A STUDY ON THE EXCITATION, ESTABLISHMENT AND TRANSITION OF MULTIPLE EQUILIBRIUM STATES PRODUCED BY NEARLY RESONANT THERMAL FORCING —PART II: THEORETICAL ANALYSIS OF THE MECHANISM OF MULTIPLE EQUILIBRIUM STATES

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ABSTRACT

Based on the results acquired in Part I of this paper, analysis is made of the theoretical mechanism of thermal forcing multiple equilibrium states (MES) and their stability. The results are as follows: 1) non-linear effect and external forcing are determinative factors for MES formation; 2) under proper "environmental conditions" the forcing can excite stable MES, particularly three types of solutions, two of which, with larger amplitude of resonance, are not sensitive to the change in the forcing intensity; while the other, i. e. the one of small amplitude, dependent significantly on it; 3) in general, the domain of parameter values for the MES existence increases, but the stability decreases, with increasing thermal forcing; 4) steady thermal forced waves are always unstable for the most part; 5) thermal driving and orographic effect act equally as dynamical triggers; 6) friction has significant influence upon the behavior of MES solutions.

Analysis shows that the changes in the "environmental parameters", such as the alteration of the shear of a basic current and intensity of the forcing, induce the transition between different equilibria.

I. INTRODUCTION

Based on the asymptotic solutions of MES and their detailed operations demonstrated in Part I (hereinafter referred to as [1]), this portion as the subsequent work is devoted to the examination, on a theoretical basis, of the mathematic-physical structures of MES produced by near-resonance instability, and to the demonstration of the dynamic effect on the excitation of the stable MES caused by thermal forcing. With the results obtained in both parts of this paper, we will propose a possible mechanism of the mutual transition of the equilibrium states (ES).

The detailed governing equations of the model may be referenced in [1].

II. THEORETICAL ANALYSIS OF EXCITATION, ESTABLISHMENT AND TRANSITION OF MES

This section deals with the dynamical mechanism of the near-resonance disturbance $\phi_n(x, y, T)$, superimposing upon the stationary thermal forced waves X_n (x, y) as given in (12) of [1], which develops a finite amplitude by linear instability and then approaches one of the MES by the nonlinearity of the motion.

1. *Excitation of ES by Thermal Forcing Instability*

Charney et al. (1979, 1980)[2,3] and Pedlosky (1982)[4] described the instability of a zonal basic flow with topographic forcing and introduced the topographic instability. As the first step of the theoretical analysis, our attention is given to the so-called thermal forcing instability, responsible for the establishment of MES, i. e., the linear instability of ϕ_n upon X_n.

Since the spatial structure of ϕ_n (x, y, T) have been prescribed in (15), (17) and (18) of [1], the undetermined part of ϕ_n is only $F(x, T)$ of $\phi_1^{(0)}$. It follws that the problem of instability is turned into one to study the time-dependent behavior of $F(x, T)$ under the linear condition $|F^2| \ll 1$.

To analyze the instability of the basic states

$$F=0 \quad \text{and} \quad K=0 \tag{1}$$

through the method of linearized perturbation by letting

$$F=0+F'=F'=\exp(\sigma T)\sum_{n=1}^{J}[A_n\cos nx+B_n\sin nx] \tag{2}$$

and

$$K=0$$

in (17) of [1], we obtain

$$a_1\frac{\partial F'}{\partial T}+a_2\frac{\partial^3 F'}{\partial x^3}+\frac{\partial F'}{\partial x}(a_4+a_7\cos 2l_0x)+F'(a_5+a_9\sin 2l_0x)$$

$$-a_{10}\int_0^x F'\cos l_0x dx\cos l_0x+a_{13}\int_0^{2\pi}F'\cos l_0xdx\cos l_0x=0, \tag{3}$$

in which such terms as a_6, a_8 and a_{11} are neglected as they are all proportional to the constant K.

