\tau^*,\mu)=1\bigg/\left(1+\frac{(1-g)\tau^*}{2\mu}\right). \tag{23}$$

Secondly, for SW model, taking $\omega_0=1$ and according to (18), (19), (2) and (3), we will derive another $a(\tau^*,\mu)$ and $t(\tau^*,\mu)$ (see Shettle and Weinman[5] for detail) as

$$a(\tau^*,\mu)=1-\frac{4}{3}\frac{B_2}{\mu I_{in}}, \tag{24}$$

$$t(\tau^*,\mu)=\frac{3}{4}\mu+\frac{1}{2}-\frac{B_tT(\tau^*)}{\mu I_{in}}+\exp(-\tau^*/\mu)\left(\frac{1}{2}-\frac{3}{4}\mu\right),$$

$$B_t=\frac{3\mu I_{in}[2+3\mu+(2-3\mu)\exp(-\tau^*/\mu)]}{4[4+3T(\tau^*)]}\qquad T(\tau^*)=(1-g)\tau^*. \tag{25}$$

We have computed the $a(\tau^*,\mu)$ for various τ^* and g with TS and SW models, and compared them with more accurate results which were obtained by Van de Hulst and Grossman and quoted by Shettle and Weinman[6]. Fig. 1 shows the results of $a(\tau^*,\mu)$ for a cloud layer, characterized by $\omega_0=1$, $g=0.75$, $\tau^*=1$, 4, 16, and Fig. 2 by $\omega_0=1$, $\mu=1$, $\tau^*=2$, 16, testing the responsibility of $a(\tau^*,\mu)$ to g. We can infer from Figs. 1 and 2 the following:

(1) Neither of TS and SW model results is good when τ^* is small.

(2) Both results improve when τ^* becomes larger, but for small μ the results for SW model are rather smaller and the results for TS model are rather larger than the exact results.

(3) For a smaller g, the SW and TS results are quite accurate, but when g gets larger, the SW results get worse than TS results. In view of conclusion (3) and the simpler computation formula for TS model, next, we will improve TS model to make it more accurate.

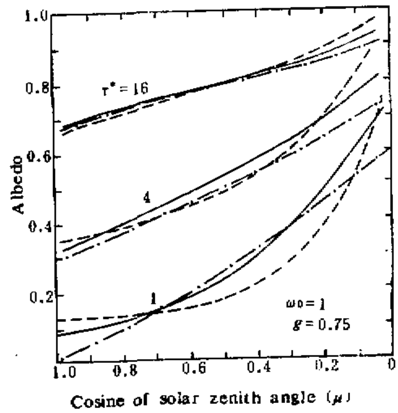

Fig. 1. Albedo as a function of total optical depth τ^* and incident sun angle arc(μ) comparison among TS model (dashed lines), SW model (dot-dashed lines), and Grossman (solid lines).

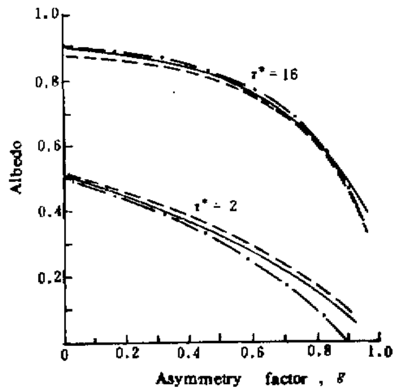

Fig. 2. Albedo as a function of the asymmetry factor g and the total optical depth: comparison among TS model (dashed lines), SW model (dot-dashed lines), and exact results of van de Hulst and Grossman (solid lines).

