EQUATORIAL SOLITARY WAVES OF TROPICAL ATMOSPHERIC MOTION IN SHEAR FLOW

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ABSTRACT

Starting from the primary equations, the author derives the KdV equation which describes solitary Rossby waves in the tropical atmosphere, and indicates that, because these waves are ageostrophic, they differ from the quasigeostrophic solitary Rossby waves studied by Redekopp et al. Owing to nonlinear action, these waves are also different from traditional linear waves of the tropical atmosphere. The author believes that the stationary tropical atmospheric waves reflect the characteristics of solitary waves in that the energy does not disperse.

I. INTRODUCTION

The intrinsic quality and the evolutive regularity of tropical atmospheric waves are the most fundamental problems of tropical synoptic and dynamic meteorology. Matsuno (1966) was the first to build a theory about tropical atmospheric waves. He demonstrated the existence of mixed Rossby-gravity waves and Kelvin waves, and indicated that tropical Rossby waves and gravity waves may still be distinguished in general cases. The Rossby number of atmospheric motion in the tropics is greater than in mid- and high-latitudes, so the order of magnitude of advective acceleration is close to that of the pressure gradient and Coriolis force. Hence the nonlinear effect is of greater significance in the tropics than in mid- and high-latitudes. In the seventies the knowledge of nonlinear waves in various physical realms has made much headway, with the scattering inverse transformation method gaining great success. These results have been used to study quasigeostrophic atmospheric motion, and it is found that there exist quasigeostrophic solitary Rossby waves in the atmosphere (Weidman and Redekopp, 1979; Redekopp, 1977; Hukuda, 1979). In the tropical area, the geostrophic relation between wind and pressure fields does not hold; thus a new theory about these nonlinear waves must be developed. In this paper, starting from the primary equations, we will derive the KdV equation describing tropical atmospheric solitary Rossby waves, and discuss some fundamental features of these ageostrophic waves.

II. ASYMPTOTIC EXPANSION OF THE SET OF TROPICAL ATMOSPHERIC DYNAMIC EQUATIONS

We define the following length and time scales:

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$$L \sim (C_g/\beta)^{1/2}, \quad T \sim (C_g\beta)^{-1/2}, \tag{1}$$

where $C_g=\sqrt{gH}$, $\beta=2\Omega/a_0$, H is the scalar depth of the atmosphere, Ω the angular velocity of the earth, and a_0 the radius of the earth. We also suppose that the scalar velocity is V. The dimensionless equation of motion of barotropic divergent atmosphere in an equatorial β-plane can be written as

$$\left.\begin{aligned}&\frac{\partial u}{\partial t}+R\left(u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}\right)-yv+\frac{\partial \phi}{\partial x}=0\\&\frac{\partial v}{\partial t}+R\left(u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}\right)+yu+\frac{\partial \phi}{\partial y}=0\\&\frac{\partial \phi}{\partial t}+R\left(u\frac{\partial \phi}{\partial x}+v\frac{\partial \phi}{\partial y}\right)+\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)=0\end{aligned}\right\}, \tag{2}$$

where t is time, x and y are eastward and northward coordinates respectively, u and v are corresponding wind velocities, ϕ is geopotential height, and $R=\frac{V}{\beta L^2}$. The magnitude of the Rossby number is one order greater in the tropics than in mid- and high-latitudes, so that direct neglect of terms involving R is not suitable. We will retain them, but reduce the latter R, and neglect their high-power terms only when the problem of eigenvalues is solved in Section III.

Suppose that the stream field consists of a basic flow $U(y)$ and a time-dependent disturbance, i.e.,

$$\left.\begin{aligned}&u=U(y)+\varepsilon u'(t,x,y)\\&v=\varepsilon v'(t,x,y)\\&\phi=\phi(y)+\varepsilon\phi'(t,x,y)\end{aligned}\right\}, \tag{3}$$

where ε is a characteristic nondimensional parameter expressing amplitude of disturbance. Substitution of (3) into (2) yields

$$\left.\begin{aligned}&\varepsilon\left[\frac{\partial u'}{\partial t}+U\frac{\partial u'}{\partial x}+(U_y-y)v'+\frac{\partial \phi'}{\partial x}\right]+\varepsilon^2\left(u'\frac{\partial u'}{\partial x}+v'\frac{\partial u'}{\partial y}\right)=0\\&yU+\phi_y+\varepsilon\left(\frac{\partial v'}{\partial t}+U\frac{\partial v'}{\partial x}+yu'+\frac{\partial \phi'}{\partial y}\right)+\varepsilon^2\left(u'\frac{\partial v'}{\partial x}+v'\frac{\partial v'}{\partial y}\right)=0\\&\varepsilon\left(\frac{\partial \phi'}{\partial t}+U\frac{\partial \phi'}{\partial x}+\phi_y v'+\frac{\partial u'}{\partial x}+\frac{\partial v'}{\partial y}\right)+\varepsilon^2\left(u'\frac{\partial \phi'}{\partial x}+v'\frac{\partial \phi'}{\partial y}\right)=0\end{aligned}\right\}, \tag{4}$$

where $U_y=\frac{dU}{dy}$ and $\phi_y=\frac{d\phi}{dy}$.