In terms of the orthogonality of the trigonometric function, a system of linear homogeneous algebraic equations for spectral coefficients A_n and B_n is obtained, whose determinant of the coefficient matrix, when a zero value is given, can prescribe an eigenvalue problem for the eigenvalue σ. By applying the "Q-R" separation method to this eigenvalue equation, we arrive at the results that, over a rather wide range of the parameter values, 1) without heating and dissipation ($Q_0=r=0$) (in this case thermal forced waves in (12) of [1] degrade into a purely zonal basic flow), all the eigenvalues obtained have negative real roots or complex roots with a negative real part; 2) with heating and dissipation, they have positive real roots or complex roots with a positive real part. Apparently, it is the incorporated external forcing that enables the steady forced wave to develop unstable cases. Since the zonal basic flow is assumed to be baroclinically metastable, and the disturbances shown in (2) has the same large-scale feature as thermal forcing, it is evident that the instability illustrated above is different from the general baroclinic instability and thus called "near-resonance thermal-forcing instability".

2. *Structures and Domains of ES Solutions*

To discuss the MES solutions obtained in [1], the relationship between self-oscillation frequency and amplitude in the nonlinear forced inharmonic system in (35) of [1] should be established together with the conditions for the occurrence of the resonance of the system. Setting $C_3=C_4=0$ (meaning that no forcing and dissipation take place) for (43) of [1], we have

$$2l_1\omega_0^2+\frac{1}{6l_0^2}(C_2\varepsilon a)^2=0\ . \tag{4}$$

Applying the near-resonance assumption $\omega_0=l_0(1+\varepsilon l_1)$, $|l_0l_1\varepsilon|\ll 1$, we get the dependence of frequency upon amplitude in the nonlinear free system

$$\omega=\omega_0(1-\varepsilon l_1)=\omega_0+\frac{\varepsilon(C_2\varepsilon a)^2}{12l_0^2\omega_0}\ . \tag{5}$$

With forcing and dissipation available and $\omega(a)=l_0$ as the frequency of external forcing, and by use of (5), we obtain the condition for nonlinear resonance due to forcing, namely,

$$a=\left(-\frac{12l_0l_1}{C_2^2\varepsilon^2}\right)^{1/2}, \tag{6}$$

which requires $l_1<0$, i. e., $\omega_0<l_0$. Since $\omega_0\propto U^{-3/2}$, the shear of a basic flow considered therein $(U_1-U_2=U)$ is supercritical as regards the threshold of linear resonance.

When $C_3=C_6=0$, the root of the algebraic equation (43) of [1] for determining the amplitude of the ES solution is in the form

$$a^2=0, \quad \text{or}, \quad a^2=-\frac{12l_0^4l_1}{C_2^2\varepsilon^2} \quad \text{(double roots)}. \tag{7}$$

Therefore, (6) corresponds to the only significant solution of (29) of [1] when no forcing and friction happen.

Alternatively, by letting $C_2=0$ in (33) of [1], we get

$$a=|C_3|\Big/\left(\frac{C_6^2}{l_0^2}+4l_1^2l_0^4\right)^{1/2}, \tag{8}$$

which implies that in a nonlinear case, (29) of [1] possesses a single solution. From the analyses above we come to the conclusion that for the system (27) and (28) of [1], none of MES solutions can exist when no forcing and nonlinear effects occur.

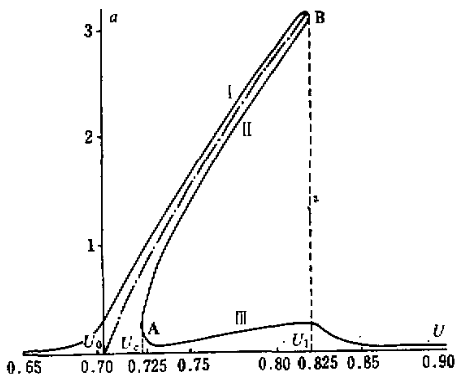

Fig. 1. The MES solution curve (solid lines) and nonlinear resonance curve (dotted-dashed line). $Q_0=0.1$; $r=0.03\,(1/7)^{2/3}$; $K=-0.75$; $\bar{\beta}=1.5$ and $\varepsilon=(1/7)^{2/3}$.

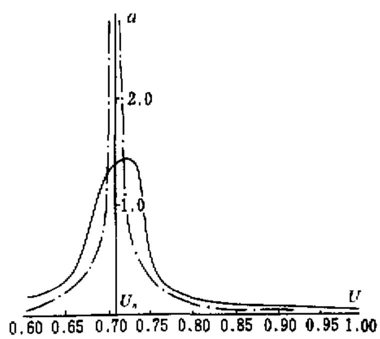

Fig. 2. The inviscid linear ES solution (dotted line) ($r=0$); the viscid ES solution (solid line) ($r=0.2\varepsilon$); $Q_0=0.1$ and the other parameters are as in Fig. 1.

