# PROBABILITY OF RECEIVED-POWER FLUCTUATION OF AN OPTICAL SYSTEM IN THE TURBULENT ATMOSPHERE

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ABSTRACT

The probability of received-power fluctuation in the turbulent atmosphere is discussed with a simple and yet reasonable model for a direct-detection optical system. Good agreement was found between the theoretical results and the field experiment. Thus the analysis in this paper may be taken as a guide for the design of atmospheric optical system.

## I. INTRODUCTION

Considering the laser application through atmospheric channel, the influence of atmospheric turbulence must be taken into account. For a direct-detection optical system, the main effects of turbulent atmosphere are intensity fluctuation, beam wander and system error[1]. Many authors discussed them individually, but for a practical system we have to consider the combined effect which has been discussed in Ref. [1] except the system error. Ref. [2] deals with probability distribution only. In this paper, we first solve the probability density of received-power fluctuation under the influences mentioned above, and then derive the probability distribution and variance. The results are expressed in an approximate form with good accuracy. Comparing it with our field experimental results, we find them in good agreement.

## II. SYSTEM MODEL

Suppose a system transmits a stationary Gaussian beam passing through the atmosphere at a distance of $L$. The beam centre deviates (caused by system error) from the receiver centre by a distance $\rho_0$ (see Fig. 1). The intensity distribution at the receiver plane can be written as

$$I(L,x,y)=\frac{I_0 W_0^2}{W^2}\exp\left[-2\frac{(x-\rho_0)^2+y^2}{W^2}\right], \qquad (1)$$

where $I_0$ is the intensity of beam centre at the transmitter; $W_0$, the initial beam radius; and $W$, the beam radius at the receiver plane. Because of system symmetry, the orientation of $\rho_0$ in Eq. (1) has no influence on our later discussions.

1) This kind of error may be a system alignment error, or beam deviation caused by daily changes of atmospheric temperature gradient.

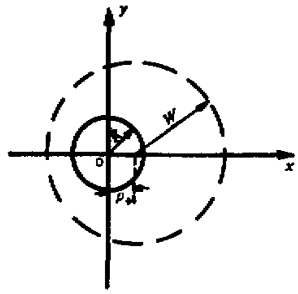


Fig. 1. Scheme of receiver plane coordinate.

For a system with an arbitrary receiving aperture denoted by $\Phi(x,y)$, the received power is given by

$$P=\iint_{-\infty}^{+\infty} I(L,x,y)\Phi(x,y)dx\,dy. \quad (2)$$

For a circular aperture with radius $R$, we have

$$\Phi(x,y)=\Phi_c(\rho,\theta)=\begin{cases}1, & 0\leqslant\rho\leqslant R,\ 0\leqslant\theta\leqslant 2\pi\\ 0. & \text{others}\end{cases} \quad (3)$$

Substituting Eqs. (1) and (3) into Eq. (2), we have

$$P_c=\frac{2\pi W_0^2}{W^2}I_0\int_0^R \exp\left[-2\frac{\rho^2+\rho_0^2}{W^2}\right]I_0\left(\frac{4\rho\rho_0}{W^2}\right)\rho d\rho, \quad (4)$$

where $I_0(x)$ is the zeroth-order imaginary variable Bessel function. Obviously, Eq. (4) is too complex to calculate further. For simplicity, we use the Gussian weighting aperture, i.e.

$$\phi_G(\rho,\theta)=\exp[-2\rho^2/R^2], \qquad 0\leqslant\rho<+\infty,\ 0\leqslant\theta\leqslant 2\pi, \quad (5)$$

and obtain

$$P_G=\frac{\pi W_0^2 I_0 R^2}{2(R^2+W^2)}\exp[-\rho_0^2/(R^2+W^2)]. \quad (6)$$

Eq. (6) is simpler than Eq. (4). The error of such replacement is given by

$$\Delta=\frac{|P_c-P_G|}{P_c}. \quad (7)$$

An analysis shows that $\Delta$ depends on the "relative aperture" $R/W$ and "relative deviation" $\rho_0/W$. For $R/W<0.3$ and $\rho_0/W<0.2$, the error is only 3%. Thus it is negligible if the statistical correlation of intensity on the receiver aperture is neglected.