V. DERIVATION OF AN IMPROVED MODEL

Usually, the Henyey-Greenstein phase function

$$P_{H\cdot G}(\theta)=\sum_{n=0}^{\infty}(2n+1)g^nP_n(\cos\theta) \tag{26}$$

is considered to be good enough to represent the phase function typical of clouds (Wiscombe 1977), where θ is the scattering angle. By addition theory of spherical harmonics, we have

$$\overline{P_{H\cdot G}}(\theta)=\frac{1}{2\pi}\int_0^{2\pi}P_{H\cdot G}(\theta)\,d\varphi=\sum_{n=0}^{\infty}(2n+1)g^nP_n(\mu)P_n(\mu'), \tag{27}$$

where φ is azimuth angle. As mentioned before, in order to take advantage of δ function and make a function more similar to the real phase function of cloud drops, we assume that

$$P(\mu,\mu')=A\delta(\mu-\mu')+B+C\mu\mu', \tag{28}$$

and to determine the constants A, B and C, we require that (27) and (28) have the same first three moments of Legendre polynomial expansion, that is,

$$\frac{1}{2}\int_{-1}^{1} P(\mu,\mu')d\mu'=1, \tag{29}$$

$$\frac{1}{2}\int_{-1}^{1} P(\mu,\mu')\mu' d\mu'=g\mu, \tag{30}$$

$$\frac{1}{2}\int_{-1}^{1} P(\mu,\mu')\,\frac{1}{2}\,(3\mu'^2-1)d\mu'=\frac{3\mu^2-1}{2}g^2. \tag{31}$$

Substituting (28) into (29), (30) and (31) yields $A=2g^2$, $B=1-g^2$, $C=3g(1-g)$ and, as a result,

$$P(\mu,\mu')=2g^2\delta(\mu-\mu')+1-g^2+3g(1-g)\mu\mu'. \tag{32}$$

Using (32) and the radiance expressed in (20) and (21) as the first-order approximation in the integral of Eq. (4), we have

$$\frac{1}{2}\int_{-1}^{1} \tilde{I}(\tau,\mu')P(\mu,\mu')d\mu'=g^2 I(\tau,\mu)+A\tau+B, \tag{33}$$

where

$$A=(1-g^2)nlI_{in},$$

$$B=\frac{1-g^2}{2}I_{in}\left[1-\frac{\mu m}{2}-nl\tau^*+nm\mu\tau^*(1-\tau^*nl)\right], \tag{34}$$

$$l=\ln[(1+\tau^*n)/\tau^*n],\qquad m=3\frac{g}{1+g},\qquad n=\frac{1-g}{2}.$$

Then, putting (33) and (34) into (4) and designating upward and downward radiance as $\tilde{I}_1$ and $\tilde{I}_2$, respectively, we will obtain

$$\mu\frac{d\tilde{I}_1(\tau,\mu)}{d\tau}=-\tilde{I}_1(1-g^2)+A\tau+B, \tag{35}$$

$$\mu\frac{d\tilde{I}_2(\tau,\mu)}{d\tau}=-\tilde{I}_2(1-g^2)+A\tau+B, \tag{36}$$

noting that because of the upward increase of τ and $\mu<0$ for the downward radiance, (35) and (36) have the same expressions. Eqs. (35) and (36) are two independently inhomogeneous linear differential equations, the inhomogeneous terms being polynomials.

Considering the boundary conditions (5), we can get the solutions to (35) and (36) as

$$\tilde{I}_1(\tau,\mu)=e^{-(1-g^2)\tau/\mu}\Big\{A\mu\frac{e^{(1-g^2)\tau/\mu}}{(1-g^2)^2}\left[\frac{(1-g^2)\tau}{\mu}-1\right]+\frac{A\mu}{(1-g^2)^2}$$

$$+\frac{B}{1-g^2}\left(e^{(1-g^2)\tau/\mu}-1\right)\Big\}, \tag{37}$$

$$I_2(\tau,\mu)=e^{-(1-g^2)\tau/\mu}\Big\{A\mu\frac{e^{(1-g^2)\tau/\mu}}{(1-g^2)^2}\left[\frac{(1-g^2)\tau}{\mu}-1\right]+\frac{A\mu}{(1-g^2)^2}$$

$$+\frac{B}{1-g^2}\left(e^{(1-g^2)\tau/\mu}-1\right)+C\Big\}, \tag{38}$$

where

$$C=I_{in}e^{k}-A\mu\frac{e^{k}}{(1-g^2)^2}(k-1)-\frac{A\mu}{(1-g^2)^2}-\frac{B}{1-g^2}(e^{k}-1),\quad k=\frac{(1-g^2)\tau^*}{\mu}. \tag{39}$$