Introduce the following multiple-scale variables:

$$\frac{\partial}{\partial t}=-\varepsilon^{1/2}c\frac{\partial}{\partial \xi}+\varepsilon^{3/2}\frac{\partial}{\partial \tau'}, \quad \frac{\partial}{\partial x}=\varepsilon^{1/2}\frac{\partial}{\partial \xi'}, \quad \frac{\partial}{\partial y}=\frac{\partial}{\partial y}. \tag{5}$$

Obviously in the case considered here, the meridional scale is smaller than the zonal scale, so that mixed Rossby-gravity waves do not exist (Li and Yao, 1981).

We expand u', v' and ϕ' with the small parameter ε:

$$\left.\begin{aligned}&u'=u_0+\varepsilon u_1+\varepsilon^2 u_2+\cdots\\&v'=\varepsilon^{1/2}v_0+\varepsilon^{3/2}v_1+\varepsilon^{5/2}v_2+\cdots\\&\phi'=\phi_0+\varepsilon\phi_1+\varepsilon^2\phi_2+\cdots\end{aligned}\right\}. \tag{6}$$

Upon substituting (5) and (6) into (4) and taking the lowest approximation of ε, we obtain

$$yU - \phi_y = 0, \tag{7}$$

which shows that the basic flow of low-latitude atmosphere is in geostrophic balance. As y in front of U is not constant, this balance relation is slightly different from the geostrophic relation of midlatitude.

Taking the first-order approximation, we obtain:

$$\left.\begin{aligned}
&(U-c)\frac{\partial u_0}{\partial \xi} + (U_y - y)v_0 + \frac{\partial \phi_0}{\partial \xi} = 0 \\
&yu_0 + \frac{\partial \phi_0}{\partial y} = 0 \\
&(U-c)\frac{\partial \phi_0}{\partial \xi} + \phi_y v_0 + \frac{\partial u_0}{\partial \xi} + \frac{\partial v_0}{\partial y} = 0
\end{aligned}\right\}. \tag{8}$$

From the second-order approximation we obtain

$$\left.\begin{aligned}
&(U-c)\frac{\partial u_1}{\partial \xi} + (U_y - y)v_1 + \frac{\partial \phi_1}{\partial \xi} = -\left\{\frac{\partial u_0}{\partial \tau} + u_0\frac{\partial u_0}{\partial \xi} + v_0\frac{\partial u_0}{\partial y}\right\} \\
&yu_1 + \frac{\partial \phi_1}{\partial y} = -(U-c)\frac{\partial v_0}{\partial \xi} \\
&(U-c)\frac{\partial \phi_1}{\partial \xi} + \phi y v_1 + \frac{\partial u_1}{\partial \xi} + \frac{\partial v_1}{\partial y} = -\left\{\frac{\partial \phi_0}{\partial \tau} + u_0\frac{\partial \phi_0}{\partial \xi} + v_0\frac{\partial \phi_0}{\partial y}\right\}
\end{aligned}\right\}. \tag{9}$$

From (8) and (9), nonlinear terms do not appear in the equations of first-order approximation, so the meridional structure of the disturbance is similar to that for linear waves. The nonlinear action makes waves vary slowly with time.

III. DERIVATION OF KdV EQUATION

Suppose (8) has solutions with the following form:

$$\left.\begin{aligned}
u_0 &= \psi(\tau,\xi)\hat{u}_0(y) \\
v_0 &= \psi(\tau,\xi)\hat{v}_0(y) \\
\phi_0 &= \psi(\tau,\xi)\hat{\phi}_0(y)
\end{aligned}\right\}. \tag{10}$$

Substituting (10) into (8) yields

$$\left.\begin{aligned}
&(U-c)\hat{u}_0 + (U_y - y)\hat{v}_0 + \hat{\phi}_0 = 0 \\
&y\hat{u}_0 + \frac{\partial \hat{\phi}_0}{\partial y} = 0 \\
&(U-c)\hat{\phi}_0 + \phi_y\hat{v}_0 + \hat{u}_0 + \frac{\partial \hat{v}_0}{\partial y} = 0
\end{aligned}\right\}. \tag{8'}$$