Because of intensity fluctuation and beam wander, the received power will be a random variable. For the weak fluctuation condition ($\sigma_x^2\sim0.2$—$0.5$, where $\sigma_x^2$ is logarithm amplitude variance of plane wave), the intensity fluctuation is described by log-normal distribution[3], which may also exist under middle or very strong turbulent conditions[5]. Therefore, if the receiver aperture of system has a radius less than the correlation distance of arrived beam, the power fluctuation caused by intensity fluctuation, $P_S$, would also follow the log-normal law, i.e.

$$f_{P_s}(P_s)=\frac{1}{2\pi\sigma^2_{\ln P_s}P_s}\exp\left[-\frac{(\ln P_s-\mu)^2}{2\sigma^2_{\ln P_s}}\right]. \tag{8}$$

Under the conditions mentioned above, we have

$$\sigma^2_{\ln P_s}\doteq\sigma^2_{\ln I}=4\sigma^2_{\chi B}(L,\rho), \tag{9}$$

where $\sigma^2_{\chi B}(L, \rho)$ is the variance of logarithm amplitude of beam wave given in Ref. [4].

From the probability theorem we know that the $K$th moment of $P_s$ is

$$\langle P_s^K\rangle=\exp[K\mu+K\sigma^2_{\ln P_s}/2], \qquad K=1,2,3,\cdots. \tag{10}$$

The symbol $\langle\ \rangle$ denotes ensemble average. When $K=1$, we have

$$\langle P_s\rangle=\int_0^{2\pi}\int_0^{\infty}\langle I(L,\rho)\rangle\exp[-2\rho^2/R^2]\rho d\rho d\theta=\frac{\pi W_0^2I_0R^2}{2(R^2+W_b^2)}. \tag{11}$$

Then, we have

$$\mu=\ln\langle P_s\rangle-\sigma^2_{\ln P_s}/2. \tag{12}$$

In Eq. (11), $\langle I(L,\rho)\rangle$ is the average distribution of intensity and $W_b$ is the broadened beam size at the receiver plane, both of which are given in Ref. [4].

Several experiments and analyses show that beam wander follows normal distribution and the wander in $x$ direction is independent of $y$ direction[5]. By taking account of the system error $\rho_0$, the probability density of beam centre at a point $(x, y)$ on the receiver plane can be written as

$$f_{x,y}(x,y)=\frac{1}{2\pi\sigma^2_\omega}\exp\left[-\frac{(x-\rho_0)^2+y^2}{2\sigma^2_\omega}\right], \qquad -\infty<{x \atop y}<\infty. \tag{13}$$

Then, we have

$$\begin{aligned}f_\rho(\rho)&=\int_0^{2\pi}f_{x,y}(x,y)d\theta\\&=\frac{\rho}{\sigma^2_\omega}\exp\left[-\frac{(\rho^2+\rho_0^2)}{2\sigma^2_\omega}\right]I_0\left(\frac{\rho\rho_0}{\sigma^2_\omega}\right),\end{aligned} \tag{14}$$

which is the probability density of beam centre at a distance $\rho_0$ from the origin, where $\sigma^2_\omega$ was given by Ref. [6].

From Eq. (7) it can be seen that the received power is a single-value function of beam centre coordinates, so that the power fluctuation caused by beam wander is given by

$$f_{P_\omega}(P_\omega)=f_\rho[\rho(P_\omega)]\left|\frac{\partial\rho}{\partial P_\omega}\right|_{\rho=g(P_\omega)}, \tag{15}$$

where $g(P_\omega)$ is the inverse function of $P_\omega(\rho)$. Defining the received power modulation index caused by beam wander as

$$m_\omega\equiv P_\omega/P_0, \qquad 0\leqslant m_\omega\leqslant 1, \tag{16}$$

we have

$$f_{m_\omega}(m_\omega)=f_\rho[\rho(m_\omega)]\left|\frac{\partial\rho}{\partial m_\omega}\right|_{\rho=g'(m_\omega)}. \tag{17}$$

In Eq. (16), $P_0$ is the received power of system through free space.

