ON THE PARAMETERIZATION OF THE VERTICAL VELOCITY AT THE TOP OF PLANETARY BOUNDARY LAYER

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ABSTRACT

In this paper, an equation of the vertical velocity at the top of PBL is derived by use of a PBL model which is based on an analytic and actual form of K. Results show that the vertical velocity is a function of geostrophic vorticity, geostrophic wind speed, Coriolis parameter and the roughness of the ground, thus improving Charney-Eliassen's formula. The order of magnitude of the vertical velocity computed from our equation is in agreement with that from the latter, but more factors affecting the vertical velocity are included.

I. INTRODUCTION

The vertical velocity at the top of PBL (planetary boundary layer) is an important parameter in large-scale and mesoscale meteorology. The so-called Ekman pumping process has often been taken into account in many numerical models in which the vertical velocity at the top of PBL must be known. So far, the widely used Charney–Eliassen formula (Charney and Eliassen, 1949) based on the Ekman model has the form

$$w=\sqrt{\frac{K}{2f}}\zeta_g\ , \tag{1}$$

where K is the constant eddy transfer coefficient, f the Coriolis parameter and ζ_g the geostrophic vorticity. According to the concept of modern micrometeorology, the Ekman solution is not accurate enough because of the constant K, especially in the lower boundary layer where the wind speeds in solutions of the Ekman model are much less than the actual winds. Hence the vertical velocity in Eq. (1), derived from the divergence of the Ekman velocity field, is not accurate enough, either. Furthermore, there has been lack of objective methods to determine the constant K in Eq. (1), which is, in turn, often determined by experience. Physically, however, K should depend on the wind speed and the vertical shear, or the profile of the wind, according to turbulence theory.

From the viewpoint of the similarity theory of PBL, the wind velocity in the neutral PBL should be a function of geostrophic wind speed G, the roughness of ground z_0 and the Coriolis parameter f, besides the height z. Obviously, the vertical velocity at the top of PBL should also be a function of G, f and z_0. However, Eq. (1) can not completely reflect the roles of these three parameters in determining the vertical velocity, i.e., the factors considered in C-E formula are not perfect, thus decreasing its accuracy.

In modern parameterization models of boundary layer (Bhumralkar, 1975; Tennekes, 1973), the boundary layer Rossby number $\mathrm{Ro}_b=\frac{G}{fz_0}$ (in order to distinguish it from the large-scale Rossby number Ro, here $\frac{G}{fz_0}$ is called as boundary layer Rossby number Ro_b instead of Rossby number Ro referred to in many literatures on boundary layer meteorology) has been used to determine the turbulent fluxes in the PBL, i.e., G, f and z_0 are used as independent variables in parameterization. The wind velocities in the neutral PBL are also determined by these parameters and height. G, f are the model variables in large-scale models. z_0 is a micrometeorological parameter, but its values are usually estimated from the characteristics of the ground in the modern models of PBL parameterization. Therefore, z_0 can be used in large-scale models. For example, Dubov (1977) took $z_0=1$ m for forest, 5 cm for plain, 4 cm for desert, etc. It is expected that w at the top of PBL can also be parameterized by G, f and z_0 in order that it may be computed from the model variables in large-scale models.

In a large-scale model, it is not convenient to apply the complicated modern boundary-layer numerical model to compute the precise wind distribution and w in a given boundary layer. The purpose of this paper is to find out the parameterized equation of w from a PBL model, in which an actual and analytic K profile is used. In this parameterized equation, the influences of G, f and z_0 can be shown and applied to large- or meso-scale models. The main method here is the same one as used by Charney and Eliassen, by which reasonable results can be obtained though they are not very strict theoretically.

II. THE PBL MODEL

The solution of PBL equation depends on the form of K. Based on the turbulence theory, Blackadar (1962) derived an expression for K, from which the solutions obtained were in agreement with observations. However, by means of his K, the PBL equations can only be solved numerically. Nieuwstadt (1983) found that the following dimensionless form of K closely fitted in with the neutral K distribution derived from Wyngaard's second-order closure model (Wyngaard et al., 1974) and approached the real distribution of K, namely,

$$K=c\eta(1-\eta)^2, \tag{2}$$

where $K=\frac{K^*}{u_*^* h^*}$ (hereinafter the dimensional variables are denoted by the superscript asterisk) is the dimensionless eddy transfer coefficient; η the dimensionless height $\frac{z^*}{h^*}$; h^* the height of boundary layer; u_*^* the friction velocity; c a constant and approximately equal to 0.2. In this paper, we shall apply K in Eq. (2) to finding out the wind profile in PBL and, finally, to givng the vertical velocity.