In this paper, we do not attempt to discuss the problem of critical layer, so we may assume that $U - c \neq 0$. Eliminating $\hat{\phi}_0$ and U_0 in (8′), we obtain an ordinary differential equation about $\hat{v}_0$:

$$\frac{d^2\hat{v}_0}{dy^2} + A_1(y)\frac{d\hat{v}_0}{dy} + A_2(y)\hat{v}_0 = 0, \tag{11}$$

in which $A_1(y) = -\bar{U}_y - \dfrac{2(U-c)U_y}{(U-c)^2 - 1}$ and

$$A_2(y)=\frac{1}{U-c}[Cy^2-U^2-UU_y y+CU-U_{yy}+1]$$

$$+\frac{2U_y}{(U-c)^2-1}[U_y(U-c)+(U_y-y)]. \tag{12}$$

The boundary condition is

$$\hat{v}_0\to 0, \qquad y\to\pm\infty. \tag{13}$$

Eqs. (11)—(13) determine meridional structure and phase speed of the waves. We will discuss the details of this eigenvalue problem in the next section. When the solution of (11)—(13), $\hat{v}_0$ and $\hat{\phi}_0$, is obtained and substituted into the equations of second-order approximation, we should have

$$\frac{\partial^2 v_1}{\partial y^2}+A_1(y)\frac{\partial v_1}{\partial y}+A_2(y)v_1$$

$$=\hat{v}_0\psi_{\xi\xi\xi}+\left\{\left[-y-\frac{2U_y}{(U-c)^2-1}\hat{u}_0+\frac{1}{(U-c)}\frac{\partial\hat{u}_0}{\partial y}\right]\right.$$

$$\left.-\left[\frac{U_y-y}{U-c}-\frac{2(U-c)U_y}{(U-c)^2-1}\right]\hat{\phi}_0-\hat{\phi}_{0y}\right\}\psi_\tau$$

$$+\left\{\left[-y-\frac{2U_y}{(U-c)^2-1}\right](\hat{u}_0^2+\hat{u}_{0y}\hat{v}_0)+\frac{1}{U-c}[2\hat{u}_0\hat{u}_{0y}+\hat{u}_{0yy}\hat{v}_0\right.$$

$$+\hat{u}_{0y}\hat{v}_{0y}]+\left(-\frac{u_y-y}{U-c}+\frac{2(U-c)U_y}{(U-c)^2-1}\right)(\hat{u}_0\hat{\phi}_0+\hat{v}_0\hat{\phi}_{0y})$$

$$\left.-(\hat{u}_{0y}\hat{\phi}_0+\hat{u}_0\hat{\phi}_{0y}+\hat{v}_{0y}\hat{\phi}_{0y}+\hat{v}_0\hat{\phi}_{0yy})\right\}\psi\psi_\xi, \tag{14}$$

in which the subscripts ξ or y represent derivatives with respect to ξ or y.

Since the homogeneous part of (14) is identical with (11), from solvable condition $\psi(\tau,\xi)$ must satisfy

$$e_1\psi_\tau+e_2\psi\psi_\xi+e_3\psi_{\xi\xi\xi}=0, \tag{15}$$

in which

$$\left.\begin{aligned}
e_1&=\int_{-\infty}^{\infty}\left\{-\frac{2U_y}{(U-c)^2-1}\hat{u}_0+\frac{1}{U-c}\frac{\partial\hat{u}_0}{\partial y}-\left(\frac{U_y-y}{U-c}-\frac{2(U-c)U_y}{(U-c)^2-1}\right)\hat{\phi}_0\right\}\hat{v}_0dy,\\
e_2&=\int_{-\infty}^{\infty}\left\{\left(-y-\frac{2U_y}{(U-c)^2-1}\right)(\hat{u}_0^2+\hat{u}_{0y}\hat{v}_0)+(2\hat{u}_0\hat{u}_{0y}+\hat{u}_{0yy}\hat{v}_0+\hat{u}_{0y}\hat{v}_{0y})\left(\frac{1}{U-c}\right)\right.\\
&\quad+\left(-\frac{U_y-y}{U-c}+\frac{2(U-c)U_y}{(U-c)^2-1}\right)(\hat{u}_0\hat{\phi}_0+\hat{v}_0\hat{\phi}_{0y})-(\hat{u}_{0y}\hat{\phi}_0+\hat{u}_0\hat{\phi}_{0y}+\hat{v}_{0y}\hat{\phi}_{0y}\\
&\quad\left.+\hat{v}_0\hat{\phi}_{0yy})\right\}\hat{v}_0dy,\\
e_3&=\int_{-\infty}^{\infty}\hat{v}_0^2dy.
\end{aligned}\right\} \tag{16}$$

Eq. (15) is a typical KdV equation. As KdV equations have permanent pulse-like travelling wave solutions, called solitary waves or solitons, (15) may show that there exist atmospheric solitary waves in the tropics. They form as a result of combined action of nonlinear and dispersive process while balancing each other. During moving and interacting with one another, the solitary waves behave like stable particles. On the contrary, linear waves exert effects of dispersion only, so their energy disperses all around. This fact shows the intrinsic difference between the nonlinear solitary waves in the tropical atmosphere and the linear waves.