From Eqs. (6), (12), (16) and (17), we obtain

$$\langle m_\omega^K\rangle=\int_0^1 m_\omega^K f_{m_\omega}(m_\omega)\,dm_\omega$$

$$=\frac{R^2+W^2}{2K\sigma_\omega^2+R^2+W^2}\exp[-K\rho_0^2/(2K\sigma_\omega^2+R^2+W^2)]. \tag{18}$$

## III. THE PROBABILITY DENSITY AND DISTRIBUTION OF RECEIVED POWER FLUCTUATION

It has been mentioned that the received power can be considered as a randomly modulated variable caused by the intensity fluctuation and beam wander, and written as

$$P=P_0\cdot m_s\cdot m_\omega \tag{19}$$

where $m_s\equiv P_s/P_0$ is the modulation index caused by the intensity fluctuation. These two random modulations are independent, because the intensity fluctuation is induced by smaller-scale turbulent eddies while the beam drift is induced by larger-scale ones (here the smaller and larger are concerned as compared with the beam size). According to this argument and the probability theorem, the probability density of received power is given by

$$f_P(P)=\int_0^1\frac{1}{|m_\omega|}f_{P_s}\left(\frac{P}{m_\omega}\right)f_{m_\omega}(m_\omega)dm_\omega. \tag{20}$$

Substituting Eqs. (9) and (16) into Eq. (20) and making use of some variable replacements, we obtain

$$f_P(P)=\frac{\exp[-\rho_0/2\sigma_\omega^2]}{(2\pi\sigma_{\ln P_s}^2)^{1/2}\sigma_\omega^2 P}\int_0^\infty\exp\left\{-\frac{\left[\ln P-\ln\langle P_s\rangle+\frac{\sigma_{\ln P_s}^2}{2}+\frac{2\rho^2}{R^2+W^2}\right]^2}{2\sigma_{\ln P_s}^2}\right\}$$

$$\times\exp(-\rho^2/2\sigma_\omega^2)I_0\left(\frac{\rho\rho_0}{\sigma_\omega}\right)\rho d\rho. \tag{21}$$

By letting

$$A=\exp(-\rho_0^2/2\sigma_\omega^2)/(2\pi\sigma_{\ln P_s}^2)^{1/2}, \tag{22-1}$$
$$B=-\ln\langle P_s\rangle+\sigma_{\ln P_s}^2/2, \tag{22-2}$$
$$C=4\sigma_\omega^2/(R^2+W^2), \tag{22-3}$$
$$D=\sqrt{2}\,\rho_0/\sigma_\omega, \tag{22-4}$$
$$t=\rho^2/2\sigma_\omega^2, \tag{22-5}$$

Eq. (21) can be written as

$$f_P(P)=\frac{A}{P}\int_0^\infty\exp\left[-\frac{(\ln P+B+Ct)^2}{2\sigma_{\ln P_s}^2}\right]I_0(D\sqrt{t})\exp(-t)dt. \tag{23}$$

One of the methods for carrying out the integration in Eq. (23) is to expand $I_0(D\sqrt{t})$ in power series, but the result may be very complex. However, the result may be very accurate and simple by using the Laguerre approximate integration formula[7]. Thus we have

$$f_P(P)=\frac{A}{P}\sum_{K=1}^{n}\lambda_K^{(n)}\exp\left[-\frac{(\ln P+B+C\xi_K^{(n)})^2}{2\sigma_{\ln P_s}^2}\right]I_0(D\sqrt{\xi_K^{(n)}}), \tag{24}$$

where $\lambda_K^{(n)}$ and $\xi_K^{(n)}$ are the $K$th integral coefficient of n-order interpolation and $K$th root of n-order Laguerre polynomials respectively.