By using (37) and (38) in (6) and making calculations, finally we will obtain the improved model to compute the albedo and transmissivity for cloud layer as

$$a(\tau^*,\mu)=e^{-k}\Big\{\frac{\mu nl}{(1-g^2)}[(k-1)e^{k}+1]+\frac{e^{k}-1}{2}$$

$$\times\left[1-\frac{\mu m}{2}-nl\tau^*+nm\mu\tau^*(1-\tau^* nl)\right]\Big\}, \tag{40}$$

$$t(\tau^*,\mu)=\frac{C}{I_{in}}=1-a(\tau^*,\mu), \tag{41}$$

where l, m, n, k and C have been given in (34) and (39), (41) itself also makes it clear that the expressions (40) and (41) derived by us tally with the law of conservation of radiative energy. To test (40), we have compared the results by (40) with those by Van de Hulst and Grossman, which were cited by Liou (7) (1973), and both results are listed in Table 1. It is shown that most of the results by (40) are very accurate except for $\tau^*=1$, for which the maximum error is 8.2 %. In Fig. 3, we compared the results of albedo among every two of by Van de Hulst, SW model and (40), where $g=0.848$, which is usually accepted as the asymmetry factor for clouds. It can be seen from Fig. (3) that in most cases, especially for smaller τ^*, our results are superior to those by SW model.

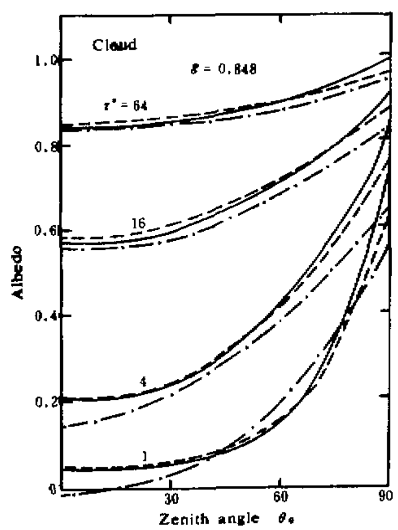

Fig. 3. Albedo as a function of total optical depth τ^* and incident sun angle: comparison among SW model (dot-dashed lines). Our model (dashed lines), and the exact results of Van de Hulst and Grossman (solid lines).

Table 1. Comparison of $a(\tau^*, \mu)$ as Computed by Our Model (40), and by Van de Hults ($g=0.75$)

Method	τ^*	0.1	0.5	0.9
V. d. H.	1	0.581	0.240	0.097
(40)		0.545	0.226	0.105
V. d. H.	4	0.733	0.519	0.348
(40)		0.726	0.530	0.364
V. d. H.	16	0.881	0.787	0.707
(40)		0.886	0.808	0.731

VI. DISCUSSION AND CONCLUSION

Lastly, there are some notes as follows:

(1) In the derivation of (40) and (41). $\omega_0=1$ is assumed, which fits the possible values in visible spectrum for clouds (Liou[7]) and, therefore, limits the (40) and (41) to correspondent spectrum. As for the nonconservative radiation ($\omega_0 \neq 1$), the expressions derived by TS model are less simple and consequently we could hardly get any analytical expressions if we put them into the integral in (4). We will try solving these problems by numerical integration and polynomial fitting.

(2) The variation of τ due to wavelengths can be neglected in visible region[7] because the cloud particles at these wavelengths approach the geometrical optics region. Consequently, the models (40) and (41) can be adapted in entirely visible region.

(3) As mentioned before, the problems concerned belong to the standard radiative problem, but the planetary problems, in which there is a reflecting ground at the bottom of the atmosphere, may be reduced to the standard ones. According to (A 16) and (A 17) of Stamnes and Swanson[4], it can be seen that once the former is solved, the latter follows immediately.

For the sake of putting the radiative factors into the model for the weather forecast, we need to simplify radiative calculations with a certain accuracy. This paper shows the computations of albedo and transmissivity can be simplified if δ function is associated with phase function properly. And on the basis of TS model, a more accurate simplified model for $a(\tau^*, \mu)$ and $t(\tau^*, \mu)$ was derived.

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