IV. MERIDIONAL STRUCTURE OF WAVES

In previous sections we have stated that the waves discussed so far in this paper belong to long waves whose meridional structure is still a linear problem, but as shown by (9) or (16), they surely influence the nonlinear characteristics of the waves. Let us assume a basic flow with a horizontal shear as follows:

$$U(y) = U_y(y - y_c) = U_0 + U_y y. \tag{17}$$

In light of the discussion in Section I, we neglect the terms involving high powers of R, so (11) and (12) are reduced to

$$\frac{d^2\Phi_0}{dy^2} + (a_0 + b_0 y + c_0 y^2)\frac{d\Phi_0}{dy} + (a_1 + b_1 y - y^2)\Phi_0 = 0, \tag{18}$$

$$\left.\begin{aligned} &a_0 = 2U_y c/[(U-c)^2 - 1], \qquad b_0 = -U_0, \qquad c_0 = -U_y \\ &a_1 = \left(\frac{1}{U_0 - c} - U_0\right), \quad b_1 = -\left[1 + \frac{2}{(U-c)^2 - 1} + \frac{1}{(U-c)^2}\right]U_y \end{aligned}\right\}, \tag{19}$$

which are more or less similar to the result of a previous paper. Thus we may use the transformation given there in this paper, i.e.

$$W = \Phi_0 \exp\left\{\frac{c_0}{b_0}y^3 + \frac{2+b_0}{4}y^2 + \frac{a_0 + c_0 - b_0}{2}y\right\}, \tag{20}$$

$$z = \frac{1}{2}\left\{\sqrt{2}\,y - \frac{\sqrt{2}}{2}(b_1 - c_0)\right\}^2. \tag{21}$$

Substitution of (20) and (21) into (18) yields

$$z\frac{d^2w}{dz^2} - \left[\frac{1}{2} - z\right]\frac{dw}{dz} + \frac{\alpha}{2}w = 0, \tag{22}$$

$$\alpha = \frac{1}{2}\left[\frac{1}{U_0 - c} - U_0 + \frac{1}{2}U_0 - 1\right]. \tag{23}$$

Eq. (22) is a confluent hyperbolic equation with a solution as

$$W = K_1 M\left(-\frac{\alpha}{2}, \frac{1}{2}, z\right) + K_2 z^{1/2} M\left(-\frac{\alpha}{2} + \frac{1}{2}, \frac{3}{2}, z\right). \tag{24}$$

Changing (13) into a boundary condition in the limited region $y = \pm y_w$, $V_0 = 0$, we can determine the eigenvalue α from this boundary condition. The dispersion relation of first order approximation (linear) is

$$c = U_0 - \frac{1}{2\alpha + 1 + \frac{U_0}{2}}, \tag{25}$$

or

$$\omega - UK = -\frac{K}{2\alpha + 1 + \frac{U_0}{2}},$$

where K is zonal wave number. If $\alpha = -1$ and $U_0 = 0$ it also represents the Kelvin wave's frequency. The Rossby wave approximation expressed by (25) had been shown (Lighthill 1969, Lau 1982) to be equivalent to the assumption of geostrophic balance in zonal direction which is in agreement with (8) (linear case).

Since the zonal wind expressed by (8) is in geostrophic balance, the gravity waves are excluded and Yanai waves disappear. This fact may be found from (25), i.e., if we let $K=0$, the wave speed formula given by Matsuno (1966), and compare that with (25), it can be seen that they are Rossby waves. So (25) is a very fair approximation for large-scale Rossby waves and Kelvin waves. But the KdV equation derived in this paper describes nonlinear process of Rossby waves only (Fig. 1).

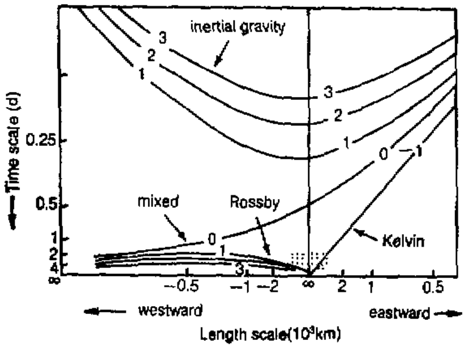

Fig. 1. The dispersion relation for equatorial waves (referred to Pedlosky, 1980).