From Eq. (23) the probability distribution can be written as

$$F_P(P\leqslant P_t)=A\int_0^{P_t}\frac{1}{P}\int_0^{\infty}\exp\left[-\frac{(\ln P+B+Ct)^2}{2\sigma^2_{\ln P_s}}\right]I_0(D\sqrt{t})e^{-t}dtdP. \tag{25}$$

Exchanging the integration order in Eq. (25) and noting that

$$\int_0^{P_t}\frac{1}{P}\exp\left[-\frac{(\ln P+B+Ct)^2}{2\sigma^2_{\ln P_s}}\right]dP$$

$$=\frac{1}{2}\sqrt{2\pi}\,\sigma_{\ln P_s}\left[1+\mathrm{erf}\left(\frac{\ln P_t+B+Ct}{\sqrt{2}\,\sigma_{\ln P_s}}\right)\right], \tag{25-1}$$

$$\int_0^{\infty}I_0(D\sqrt{t})e^{-t}dt=\exp\left(\frac{\rho_0^2}{2\sigma_\omega^2}\right), \tag{25-2}$$

we obtain

$$F_P(P\leqslant P_t)=\frac{1}{2}+A'\int_0^{\infty}\mathrm{erf}\left[\frac{\ln P_t+B+Ct}{\sqrt{2}\,\sigma_{\ln P_s}}\right]I_0(D\sqrt{t})e^{-t}dt, \tag{26}$$

where $A'=A\sqrt{2\pi\sigma^2_{\ln P_s}}$ erf $(x)$ is the error function. Similarly, the probability distribution is given by

$$F_P(P\leqslant P_t)=\frac{1}{2}+A'\sum_{K=1}^{n}\lambda_K^{(n)}\mathrm{erf}\left[\frac{\ln P_t+B+C\xi_K^{(n)}}{\sqrt{2}\,\sigma_{\ln P_s}}\right]I_0\left(D\sqrt{\xi_K^{(n)}}\right). \tag{27}$$

Figs. 2—4 show the calculated results from Eqs. (23) and (27) for different parameters. In our calculations the order of Laguerre approximate intergration is nine, transmitted power is 1W, and $\sigma^2_{\ln P_s}(L,\rho)$ is taken to be $\sigma^2_{\ln P_s}(L,O)$ as an approximation.

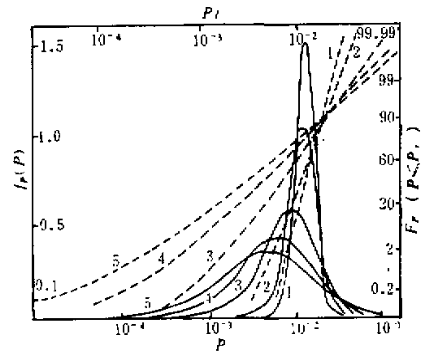


Fig. 2. $f_P(P)$ and $F_P(P\leqslant P_t)$as a function of $C_n^2$. The parameters are $L=5$ km, $W_c=0.028$m, $R=0.1$ m, $\rho_0=0.05$ m, $\lambda=10.6$ $\mu$m. Curves 1—5 are respectively for $C_n^2$ equal to $(0.5, 1, 3, 5, 7)\times10^{-14}$ $m^{-2/3}$.

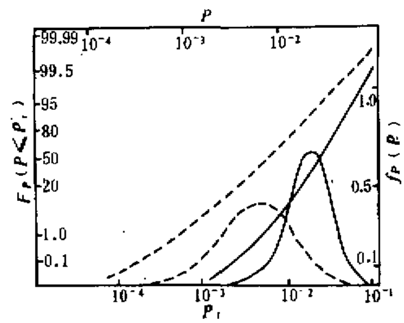


Fig. 3. $f_P(P)$ and $F_P(P\leqslant P_t)$ as a function of $L$ (solid lines for $L=3$ km, dashed for $L=5$ km). The parameters are $C_n^2=5\times10^{-14}$ $m^{-2/3}$, $R=0.1$m, $W_c=0.028$m, $\lambda=10.6$ $\mu$m, $\rho_0=0.05$ m.