If all the velocities are scaled by u_*^*, and K^* and z^* are replaced by their dimensionless values, then the dimensionless motion equations can be written as:

$$\frac{d}{d\eta}K\frac{du}{d\eta}+\frac{fh^*}{u_*^*}(v-v_g)=0, \tag{3}$$

$$\frac{d}{d\eta}K\frac{dv}{d\eta}-\frac{fh^*}{u_*^*}(u-u_g)=0, \tag{4}$$

where u and v are dimensionless wind components; u_g and v_g, dimensionless geostrophic wind components. By defining the complex velocities $W=u+iv$ and $W_g=u_g+iv_g$, the motion equations (3) and (4) become

$$\frac{d}{d\eta}K\frac{dW}{d\eta}-i\frac{fh^*W}{u_*^*}=-i\frac{fh^*}{u_*^*}W_g \quad . \tag{5}$$

The upper boundary ($\eta=1$) condition is

$$W=W_g, \tag{6}$$

and the lower boundary ($\eta=\eta_0$, $\eta_0=\frac{z_0^*}{h^*}$ is dimensionless roughness) condition is

$$W=0 \quad . \tag{7}$$

For the lower boundary condition (7), $\eta=\eta_0$ instead of $\eta=0$ is chosen due to the fact that the solution of Eq. (5) is divergent at $\eta=0$ when the K in Eq. (2) is used. In fact, from Eq. (2), when η is small, K is proportional to η. In this case, it is well known in micrometeorology that wind velocity should be a linear function of $\ln\eta$; in other words, wind can not be defined with $\eta=0$, and the wind speed should vanish at η_0. Consequently, our Eq. (5) has no solution at $\eta=0$. Then, this model implicitly includes the existence of roughness which is naturally introduced in the model. For the upper boundary condition (6), taking $\eta=1$ to be the upper boundary is closer to the reality than taking $\eta\to\infty$ as in the Ekman model and the former is more convenient to be used in large-scale models.

We define the frictionally induced velocity $W^B=W-W_g$, then the equation for W^B becomes

$$\frac{d}{d\eta}K\frac{dW^B}{d\eta}-i\frac{fh^*W^B}{u_*^*}=0 \quad . \tag{8}$$

The boundary conditions are:

$$W^B=0 \qquad \text{where} \qquad \eta=1, \tag{9}$$

and

$$W^B=-W_g \qquad \text{where} \qquad \eta=\eta_0. \tag{10}$$

Eq. (8) may be written as

$$\frac{d^2W^B}{d\eta^2}+\frac{1-3\eta}{\eta(1-\eta)}\frac{dW^B}{d\eta}-i\frac{fh^*W^B}{c\eta(1-\eta)^2u_*^*}=0 \quad . \tag{11}$$

Eq. (11) is the Fuchs-type equation with regular singularities 0, 1 and ∞ which can be solved in series form by standard method. The solution of Eq. (11) satisfying condition (9) is

$$W^B=C(1-\eta)^{\alpha-1}F(\alpha+1,\alpha-1;2\alpha;1-\eta), \tag{12}$$

where C is a constant, $F(a,b;c;x)$ the hypergeometric series, $\alpha=\frac{1}{2}+\frac{1}{2}\sqrt{1+4iQ}$ and $Q=\frac{fh^*}{cu_*^*}$. From condition (10), we have

$$C=\frac{-W_g}{(1-\eta_0)^{\alpha-1}F(\alpha+1,\alpha-1;2\alpha;1-\eta_0)} \quad , \tag{13}$$

and thus we get the solution of Eq. (5)

$$W=W_g\left[1-\frac{(1-\eta)^{\alpha-1}F(\alpha+1,\alpha-1;2\alpha;1-\eta)}{(1-\eta_0)^{\alpha-1}F(\alpha+1,\alpha-1;2\alpha;1-\eta_0)}\right] \quad . \tag{14}$$

Eq. (14) is the horizontal wind distribution that we need to find out the vertical velocity, and can be applied in the region $\eta > \eta_0$. But no solution can be found at $\eta=0$ because the hypergeometric series does not converge there (for $c=a+b$, $F(a,b;c;x)$ does not converge at $x=1$).

III. THE VERTICAL VELOCITY

The wind distribution in Eq. (14) is used to find out the vertical velocity. In general, we should utilize the solution of the boundary layer equation under the horizontally inhomogeneous condition to compute the vertical velocity just as Wu (1984) did. However, on account of the complexity of K form in this paper, we shall simply use Eq. (14), i.e., the solution of the motion equation without the horizontal advection terms, to compute the vertical velocity as Charney and Eliassen (1949) did.

