ON GRAVITY WAVE–MEAN FLOW INTERACTIONS IN A THREE DIMENSIONAL STRATIFIED ATMOSPHERE

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ABSTRACT

By using multiple scale analysis method, two coupled models of gravity wave-mean flow interactions in a three-dimensional stratified atmosphere are derived. One model can be applied to the inertio-gravity wave-mean flow interactions in a global scale. The other one treats the meso-scale gravity wave forcing in a local region.

I. INTRODUCTION

Gravity wave-mean flow interactions play a crucial role in the middle atmospheric circulations. Generally speaking, such interactions involve two different kinds of processes. One is the so called "dissipation and damping" processes mainly caused by eddy diffusion and infrared radiative cooling. The other one is the wave transience caused by inhomogeneity of mean states.

In order to study such interactions we usually need two sets of coupled equations. One is for the propagation of the waves as influenced by the mean flow, and the other is for the mean flow evolution caused by the waves. Coy (1983) derived these coupled equations for a two dimensional (x-z plane) model of the middle atmosphere in which the earth's rotation effects were neglected.

An elegant form of representing linear wave dynamics is called wave action equation, which yields a strong insight into how the waves are dynamically constrained. A very general form of the wave action equation has been derived by many authors in different cases (e. g. Bretherton and Garrett, 1968; Hayes, 1970; Andrews and McIntyre, 1978). However, their derivations, based on the variational method or the generalized Lagrange-mean, are excessively abstract for most purposes. Furthermore, most of these derivations assume that the x-direction averagings are taken along the whole latitude.

In this paper we give an explicit derivation of the inertio-gravity wave-mean flow interaction relations by the two-scale method for a special three dimensional, hydrostatic, quasilinear model, which is suitable to the middle atmospheric wave-mean flow interaction dynamics. We take the x-direction averaging arbitrarily so that the result can be applied to local wave forcing problems. This work may be considered as a counterpart of the works by Straus (1983) and Chen and Chao (1983), where wave action conservation laws for Rossby waves were developed explicitly and comprehensively by the two-scale method.

II. EQUATIONS FOR WAVE PROPAGATION

It is reasonable to assume that gravity waves in middle atmosphere satisfy the hydrostatic approximation; however, they are influenced by vertical stratification and rotation. We thus propose the following governing equations for waves on an f-plane (Holton, 1975)

$$\frac{\partial u'}{\partial t}+\bar{u}\frac{\partial u'}{\partial x}+\bar{v}\frac{\partial u'}{\partial y}+u'\frac{\partial \bar{u}}{\partial x}+v'\frac{\partial \bar{u}}{\partial y}+w'\frac{\partial \bar{u}}{\partial z}-fv'+\frac{\partial \phi'}{\partial x}=d'_x, \tag{1}$$

$$\frac{\partial v'}{\partial t}+\bar{u}\frac{\partial v'}{\partial x}+\bar{v}\frac{\partial v'}{\partial y}+u'\frac{\partial \bar{v}}{\partial x}+v'\frac{\partial \bar{v}}{\partial y}+w'\frac{\partial \bar{v}}{\partial z}+fu'+\frac{\partial \phi'}{\partial y}=d'_y, \tag{2}$$

$$\frac{\partial \phi'_z}{\partial t}+\bar{u}\frac{\partial \phi'_z}{\partial x}+\bar{v}\frac{\partial \phi'_z}{\partial y}+f\frac{\partial \bar{v}}{\partial z}u'-f\frac{\partial \bar{u}}{\partial z}v'+N^2w'=q', \tag{3}$$

$$\frac{\partial u'}{\partial x}+\frac{\partial v'}{\partial y}+\frac{1}{\rho_s}\frac{\partial}{\partial z}(\rho_s w')=0, \tag{4}$$

where d'_x, d'_y and q' represent dissipation and damping terms, N^2 is the squared buoyancy frequency

$$N^2=\frac{R}{H}\left(\frac{dT_0}{dz}+\frac{\kappa T_0}{H}+\frac{\partial \bar{T}}{\partial z}\right), \tag{5}$$

and ρ_s is the mean density of the atmosphere

$$\rho_s=\rho_{0s}\exp\left(-\frac{z-z_0}{H}\right). \tag{6}$$

The $\overline{(\ )}$ denotes here the Eulerian average over a distance L along the x-direction