Fig. 1 shows again the MES solution curves and the nonlinear resonance curve dominated by (6). It is interesting that very small thermal forcing can excite two ES solutions whose order of magnitude is twice or more as large as that of the forcing, which is evidently due to the resonant effect. This is illustrated as follows. Suppose DET$=0$ for (50) of [1] gives U_c as the critical "bifurcation" point for the appearance of multiple solutions and U_t as the point for their disappearance, then, when $U_c<U<U_t$, (43) of [1] has three solutions, two of them are of large amplitude and resonant in nature and correspond to the solutions of its own with $C_3=C_6=0$, and can be approached by (6); the other, a smaller-amplitude solution, to the linearity solution of the equation when $C_2=0$ and by (8). From (6) and (8) it is clear that large-amplitude resonance solutions of (43) of [1] are not sensitive to the change in the thermal forcing intensity Q_0 and the reverse is true for the linearity solution.

The distribution of the solution of (8) in the linear case as the function of U is depicted in Fig. 2, where the "skeleton curve" in Fig. 1 becomes symmetric to the straight line $l_1=0$ ($U_0=0.71$), and particularly in the case $C_6=0$, $a\to+\infty$ with $l_1\to 0$, the linear resonance takes place. It is noted that the result of (5) agrees well with that of Trevisan and Buzzi (1980)[5], which states that the presence of nonlinearity modifies the natural frequency of the linear system by a correction term proportional to the square of the amplitude. It is apparent by referring to Figs. 1 and 2 that the nonlinear system shows an intuitive figure of the modifying effect, where the resonance curve $l_1=0$ (if $C_2=0$) originally in the problem of linearity is made to deviate to the region $l_1<0$ ($U>U_0=0.71$) providing possibility of the existence of multiple solutions. And the stronger the nonlinearity, the larger the deviation, the wider the range for the multiple solutions and the smaller the difference in their values.

As shown in Fig. 1, the lower-branch small-amplitude solution (curve III) does not decrease with increasing U as illustrated by Trevisan and Buzzi (1980)[5] but increases with U as indicated by Pedlosky (1982)[4] for the baroclinic problem. Since, as will be shown, Fig. 5 corresponding to Fig. 1 still shows that the behaviors described above after dissipation has been removed, the comparison of the solutions with fixed Q_0 in Fig. 1 to the ES solution structures by the barotropical model of Trevisan and Buzzi (1980)[5] indicates that the growth of the small-amplitude solution with U in the multi-solution domain has resulted neither from frictional dissipation nor from the difference between forcing and orographic effects; but, most probably, from the baroclinicity of our model. This is the manifestation of the influence upon the MES solutions of such baroclinicity in the atmosphere as thermal-wind vorticity advection caused by the vertical shear of a basic current.

Now that the spatial structure of the solution over the domain of the parameter values is examined, with DET$=0$ in (40) of [1], we have a critical relation of Q_0 to l_0 (hence to U) for the multiple solutions of a^2 in (43) of [1], that is,

$$C_3^2=\left[-\frac{1}{4\sqrt{3}}\left(\frac{13}{3}l_1^2-\frac{C_6^2}{l_0^4}\right)^{3/2}-\frac{2}{9}l_1^3\frac{C_6^2}{2l_0^4}\right]\Big/\left(\frac{C_2\varepsilon}{36l_0^4}\right)^2, \tag{9}$$

and, by setting $C_3=C_6=0$ in (49) of [1], we obtain

$$\mathrm{DET}\propto\left(\frac{2}{9}\right)^2-\frac{1}{48}\left(\frac{13}{3}\right)<0,$$

meaning that (43) of [1] still has multiple solution for a^2. Yet (6) or (7) in this Part indicates that (43) of [1] gives a unique solution for a only.

It it evident from Fig. 3 that the multi-solution region is widened as U grows, and within the near-resonance section of U, the upper-boundary value of Q_0 is far higher than its given value for solving (29) of [1]. In the case $Q_0=0$ (i. e. $C_3=0$, and $C_4=0$ as required by the compatibility of the model), as (29) of [1] does not give multi-solution for a, $Q_0=0$ as a "singular line" can be set to be the lower boundary for the multi-solution section of a within the region of a^2. It follows that even if $Q_0\neq 0$, (29) of [1] always yields more than one solutions for a, however small Q_0 may be. The result was also obtained by Trevisan and Buzzi (1982)[5] in studying the stationary response of barotropic Rossby waves to orographic forcing.