From (24), we may further obtain

$$\left.\begin{aligned} U_0 &= \frac{1}{(U-c)^2-1}\left[-(U_y-y)(U-c)\vartheta_0 - yu\vartheta_0 + \frac{\partial\vartheta_0}{\partial y}\right] \\ \phi_0 &= \frac{1}{(U-c)^2-1}\left\{[(U-c)yU-(U_y-y)]\vartheta_0 - (U-c)\frac{\partial\vartheta_0}{\partial y}\right\} \end{aligned}\right\}. \tag{26}$$

The meridional structure of the waves is similar to that in the linear case, yet they are modulated by nonlinear effects in the zonal direction and their amplitudes vary with time. The shear of basic flow affects both the linear dispersive relation and the meridional structure of waves. Thus from (16) it inevitably affects coefficients of the KdV equation. This fact is an important aspect of the effect of the shear of basic flow on the tropical atmosphere. In this paper we cannot discuss this problem in detail. The author will discuss this problem in a special work.

V. CHARACTERISTICS OF TROPICAL SOLITARY ROSSBY WAVES

For convenience, we first transform (15) into the following normal form:

$$u_\tau - 6uu_\eta + u_{\eta\eta\eta} = 0 \tag{27}$$

$$u = e_2 e_3^{-1/3} e_1^{-2/3}\psi/6, \quad \eta = (e_1/e_3)^{1/3}\xi. \tag{28}$$

In section II, we indicate that, because of the combined actions of dispersive and nonlinear terms, the solutions of (15) may be solitary waves. This is its most distinctive feature. We call the solitary waves solution of (15) or (27) a tropical solitary Rossby wave. Its mathematical form is

$$u(\tau,\eta)=a\,\mathrm{sech}^2(k\eta-4k^3\tau+\delta), \quad a=2k^2, \tag{29}$$

k and δ being determined by distribution of the initial disturbance. Eq. (29) shows that the speed of this wave is proportional to its amplitude and inversely proportional to the square root of its width. Unlike the linear dispersive waves, these waves do not disperse energy when they advance forward. This fact implies that waves with characteristics of permanent pulse-like travelling waves can also exist in the tropical atmosphere. Furthermore, as an arbitrary initial condition can not simultaneously excite two solitary waves with the same wave speed, when there are two such solitary waves in the atmosphere, they inevitably meet because of their different wave speeds. In the interacting process, they vary only in phase, but still advance with their initial speeds after interaction. This fact unveils the great difference between the tropical atmospheric solitary waves and pure linear modes in the aspect of interaction. For example, if two waves in the initial time are represented by

$$u(x,o)=12K_i\,\mathrm{sech}^2[K_i+\delta]$$

and

$$i=1,\ 2,\quad K_1=1,\quad K_2=2,$$

then the solution with two disturbances superimposed is

$$u(x,t)=\frac{3+4\cosh[(2x-8t)+\cosh(4x-64t)^2]}{[3\cosh(x-28t)+\cosh(3x-36t)]}.$$

This result shows that the amplitude is enlarged. The superimposed waves will separate from each other after some time and the larger wave will run in front of the smaller one. So the solitary waves will automatically rearrange themselves. This interaction has been shown by numerical computation with the method of characteristic line (Li and Yao, 1982). Fig. 2 is the case of two solitary waves. When $T=0.75$ then the two waves combine with each other, and the intensity of amplitude increases. When $T=1.5$ they separate again and at this time the larger solitary wave is in front of the smaller one.

This situation can also be qualitatively represented in a two-dimensional map. Fig.3 shows the geopotential height evolution in the equatorial area for

$$U=U_y y+u_0,\quad (u_0<0).$$

It is seen from the figure that the larger soliton matches the smaller one. When they are combined with each other, at last the larger one lies in front of the smaller one. In the equatorial area we often observed that the synoptic systems combine and separate from each other in the weather maps. We think that these may be the behaviors of interaction of the solitary Rossby waves.

The results of spectral and synoptic analyses of tropical atmospheric motion show that there exist slowly moving atmospheric waves in the tropics, the frequency of which falls in the category of Rossby waves. Their lives are usually long and they can keep on advancing after crossing the whole ocean. But the linear theory about tropical atmospheric waves holds that the energy of tropical Rossby long waves propagates with a group speed different from their phase speed, i.e., waves will not remain stable when they match a long distance. Therefore, the linear theory cannot explain these waves completely. In the light of our work, the above behavior is just the characteristic of solitary Rossby waves which is formed under balance

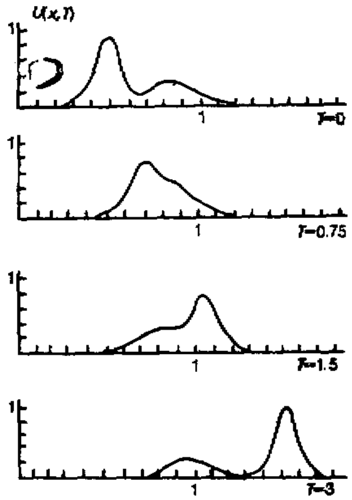

Fig. 2. Mergence and separation of two equatorial solitary waves.