$$\bar{\psi}(x,y,z,t)=\frac{1}{L}\int_{x-\frac{L}{2}}^{x+\frac{L}{2}}\psi(x,y,z,t)\,dx, \tag{7}$$

where ψ stands for any dependent quantity. Sometimes it is hard to give a value (or even if a range) of L. However, identifying "perturbations" ($\psi'=\psi-\bar{\psi}$) from "mean" $\overline{(\ )}$ is basically required for the purpose of studying wave-mean flow interactions. We will especially consider two kinds of averaging— local and global in the following sections. The former one refers to $L\sim 1000$ km and the later one is the averaging over the whole latitude circle. Note that for the later case all the mean quantities are independent on x, i.e.,

$$\frac{\partial \bar{\psi}}{\partial x}=0. \tag{8}$$

In the equations (1)—(4) we have assumed that $\bar{u}(x,y,z,t)$ and $\bar{v}(x,y,z,t)$ are in quasi-geostrophic balance and satisfy thermal wind balance. In the real atmosphere it is a very good approximation to assume that the static stability, which measures vertical gradient of mean temperature field, is only a function of z

$$N^2=N^2(z). \tag{9}$$

Since the ageostrophic components are usually small for large-scale flows, we can let mean flow satisfy quasi-geostrophic relations in the wave equations. And the adjustment process for hydrostatic equilibrium is much more efficient than geostrophic adjustment, which adjusts the mean state to thermal wind balance (Yeh and Li, 1965; Blumen, 1972). Therefore the above assumptions are not contradictory.

1. *Wave Kinematics*

Following the standard procedure of WKB approximation (e.g. Zeng, 1981), we introduce the slow variables

$$X=\epsilon x, Y=\epsilon y,\ Z=\epsilon z,\ T=\epsilon t, \tag{10}$$

where ϵ is a small parameter. We assume that the dissipation and damping are small and the mean states are slowly varying functions of space and time:

$$|d_x'|/\left|\frac{\partial u'}{\partial t}\right|\leq\epsilon,\ |d_y'|/\left|\frac{\partial v'}{\partial t}\right|\leq\epsilon, |q'|/\left|\frac{\partial \phi_z'}{\partial t}\right|\leq\epsilon, \tag{11a,b,c}$$

$$\left|\frac{\partial \psi}{\partial x_i}\right|/\left|\frac{\partial \psi'}{\partial x_i}\right|\leq\epsilon, \tag{12}$$

where ψ and x_i denote any dependent and independent variables respectively. From (12) we can denote

$$\bar{u}(x,y,z,t)=U(X,Y,Z,T), \bar{v}(x,y,z,t)=V(X,Y,Z,T), N^2(z)=N^2(Z). \tag{13a,b,c}$$

We assume the following forms of wave packet solution:

$$u'=u(X,Y,Z,T)\exp\left[(i\theta/\epsilon)+\frac{z-z_0}{2H}\right], \tag{14a}$$

$$v'=v(X,Y,Z,T)\exp\left[(i\theta/\epsilon)+\frac{z-z_0}{2H}\right], \tag{14b}$$

$$w'=w(X,Y,Z,T)\exp\left[(i\theta/\epsilon)+\frac{z-z_0}{2H}\right], \tag{14c}$$

$$\phi'=\phi(X,Y,Z,T)\exp\left[(i\theta/\epsilon)+\frac{z-z_0}{2H}\right], \tag{14d}$$

where θ is slowly varying phase $\theta=\theta(X,Y,Z,T)$. In the special case of averaging over the whole latitude circle we consider a single zonal harmonic wave modulated in t, y and z i.e., replace $\theta(X,Y,Z,T)$ by $kX+\theta(Y,Z,T)$ with the constant zonal wave number k.