For a large-scale motion, the horizontal velocity should be nondimensionalized by the velocity scale U in such motion. If the horizontal length scale in a large-scale motion is L, the vertical velocity scale in the boundary layer is Uh^*/L according to Pedlosky (1979). Consequently, the dimensionless contiunity equation has the form

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial \eta}=0\,. \tag{15}$$

Integrating Eq. (15) yields the vertical velocity at the top of boundary layer as follows

$$w=-\int_{\eta_0}^{1}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)d\eta\,. \tag{16}$$

In Eq. (14), the dimensional velocities are nondimensionalized by the friction velocity u_*^*. If all the dimensionless velocities in Eq. (14) are changed to be nondimensionalized by U, then Eq. (14) will not be altered. Assume all the velocities in Eq. (14) to be nondimensionalized by U, and substitute Eq. (14) into Eq. (16). Then the vertical velocity

$$w=-\int_{\eta_0}^{1}\left(\frac{\partial}{\partial x}\mathrm{Re}W+\frac{\partial}{\partial y}\mathrm{Im}W\right)d\eta=-\frac{\partial}{\partial x}\mathrm{Re}\int_{\eta_0}^{1}Wd\eta-\frac{\partial}{\partial y}\mathrm{Im}\int_{\eta_0}^{1}Wd\eta$$

$$=-\mathrm{Re}\frac{\partial}{\partial x}\int_{\eta_0}^{1}Wd\eta-\mathrm{Im}\frac{\partial}{\partial y}\int_{\eta_0}^{1}Wd\eta \tag{17}$$

can be yielded. Substituting Eq. (14) into (17) and using the formula

$$\int(1-\eta)^{a-1}F(a+1,a-1;2a;1-\eta)d\eta=-\frac{1}{a}(1-\eta)^{a}F(a,a-1;2a;1-\eta) \tag{18}$$

to calculate the integral in Eq. (17), we obtain

$$\int_{\eta_0}^{1}(1-\eta)^{a-1}F(a+1,a-1;2a;1-\eta)d\eta\doteq\frac{1}{a}F(a,a-1;2a;1-\eta_0)\,. \tag{19}$$

Usually $\eta_0<10^{-3}$; thus $1-\eta_0\to1$. From the properties (Abramowitz and Stegun, 1965) of the hypergeometric series $F(a,a-1;2a;1-\eta_0)$ in which $c=a+b+1$ for $F(a,b;c;x)$, it is accurate enough to retain the first term in the expansion. Consequently,

$$\int_{\eta_0}^{1}(1-\eta)^{a-1}F(a+1,a-1;2a;1-\eta)d\eta\doteq\frac{1}{a}\,\frac{\Gamma(2a)}{\Gamma(a)\Gamma(a+1)}=\frac{\Gamma(2a)}{a^2\Gamma^2(a)}\,, \tag{20}$$

where $\Gamma(x)$ is the Gamma function. The value of the series in the denominator of Eq. (14) can be calculated by the following approximate equation (Whittaker and Watson, 1927)

$$F(a,b;c;x)=\frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}\ln\frac{1}{1-x},\text{where } c=a+b \text{ and } x\to 1\ . \tag{21}$$

Then Eq. (17) turns into

$$\begin{aligned}
w&=\left(-\mathrm{Re}\frac{\partial}{\partial x}-\mathrm{Im}\frac{\partial}{\partial y}\right)\left[W_g\left(1-\frac{\dfrac{\Gamma(2a)}{a^2\Gamma^2(a)}}{\dfrac{\Gamma(2a)}{\Gamma(a+1)\Gamma(a-1)}\ln\dfrac{1}{\eta_0}}\right)\right]\\
&=\left(-\mathrm{Re}\frac{\partial}{\partial x}-\mathrm{Im}\frac{\partial}{\partial y}\right)\left[W_g\left(1+\frac{\Gamma(a-1)}{a\Gamma(a)\ln\eta_0}\right)\right]\\
&=\left(-\mathrm{Re}\frac{\partial}{\partial x}-\mathrm{Im}\frac{\partial}{\partial y}\right)\left[W_g\left(1+\frac{1}{a(a-1)\ln\eta_0}\right)\right]\\
&=\left(-\mathrm{Re}\frac{\partial}{\partial x}-\mathrm{Im}\frac{\partial}{\partial y}\right)(W_g\varphi),
\end{aligned}\tag{22}$$

where

$$\varphi=1-i\frac{1}{Q\ln\eta_0}=1-i\frac{cu_*^*}{fh^*\ln\eta_0}. \tag{23}$$

In Eq. (15), the vertical velocity is scaled by $\frac{h^*U}{L}$, i.e., h^* has been treated as a constant. Taking the boundary layer height h^* to be a constant is a method usually used in large-scale models (Bhumralkar, 1976). Therefore h^* in Eq. (23) deos not change with the horizontal coordinate. We assume that W_g in Eq. (22) depends on the horizontal coordinate, and that u_*^* is independent of x and y when $W_g\varphi$ is differentiated with respect to x and y. This treatment is similar to Charney-Eliassen's work, in which W_g depends on x and y but K is independent of the horizontal coordinate. It is, of course, not theoretically strict and introduces some approximations into the results. Thus Eq. (22) becomes