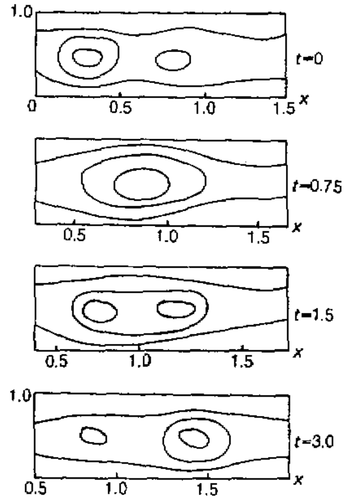

Fig. 3. The evolution of geopotential height in the equatorial area showing the mergence and separation of two solitary waves.

between nonlinear and dispersive effects. In addition, the stream field of disturbance is like a pulse, whereas troughs (or lows) alternating with ridges (or highs) are more often seen in the tropical atmosphere. Although great parts of waves are severely affected by diabatic heating, the nonlinear effect is still playing an important part in maintaining the wave shape to look like a pulse.

Solitary waves are not the unique solution of the KdV equation. Generally speaking, the stream field described by the KdV equation consists of a number of solitary waves with different speeds and an oscillating wave train. Especially when the initial stream field satisfies the inequality $U(0,\eta)\geqslant 0$, the solution involves only the oscillating wave train. Now mathematicians have an intimate understanding of non–soliton solutions of the KdV equation. Its stream field consists of waves involving troughs alternating with ridges, especially a tri-angular leading wave head with an oscillating wake. They disperse energy, so all the wave trains decay with time. It can be seen that these waves are different from the solitary waves discussed above, hence we call them nonlinear Rossby wave trains. This type of wave train is different from linear waves (i.e., the solution for the case $e_1=0$). Its leading wave decays more quickly than the tail one in contrast with linear waves. After a long time, oscillating wave tails form a number of wave groups. No matter how long the wavelength of the initial disturbance is, it may be short in wave groups. In other words, the shorter wave may be excited by long initial disturbance with a nonlinear effect. We should describe the evolution of these short waves with the nonlinear Schrodinger equation.

VI. CASE WITHOUT BASIC FLOW

Let $U(y)$ in (11)—(13) vanish; thus the corresponding eigenvalue problem is transformed into

$$\frac{\partial^2 \hat{v}_0}{\partial y^2}+\left[-\frac{1}{c}-y^2\right]\hat{v}_0=0. \tag{30}$$

Its solutions are

$$\left.\begin{aligned} \hat{v}_0 &= e^{-\frac{y^2}{2}}H_n(y) \\ \hat{u}_0 &= \frac{2n+1}{2}e^{-y^2/2}\left\{\frac{1}{2n+1}H_{n+1}(y)-H_{n-1}(y)\right\} \\ \hat{\phi}_0 &= \frac{2n+1}{2}e^{-y^2/2}\left\{\frac{1}{2n+1}H_{n+1}(y)\right\} \end{aligned}\right\}, \tag{31}$$

where $H_n(y)$ is the nth order Hermitian polynomial and

$$c=-\frac{1}{2n+1}.$$

The coefficients of the KdV equation are

$$\left.\begin{aligned} e_1 &= \int_{-\infty}^{\infty}\frac{1}{2}\hat{v}_0^2 dy \\ e_2 &= \int_{-\infty}^{\infty}\left\{\left[-y-\frac{1}{c}\frac{\partial}{\partial y}\right](\hat{u}_0^2+\hat{u}_{0y}\hat{v}_0)-\left(\frac{y}{c}+\frac{\partial}{\partial y}\right)(\hat{u}_0\hat{\phi}_0+\hat{v}_0\hat{\phi}_{0y})\right\}\hat{v}_0 dy \\ e_3 &= \int_{-\infty}^{\infty}\hat{v}_0^2 dy \end{aligned}\right\} \tag{32}$$

For quasigeostrophic motion in mid or high latitudes, solitary Rossby waves cannot exist if the shear of basic flow vanishes. In the tropical atmosphere, solitary Rossby waves may exist even if the shear of basic flow vanishes. Here we are going to show that this difference is due to the ageostrophic characteristic of the tropical atmospheric motion.