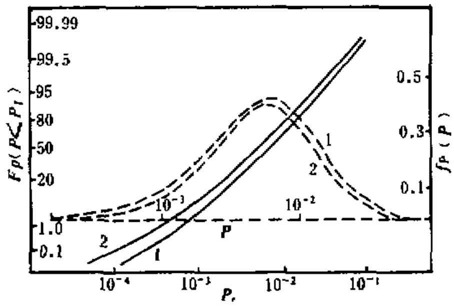


Fig. 4. $f_P(P)$ and $F_P(P\leq P_t)$ as a function of $\rho_0$ (curves 1 for $\rho_0=0.0$ m and 2 for $\rho_0=0.25$ m). The parameters are $L=5$ km, $R=0.1$ m, $W_0=0.028$ m, $\lambda=10.6$ $\mu$m, $C_n^2=5\times10^{-14}$ $m^{-2/3}$.

## IV. THE VARIANCE

As the intensity fluctuation and the fluctuation due to beam wander are independent, the moments of received power fluctuation should be

$$\langle P^K\rangle=\langle P_S^K\rangle\langle m_\omega^K\rangle \tag{28}$$

From Eqs. (10) and (18) we obtain

$$\langle P^K\rangle=\frac{R^2+W^2}{2K\sigma_\omega^2+R^2+W^2}\langle P_S^K\rangle\exp\left[K(K-1)\frac{\sigma_{\ln P_S}^2}{2}-\frac{K\rho_0^2}{K2\sigma_\omega^2+R^2+W^2}\right]. \tag{29}$$

Thus the variance is

$$\sigma_P^2\equiv\frac{\langle P^2\rangle-\langle P\rangle^2}{\langle P\rangle^2}=\frac{(2\sigma_\omega^2+R^2+W^2)^2}{(4\sigma_\omega^2+R^2+W^2)(R^2+W^2)}\exp(\sigma_{\ln P_S}^2)$$

$$\times\exp\left[\frac{4\rho_0^2\sigma_\omega^2}{(2\sigma_\omega^2+R^2+W^2)(4\sigma_\omega^2+R^2+W^2)}\right]-1. \tag{30}$$

## V. DISCUSSION AND COMPARISON WITH FIELD EXPERIMENT

The calculated results show that, generally speaking, the distribution is not log-normal because of the interference of beam wander. The significant difference is that in small value range the probability value is larger. This can be explained by the fact that beam wander generally gives rise to slight fluctuation. When the turbulence is very weak, the distribution is still close to log-normal (see dashed lines 1 and 2 in Fig. 2).

Fig. 4 shows that the influence of system error on the distribution is not important. Thus we conclude that the main factor that determines the power fluctuation is the intensity fluctuation. It should be pointed out that the extent of influence of beam wander can not be measured by $\sigma_{\chi_B}$. For instance, for a 5 km path, $\sigma_{\chi_B}^2$ is 0.0501 when $C_n^2=5\times10^{-15}$ $m^{-2/3}$, the distribution is very close to log-normal (see dash line 1 in Fig. 2). But for a 800 m path, $\sigma_{\chi_B}^2$ is 0.0177 when $C_n^2=5\times10^{-14}$ $m^{-2/3}$, the distribution deviates from log-normal (see curve 1 in Fig. 5).