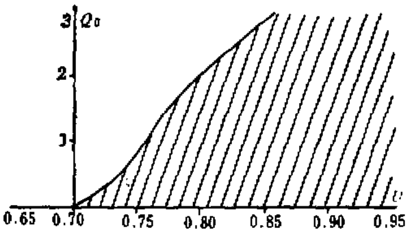

Fig. 3. The multi-solution region (hatched) for a^2 given by (43) of [1]. The upper limit is set by (9) and the lower limit is $Q_0=0$, with the other parameters as in Fig. 2.

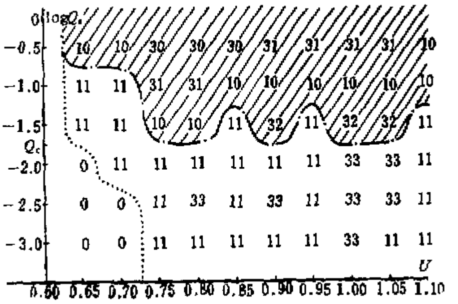

Fig. 4. Distribution of the ES solutions together with their stability on the Q_0-U plane. 0 denotes a region with no solution; 11 with one stable solution; 31 with one of the three solutions being stable; 32 with two of the three solutions being stable; 33 with three stable solutions. The region with at least one of the solutions being unstable is shaded. $Q_0=-2+\log 2$ and $r=0.02\varepsilon$.

Fig. 4. demonstrates the numerical results of the number of a as the solutions of (43) of [1] on the Q_0-U plane. It is evident that although the domain of the multiple solutions of a is narrowed as Q_0 diminishes, its MES solutions exist even if Q_0 gets considerably small.

3. *Stability Analysis*

Of all solutions obtained in Part I only the stable one is of significance to the ES of the actual atmosphere. For this reason stability analysis should be made of all the solutions acquired.

Suppose that for (27) of [1],

$$F(x,T)=f(x)+F'(x,T) \tag{10}$$

is set, where $f(x)$ is the ES solution found in Part I and $F'(x, T)$ the disturbance. Putting (10) into (27) of [1] and linearizing the resulting equation, we have the equation of disturbance

$$a_1\frac{\partial F'}{\partial T}+a_2\frac{\partial^3 F'}{\partial x^3}+a_3\frac{\partial fF'}{\partial x}+\frac{\partial F'}{\partial x}(a_4+a_6\sin l_0x+a_7\cos 2l_0x)$$

$$+F'(a_5+a_8\cos l_0x+a_9\sin 2l_0x)+a_{10}\int_0^x F'\cos l_0x dx\cos l_0x$$

$$+a_{13}\int_0^{2\pi} F'\cos l_0x dx\cos l_0x=0. \tag{11}$$

Assuming

$$F'(x,T)=\exp(\sigma T)\sum_{n=1}^{3}(A_n\cos nx+B_n\sin nx), \tag{12}$$

where A_n, B_n are constants which limit the analysis to the first three sine and cosine harmonics, and inserting it into (11), we obtain the following eigenvalue problem for:

$$\begin{vmatrix}
a_1\sigma+a_5+\frac{a_3}{2}N_2+(a_8-a_6)/2 & 0 & (a_8-3a_6)/2-\frac{a_3}{2}(N_2-N_4) & a_4-a_2-\frac{1}{2}a_3M_2 & 0 & \frac{1}{2}a_3(M_2-M_4) \\
0 & a_1\sigma+a_5+\frac{1}{2}a_{13}+a_3N_4 & 0 & 0 & 2a_4-8a_2-a_3M_4 & 0 \\
(a_8+a_6)/2+\frac{3}{2}a_3(N_2+N_4) & 0 & a_1\sigma+a_5 & \frac{3}{2}a_3(M_2-M_4) & 0 & 3a_4-27a_2 \\
a_2-a_4-\frac{a_3}{2}M_2 & 0 & -\frac{a_3}{2}(M_2+M_4) & a_1\sigma+a_5+\frac{1}{2}a_6-\frac{1}{2}a_3N_2 & 0 & -a_3N_2-\frac{3}{2}a_6 \\
0 & 8a_2-2a_4-a_3M_4 & 0 & 0 & a_1\sigma+a_5-a_3N_2 & 0 \\
-\frac{3}{2}a_3(M_2+M_4) & 0 & 27a_2-3a_4 & \frac{1}{2}a_6 & 0 & a_1\sigma+a_5
\end{vmatrix}=0, \tag{13}$$