We suppose that the first order approximate stream field obeys geostrophic balance; thus the terms involving $(U-c)$ in (8) and (9) disappear and (14) becomes a conservative form of quasigeostrophic potential vorticity. It is easy to demonstrate that (32) becomes

$$\begin{aligned} e_2' &= -\frac{1}{c}\int_{-\infty}^{\infty}\left[\frac{\partial}{\partial y}(\hat{u}_0^2+\hat{u}_{0y}\hat{v}_0)+y(\hat{u}_0\hat{\phi}_0+\hat{v}_0\hat{\phi}_{0y})\right]\hat{v}_0 dy \\ &= -\frac{1}{c}\int_{-\infty}^{\infty}\{\hat{u}_0(\hat{u}_{0y}+y\hat{\phi}_0)+\hat{v}_0(\hat{u}_{0y}+y\hat{\phi}_0)y+\hat{u}_{0y}(\hat{u}_0+\hat{v}_{0y})-\hat{v}_0\hat{\phi}_0\}\hat{v}_0 dy. \end{aligned} \tag{33}$$

The first and second terms on the right represent the advection of potential vorticity. The third is a divergence term. The fourth originates from the variation of y, i.e. the variation of the Coriolis parameter. Keeping in mind that geostrophic flow is nondivergent and y should be treated as constant in geostrophic relation between wind and pressure, we may regard the last two terms as zero. Thus we have

$$c_2'=\frac{1}{c}\int_{-\infty}^{\infty}\vartheta_0^3\left(\frac{\vartheta_{0yy}}{\vartheta_0}\right)_y dy. \tag{34}$$

To the first-order approximation, the stream field derived from the linear equation is

$$\vartheta_{0yy}\propto\vartheta_0, \tag{35}$$

so that

$$e_2'=0. \tag{36}$$

To ageostrophic flow, the last equality of (33) still holds. However, divergence of stream field and advection of potential vorticity are both nonzero. In addition, e_2 must also involve the influence of ageostrophic acceleration, i.e., the terms involving $(U-c)$ in (8) and (9). Therefore even there is no basic flow with zonal shear, the effect of nonlinear advection still exists, and solitary Rossby waves may still emerge. This is exactly the difference between tropical ageostrophic solitary Rossby waves and quasigeostrophic solitary Rossby waves. This also shows that nonlinear effects are not negligible for ageostrophic flow.

VII. THE MKdV EQUATION

From (32), as a result of symmetry of the Hermitian polynomial, the coefficient of the KdV equation vanishes when n is even. This case is different from geostrophic flow discussed in previous sections. At present, either the advection of potential vorticity or the divergence of stream field still exists, but due to the symmetry of the stream field their integrative effect in the y direction is zero. In these cases (not only for even-numbered meridional waves, but also for other cases with $e_2=0$), we must introduce the following new time and length scales in order to derive the nonlinear evolution equation, i.e., let

$$\frac{\partial}{\partial t}=\varepsilon(-c)\frac{\partial}{\partial\xi}+\varepsilon^3\frac{\partial}{\partial\tau},\quad \frac{\partial}{\partial x}=\varepsilon\frac{\partial}{\partial\xi}. \tag{37}$$

Furthermore, let

$$\left.\begin{aligned}u'&=u_0+\varepsilon u_1+\varepsilon^2u_2+\cdots\\ v'&=\varepsilon v_0+\varepsilon^2v_1+\varepsilon^3v_2+\cdots\\ \phi'&=\phi_0+\varepsilon\phi_1+\varepsilon^2\phi_2+\cdots\end{aligned}\right\}. \tag{38}$$

Substituting (37) and (38) into (4), we derive the first-order approximation equations identical to (8). The second-order approximation equations are

$$\left.\begin{aligned}&(U-c)\frac{\partial u_1}{\partial\xi}+(U_y-y)v_1+\frac{\partial\phi_1}{\partial\xi}=-u_0\frac{\partial u_0}{\partial\xi}-v_0\frac{\partial u_0}{\partial y}\\ &yu_1+\frac{\partial\phi_1}{\partial y}=c\\ &(U-c)\frac{\partial\phi_1}{\partial\xi}+\phi_yv_1+\frac{\partial u_1}{\partial\xi}+\frac{\partial v_1}{\partial y}=-u_0\frac{\partial\phi_0}{\partial\xi}-v_0\frac{\partial\phi_0}{\partial y}.\end{aligned}\right\}, \tag{39}$$