Our field experiment was carried out in April, 1984 in the Tengger Desert, northern China, with a 800 m optical path. The beam was transmitted from a 10.6 $\mu$m $CO_2$ laser with 10 W output. The transmitter of three-channel $CO_2$ Laser Communication System with an aperture diameter of 100 mm was used, which was placed 1.3 m above the ground. The beam had a divergent angle of 0.2 mrad and a depression angle of 2°. The terrain under the propagation path was dunes and pebbles. The detectors were two $LiTaO_3$ pyroelectric detectors which had 1 mm diameter of sensitive area and were placed 8 cm apart. The detected signal, after amplification, filteration and demodulation, were recorded on tape separately. At the same time, a two-probe platium fine-wire microtemperature meter was used to measure the strength of atmospheric turbulence. Recorded data were processed by PDP-11/24 computer, some of which are given in Figs. 5 and 6. Fig. 5 shows a good agreement between the theoretical and experimental results, in which curves 3 and 4 are close to log-normal distribution (they are moved by some distance in $P_t/<P>$ axis). This verifies our conclusion that the power fluctuation is mainly determined by the intensity fluctuation under very weak turbulent conditions.

As the diameter of sensitive area of the detector was only 1 mm, the detection could be considered as a point detection comparing with the beam width (about 160 mm). Therefore it is inferred that the intensity fluctuation at a point of receiver plane is not log-normal.

Fig. 6 shows the comparison between variances, where the solid line and dashed line depict $\sigma_P^2$ and $\sigma_{1nP_t}^2$ respectively. The difference between them can be explained by the use of Eq. (29). If $\sigma_a^2 \ll W^2$ and $\sigma_{1nP_t}^2 \ll 1$, Eq. (29) approximates to

$$\sigma_P^2 \doteq \sigma_{1nP_t}^2 + 4\rho_0^2\sigma_a^2/(R^2+W^2)^2, \tag{30}$$

which means that beam drift and system error make $\sigma_P^2$ increase.

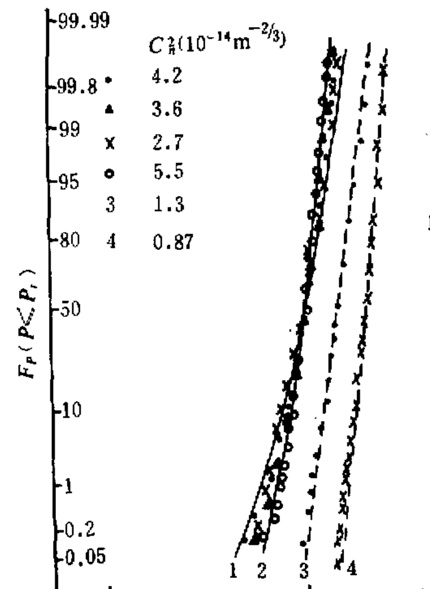


Fig. 5. Comparison between the theoretical and experimental results. Solid lines 1 and 2 are theoretical values for $C_n^2=5\times10^{-14}$ m$^{-2/3}$ and $3\times10^{-14}$ m$^{-2/3}$, and $\rho_0=0.04$ m. Other parameters are the same as in experiment. Dashed lines 3 and 4 are close to log-normal distribution.

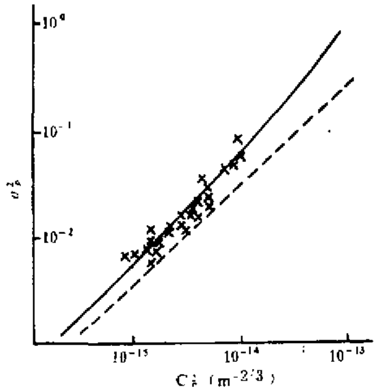


*Fig. 6.* Comparison between the theoretical and experimental variance.

## VI. CONCLUSIONS

We have discussed the statistical properties of received power fluctuation for a direct-detection system which was interfered with intensity fluctuation, beam wander and system error simultaneously. The intensity fluctuation is proposed to be in log-normal law and beam drift in normal distribution. A field experiment was carried out to validate the theoretical results. It showed that the distributions of received power fluctuation of the system and the intensity fluctuation of a point on the receiver plane are not log-normal, but under very weak turbulent conditions the distributions are log-normal.

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