where M_2, M_4, N_2 and N_4 are the coefficients of $\cos 2x$, $\cos 4x$, $\sin 2x$, and $\sin 4x$, respectively, of the ES solution $f(x)$ determined by (48) of [1]. Within the range of the parameter values considered, calculating σ by the "Q–R" separation method yields the results as shown in Fig. 4. When thermal forcing is less intense, all solutions are stable. Especially over the range of the parameter values in the neighbourhood of $U=1.0$ and $Q_0=10^{-2.5}-10^{-2}$, three stable solutions are likely to exist. They are, however, different in stability. Usually, the middle-branch solution is less stable. When thermal forcing is strengthened, reaching $Q_{0a}=0.02$, unstable solution occurs, but at some parameter points two stable ones can still exist. In the latter case, the upper- (large-) and lower-branch (small amplitude) solutions are bound to be stable and the middle one unstable (refer to Fig. 1). If only one stable solution happens, it may be either the upper or the lower one.

In general, with increased forcing, greater as the likelihood of MES solutions is, that of their actual stable ones gets less owing to the growth of instability. In the case of appropriate heat-source intensity there is much possibility of two or three stable solutions, which suggests a new conclusion of this problem.

4. *Influence of Dissipation upon Existence and Stability of MES*

In the absence of friction, i. e. $C_6=0$, we get from (49) of [1]

$$\mathrm{DET}\propto l_1^4\left\{\left[\frac{2}{9}+\left(\frac{C_2C_3\varepsilon}{36l_0^4}\right)^2\Big/l_1^3\right]^2-\frac{1}{48}\left(\frac{13}{3}\right)^3\right\}.$$

When $l_1^3<-\left(\frac{C_2C_3\varepsilon}{36l_0^4}\right)\Big/\left[\frac{1}{\sqrt{48}}\left(\frac{13}{3}\right)^{3/2}+\frac{2}{9}\right]$, DET$<0$ and (43) of [1] always has more solutions. That is, U_1 in Fig. 1 will get $\gg 1$, and in any finite section where $U>U_c$, MES solutions are available.

On the other hand, if friction is stronger, say $C_6^2\geqslant\frac{13}{3}l_1^{*2}l_0^4$ (where l_1^* is a maximum of l_1 over the region in accordance with near-resonance assumption), the DET>0 is always true so that (43) of [1] gives a single solution. Figs. 2 and 5 display the curves of (43) of [1] as the function of U for $r=0.2$ ($Q_0^*/0.7$) and $r=0$, respectively.

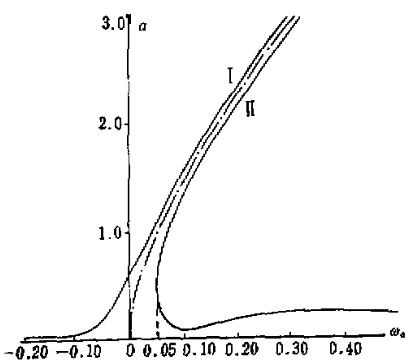

Fig. 5. Curves of MES solutions and nonlinear resonance. The parameters are as those in Fig. 2 except $r=0$, and $\omega_1=-\varepsilon l_1=1-\omega_0/l_0$.

To analyze the influence of friction on the solution stability, eigenvalue σ, at various degrees of friction, is calculated. The results are: 1) for stronger friction a single solution obtained is always stable; 2) with no friction the upper-branch (large-amplitude) solution is stable only within the regions $\omega_1\geqslant 0.2$ and $\omega_1\leqslant 0.03$ ($\omega_1=-\varepsilon l_1=1-\omega_0/l_0$) while no stable solutions occur in the bifurcation section where three solutions undergo transition, i. e., a single solution passes into multiple ones, as shown in Fig. 5.