$$\begin{aligned}
w&=-\frac{\partial u_g}{\partial x}\mathrm{Re}\varphi+\frac{\partial v_g}{\partial x}\mathrm{Im}\varphi-\frac{\partial u_g}{\partial y}\mathrm{Im}\varphi-\frac{\partial v_g}{\partial y}\mathrm{Re}\varphi\\
&=\zeta_g\mathrm{Im}\varphi=-\zeta_g\frac{cu_*^*}{fh^*\ln\eta_0}.
\end{aligned}\tag{24}$$

Transforming Eq. (24) into a dimensional form and omitting the asterisk superscripts, we obtain

$$w=\zeta_g\frac{cu_*}{f\ln\dfrac{h}{z_0}} \tag{25}$$

Because u_* in Eq. (25) is not the variable in large-scale models, we try to express it in terms of the latter. In Eq. (25), there is an identity $cu_*=c\frac{u_*}{G}G$. Noting that $c=0.2$ is only an estimated value, thus we can also substitute the estimated value of u_*/G into Eq. (25) to calculate w. The quantity u_*/G has been investigated in detail in the modern PBL parameterization technique. Here we use its typical value as its estimated value. In a large-scale motion, $Ro_b=10^6$ is a value typical of Ro_b. In this case, u_*/G may take a value 0.036 (Tenneks, 1973), and then Eq. (25) may be written as

$$w=\frac{0.007G}{f\ln\frac{h}{z_0}}\zeta_g\,. \tag{26}$$

Formula (26) is our final result, showing that w is proportional to geostrophic vorticity, in agreement with Eq. (1). Eq. (26) also shows the effects of G, f and z_0 on the vertical velocity: the larger the G and z_0 are taken, the larger the w will be, which is reasonable. The effect of K on the w is expressed from the G in the numerator. Eq. (26) can be used in large-scale models to parameterize the vertical velocity at the top of the PBL.

As an application, we calculate w in the following geopotential field in a cyclone (Wu and Blumen, 1982):

$$\phi=-\left(1-\frac{1}{4}r^2\right)e^{-\frac{1}{4}r^2}, \tag{27}$$

where ϕ is the dimensionless geopotentialdeviation, r the dimensionless radius, and the horizontal length scale $L=10^3$ km. Assuming $Ro=0.3$, $z_0=10$ cm, $h=1$ km, then $U=30$ m/s if $f=10^{-4}\text{s}^{-1}$ is taken. From Eq. (26), $w=0.46$ cm/s at $r=1$. If we take $K=5$ m^2/s[1], the C–E formula (1) gives $w=0.39$ cm/s. In this example, our result is slightly larger than C–E's value. If we choose $Z_0=1$ cm, then $w=0.37$ cm/s is obtained, i. e., it is slightly smaller than C–E's value. In general, the orders of magnitude are the same as the latter. The common shortcoming of C–E's work and our work is that the wind profiles are calculated by menas of horizontally homogeneous PBL models, but the vertical velocities are calculated in the case of horizontally inhomogeneous conditions. Both have some approximations. However, our result can give the effects of G and z_0 and include more governing factors, so that it is an improvement to C–E's formula.

IV. CONCLUDING REMARKS

In this paper, the parameterized expression of vertical velocity at the top of PBL has been derived on the basis of the PBL model in which the actual height-dependent eddy coefficient K is adopted. It expresses w as a function of ζ_g, G, f, z_0 and h and can be used in large-scale models. The expression derived above is able to compute the vertical velocity under various cases of the geostrophic wind, geostrophic vorticity, Coriolis parameter and roughness. The order of magnitude for our computations is in agreement with that of Charney–Eliassen's classical formula. Although our result improves the classical result of Charney and Eliassen, it still has some approximate treatments, and some constant values are not very accurate. The calculation of accurate vertical velocity should adopt the modern

1) According to Holton's work (1979), $h=\frac{\pi}{\sqrt{\frac{2K}{f}}}$ in Ekman model, then $K=5$ m^2/s if $h=1$ km and $f=10^{-4}$ s^{-1} are taken.

numerical model of PBL. It is, however, too complicated to be used in large-scale models. Therefore our result has its practical significance.

The other shortcoming in this paper is that the K expression only in neutral condition is used. It is necessary to consider the effects of thermal stability, and to find out the K expressions for different kinds of stability in order that accurate PBL model can be built. At the same time, the effects of topography should also be taken into account. All of these need to be further investigated.

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