The third-order approximation equations are

$$\left.\begin{aligned}&(U-c)\frac{\partial u_2}{\partial\xi}+(U_y-y)v_2+\frac{\partial\phi_2}{\partial\xi}=-\frac{\partial u_0}{\partial\tau}-u_0\frac{\partial u}{\partial\xi}-v_0\frac{\partial u_1}{\partial y}-u_1\frac{\partial u_0}{\partial\xi}-v_1\frac{\partial u_0}{\partial y}\\ &yu_2+\frac{\partial\phi_2}{\partial y}=-(U-c)\frac{\partial v_0}{\partial\xi}\\ &(U-c)\frac{\partial\phi_2}{\partial\xi}+\phi_yv_2+\frac{\partial u_2}{\partial\xi}+\frac{\partial v_2}{\partial y}=-\frac{\partial\phi_0}{\partial\tau}-u_0\frac{\partial\phi_1}{\partial\xi}-v_0\frac{\partial\phi_1}{\partial y}-v_1\frac{\partial\phi_0}{\partial y}-u_1\frac{\partial\phi_0}{\partial\xi}\end{aligned}\right\}. \tag{40}$$

Solving (8), (39), and (40), we may obtain:

$$\begin{aligned} &u_0=\phi(\tau,\xi)\hat{u}_0(y), \quad v_0=\psi_\xi(\tau,\xi)\hat{v}_0(y), \quad \phi_0=\phi(\tau,\xi)\hat{\phi}_0(y), \\ &u_1=\psi^2(\tau,\xi)\hat{u}_1(y), \quad v_1=\psi\psi_\xi(\tau,\xi)\hat{v}_1(y), \quad \phi_1=\psi^2(\tau,\xi)\hat{\phi}_1(y), \end{aligned} \tag{41}$$

and $\psi(\tau,\xi)$ satisfies

$$\tilde{e}_1\psi_\tau+\tilde{e}_2\psi^2\psi_\xi+\tilde{e}_3\psi_{\xi\xi\xi}=0, \tag{42}$$

in which

$$\begin{aligned} \tilde{e}_2=\int_{-\infty}^{\infty}\Big\{&\Big(-y-\frac{2U_y}{(U-c)^2-1}\Big)(2\hat{u}_0\hat{u}_1+\hat{v}_0\hat{u}_{1y}+\hat{v}_1\hat{u}_{0y}) \\ &-\frac{1}{U-c}(2\hat{u}_0\hat{u}_{1y}+2\hat{u}_1\hat{u}_{0y}+\hat{v}_{0y}\hat{u}_{1y}-\hat{u}_{1yy}\hat{v}_0+\hat{v}_{1y}\hat{u}_{0y}+\hat{v}_1\hat{u}_{0yy}) \\ &+\Big[-\frac{U_y-y}{U-c}+\frac{2(U-c)U_y}{(U-c)^2-1}\Big](\hat{u}_0\hat{\phi}_1+\hat{u}_1\hat{\phi}_0+\hat{v}_0\hat{\phi}_{1y}+\hat{v}_1\hat{\phi}_{0y}) \\ &-[\hat{u}_{0y}\hat{\phi}_1+\hat{u}_0\hat{\phi}_{1y}+\hat{u}_{1y}\hat{\phi}_0+\hat{u}_1\hat{\phi}_{0y}+\hat{v}_{0y}\hat{\phi}_{1y}+\hat{v}_0\hat{\phi}_{1yy}+\hat{v}_{1y}\hat{\phi}_{0y}+\hat{v}_1\hat{\phi}_{0yy}]\Big\}\hat{v}_0\,dy. \end{aligned} \tag{43}$$

Eq. (42) is a well-known modified KdV equation (MKdV equation). Its permanent travelling wave solution has the following form:

$$\psi(\tau,\ \xi)=\pm\operatorname{sech}\{|\tilde{e}_2/6\tilde{e}_3|^{1/2}(\xi-c_m\tau)\}, \tag{44}$$

$$c_m=\frac{1}{6}\operatorname{sgn}(\tilde{e}_3)\tilde{e}_2.$$

Its nonpermanent wave solution also corresponds to a nonlinear wave train, but the ratio of its nonlinear term to its dispersive term is not variable over time. In this respect the nonlinear wave is described by an MKdV equation. No matter how small is the initial amplitude (and the nonlinear term) of the wave described by the KdV equation, the relative effect of the nonlinear term can match the dispersion term after all, which implies great differences between this kind of wave and linear waves.

VIII. CONCLUSION

A kind of nonlinear ageostrophic solitary Rossby wave may exist in the tropical atmosphere. These waves are governed by the KdV equation or the MKdV equation. They are different from either geostrophic solitary Rossby waves in mid- and high-latitudes or linear waves in the tropics. They are generally affected by the shear of basic flow, but can also emerge without basic flow. We think that some stable slowly-moving atmospheric waves observed in the tropics belong to this kind of nonlinear solitary wave according to their characteristic of energy dispersion. Distinguishing these waves produced under the combined effects of nonlinear and dispersion processes in synoptic and statistical analyses is undoubtedly of great significance for understanding the intrinsic quality of evolution of tropical weather and holding its regularity.

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