The comparison of the results of Fig. 2 with those from the orographic forcing study by Trevisan and Buzzi (1980)[5] indicates that in the absence of friction, the likelihood of stable MES solutions is less for our baroclinic model with thermal forcing than for a barotropic model with orographic effect and that the transition among MESs is hard to happen. However, for the rational model atmosphere always with the simultaneous occurrence of thermal forcing and frictional dissipation under consideration, the above situations would not occur.

5. *Mutual Transition among MESs by Altering External Parameters*

The inter-state transition mechanism is rather complicated. A possible approach is by altering environmental parameters to bring about an abrupt transition. This idealized eruption model can be described with the aid of Fig. 1, where as U decreases, a state corresponding to the system is moved towards U_c along the path of the lower-branch solution and, as soon as U arrives at U_c, the state experiences a sudden shift at point A, going up the path of the upper. And if U "marches" from a single-solution region to a multiple-solution one, then the solution of the system will grow correspondingly, entering the upper solution section without changing its phase, and shift to the lower section on reaching point B in relation to U_1. In our subsequent work we shall provide a more complete model to simulate this sudden-transition mechanism numerically.

In addition, change in forcing intensity can also bring about such abrupt transition. Refer to Fig. 4. When $U=0.75$, (29) of [1] yields a single stable solution for $Q_0=10^{-2}$, the states become unstable for $Q_0=10^{-3/2}$ and the equation gives three solutions for $Q_0=10^{-1}$. Calculations show that only the maximum-amplitude solution is stable, which can be regarded as the single one only when $Q_0=10^{-1}$, and is the product of the abrupt transition caused by the change of Q_0. This transition model is constructed on the so-called "catastrophe" theory in mathematics, described qualitatively by the Hopf theorem. Dutton (1981)[6] and Vickory et al. (1979)[7] showed that in a thermal forced and dissipative system of the atmosphere nonlinear quasi-geostrophic Rossby waves are subject to such behavior as given above, which is also indicated by Li Maicun et al. (1983)[8] in the study on a nonlinear mechanism of the atmospheric circulation in a similar system.

III. CONCLUDING REMARKS

Based on the results shown in Part I analyses are made of the mathematical-physical structures, domains and stability of the thermal forced ES solutions. The results illustrate that over some domains of the parameter values there can exist three ES solutions, two of which have extremely large amplitude, exhibiting near-resonance but are not sensitive to the change in thermal-forcing intensity, the other, of smaller amplitude, being entirely dependent on forcing. Stability analysis shows that 1) the stability of the ES decreases versus increasing forcing; 2) with an appropriate Q_0 more stable ES can occur, as indicated by the more steady balance between the advection and nonlinearity term in (29) of [1] with weak forcing; 3) if only one of the three ES solutions is stable, then it may be either the maximum- or the minimum-amplitude one and, if two are stable, then they must be the upper- (large-) and lower-branch (small-amplitude) ones, the middle one being unstable, as demonstrated in Fig. 1; 4) in particular, when Q_0 is small, the three stable ES are possible only with somewhat different stabilities. It is noted that the upper and lower solutions can be utilized to simulate a persistent blocking (a low-index flow) and a quasi-zonal circulation with a strong west-wind component (a high-index flow) in the mid-latitude troposphere, respectively.

The study of the theoretical mechanism of MES indicates that both nonlinear effect and near-resonance external forcing are critical for the occurrence of MES while dissipation is important only in a relative sense. The nonlinearity referred to includes both the interaction between forced waves and the disturbance and self-coupling of perturbations. Only with the aid of the former can near-resonance external forcing exhibit its effect as a trigger, and only through that of the latter can the linear structure within the system be modified, thus making it possible for the occurrence of MES. Moreover, the nonlinear resonance of the forced and free waves results in an increasing but limited amplitude of ES. As far as dynamic triggering is concerned, purely thermal effect is as important as orographic forcing in the excitation of MES. Consequently, although the situation of multiple solutions seldom appears when thermal forcing is little, there does exist the possiblility for its occurrence so long as such thermal effect is available, no matter how small it might be.

Further analysis and discussion show that the change in the atmospheric environmental parameters can lead to the sudden transition among MES, which, however , generally depends on an unstable or poorly stable ES as a "bridge" for the shift. Such transition can be brought about by altering either U or Q_0. There are many types of mechanisms for transition. We shall propose another possible mechanism in dealing with the establishment of MES in another paper.

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