With the assumed solution form of (14), slowly varying wave numbers and frequency are defined as

$$\frac{\partial}{\partial x}\left(\frac{\theta}{\epsilon}\right)=\frac{\partial\theta}{\partial X}=k(X,Y,Z,T),\ \frac{\partial}{\partial y}\left(\frac{\theta}{\epsilon}\right)=\frac{\partial\theta}{\partial Y}=l(X,Y,Z,T), \tag{15a,b}$$

$$\frac{\partial}{\partial z}\left(\frac{\theta}{\epsilon}\right)=\frac{\partial\theta}{\partial Z}=m(X,Y,Z,T), \frac{\partial}{\partial t}\left(\frac{\theta}{\epsilon}\right)=\frac{\partial\theta}{\partial T}=-\omega(X,Y,Z,T). \tag{15c,d}$$

They are related by the following identities which can be obtained by exchanging the orders of the derivatives:

$$\frac{\partial k}{\partial T}=-\frac{\partial \omega}{\partial X},\ \frac{\partial l}{\partial T}=-\frac{\partial \omega}{\partial Y},\ \frac{\partial m}{\partial T}=-\frac{\partial \omega}{\partial Z}, \tag{16a,b,c}$$

$$\frac{\partial k}{\partial Y}=\frac{\partial l}{\partial X},\ \frac{\partial l}{\partial Z}=\frac{\partial m}{\partial Y},\ \frac{\partial m}{\partial X}=\frac{\partial k}{\partial Z}. \tag{16d,e,f}$$

Substituting (14) into (1)—(4) and using (15), we have equations corrected to the zeroth order in:

$$\boldsymbol{M}(u,v,w,\phi)^{t}=0, \tag{17}$$

where

$$\boldsymbol{M}=\begin{pmatrix} -i\hat{\omega} & -f & 0 & ik \\ f & -i\hat{\omega} & 0 & il \\ 0 & 0 & N^2 & -i\hat{\omega}\left(im+\frac{1}{2H}\right) \\ ik & il & im-\frac{1}{2H} & 0 \end{pmatrix}, \tag{18}$$

in which

$$\hat{\omega}=\omega-kU-lV$$

is Doppler-shifted frequency. The dispersion relation is determined by det $\boldsymbol{M}=0$, i.e.,

$$\omega=W(X,Y,Z,T;k,l,m)=kU+lV\pm\left[f^2+\frac{K^2N^2}{m^2+\frac{1}{4H^2}}\right]^{1/2}, \tag{19}$$

where $K^2=k^2+l^2$. From (19) we can obtain the group velocity of the wave packet

$$\boldsymbol{c}_g=(c_{gX},c_{gY},c_{gZ})=\left(\frac{\partial W}{\partial k},\ \frac{\partial W}{\partial l},\ \frac{\partial W}{\partial m}\right). \tag{20}$$

It is known that the wave packet propagates along the path determined by local group velocity, i.e.,

$$\frac{dX}{dT}=U+\frac{kN^2}{\hat{\omega}\left(m^2+\frac{1}{4H^2}\right)}\quad(=c_{gX}), \tag{21a}$$

$$\frac{dY}{dT}=V+\frac{lN^2}{\hat{\omega}\left(m^2+\frac{1}{4H^2}\right)}\quad(=c_{gY}), \tag{21b}$$

$$\frac{dZ}{dT}=-\frac{mK^2N^2}{\hat{\omega}\left(m^2+\frac{1}{4H^2}\right)^2}\quad(=c_{gZ}). \tag{21c}$$

Solving (17) gives polarization relations for u, v, w and ϕ

$$u=(k\hat{\omega}+ilf)\phi/(\hat{\omega}^2-f^2), \tag{22a}$$

$$v=(-ikf+l\hat{\omega})\phi/(\hat{\omega}^2-f^2), \tag{22b}$$

$$w=i\hat{\omega}\left(im+\frac{1}{2H}\right)\phi/N^2. \tag{22c}$$

Note that ω, k, l and m here are all the functions of X, Y, Z and T. Their variations with space and time are determined by the wave kinematics which describes these quantities changing along the rays. Using (20) and (16) we have

$$\frac{D_g k}{DT} \equiv \frac{\partial k}{\partial T} + c_{gx}\frac{\partial k}{\partial X} + c_{gy}\frac{\partial k}{\partial Y} + c_{gz}\frac{\partial k}{\partial Z}$$

$$= -\frac{\partial \omega}{\partial X} + \frac{\partial W}{\partial k}\frac{\partial k}{\partial X} + \frac{\partial W}{\partial l}\frac{\partial l}{\partial X} + \frac{\partial W}{\partial m}\frac{\partial m}{\partial X}, \tag{23}$$

but dispersion relation (19) gives, by chain rule,

$$\frac{\partial \omega}{\partial X} = \frac{\partial W}{\partial X} + \frac{\partial W}{\partial k}\frac{\partial k}{\partial X} + \frac{\partial W}{\partial l}\frac{\partial l}{\partial X} + \frac{\partial W}{\partial m}\frac{\partial m}{\partial X}. \tag{24}$$

Combining (23) with (24) gives ray tracing equation of wave number k

$$\frac{D_g k}{DT} = -\frac{\partial W}{\partial X} = -k\frac{\partial U}{\partial X} - l\frac{\partial V}{\partial X}. \tag{25a}$$

Ray tracing equations for wave number l, m and frequency ω can be obtained in similar ways:

$$\frac{D_g l}{DT} = -\frac{\partial W}{\partial Y} = -k\frac{\partial U}{\partial Y} - l\frac{\partial V}{\partial Y}, \tag{25b}$$

$$\frac{D_g m}{DT} = -\frac{\partial W}{\partial Z}, \tag{25c}$$

$$\frac{D_g \omega}{DT} = \frac{\partial W}{\partial T} = k\frac{\partial U}{\partial T} + l\frac{\partial V}{\partial T}. \tag{25d}$$

Now we have obtained the evolution equations for k, l, m and ω (25). We have also obtained the trajectory equations of the wave packet (21). In order to complete the problem we need evolution equations of the wave amplitudes u, v, w and ϕ.

2. *Wave Action Equation*

From (1)—(4) and (14) we can get the wave energy equation

$$\frac{\partial E'}{\partial t} + \bar{u}\frac{\partial E'}{\partial x} + \bar{v}\frac{\partial E'}{\partial y} + \boldsymbol{F}'_u \cdot \nabla' \bar{u} + \boldsymbol{F}'_v \cdot \nabla' \bar{v} + \nabla' \cdot \boldsymbol{F}'_\phi$$

$$-\frac{f}{N^2}\mathrm{Re}\{\overline{\rho_0 v' \phi'_z}\}\frac{\partial \bar{u}}{\partial z} + \frac{f}{N^2}\mathrm{Re}\{\overline{\rho_0 u' \phi'_z}\}\frac{\partial \bar{v}}{\partial z} = d', \tag{26}$$

where

$$E' = \frac{1}{2}\rho_0(u'v'^* + v'v'^* + \phi'_z\phi'^*_z/N^2), \tag{27}$$

$$\nabla' = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right), \tag{28}$$

$$\boldsymbol{F}'_u = (\rho_0\overline{u'u'^*}, \rho_0\mathrm{Re}\{\overline{u'v'}\}, \rho_0\mathrm{Re}\{\overline{u'w'}\}), \tag{29a}$$

$$\boldsymbol{F}'_v = (\rho_0\mathrm{Re}\{\overline{u'v'}\}, \rho_0\overline{v'v'^*}, \rho_0\mathrm{Re}\{\overline{v'w'}\}), \tag{29b}$$

$$\boldsymbol{F}'_\phi = (\rho_0 \mathrm{Re}\{\overline{u'\phi'}\},\ \rho_0 \mathrm{Re}\{\overline{v'\phi'}\},\ \rho_0 \mathrm{Re}\{\overline{w'\phi'}\}), \tag{29c}$$

$$d' = \rho_0 \mathrm{Re}\{\overline{d'_x u'} + \overline{d'_y v'} + \overline{q'\phi'_z}/N^2\}, \tag{30}$$

in which $(\ )^*$ means conjugate of $(\ )$ and

$$\mathrm{Re}\{ab\} = \frac{1}{2}(ab^* + ba^*). \tag{31}$$

Again with the assumptions of the slowly varying approximations we get the first order expansion of the energy equation:

$$\frac{\partial E}{\partial T} + U\frac{\partial E}{\partial X} + V\frac{\partial E}{\partial Y} + \boldsymbol{F}_u \cdot \nabla U + \boldsymbol{F}_v \cdot \nabla V + \nabla \cdot \boldsymbol{F}_\phi$$

$$-\frac{f}{N^2}\rho_{00}\mathrm{Re}\{v\phi_z\}\frac{\partial U}{\partial Z} + \frac{f}{N^2}\rho_{00}\mathrm{Re}\{u\phi_z\}\frac{\partial V}{\partial Z} = d'/\epsilon, \tag{26'}$$

where

$$\nabla \equiv \left(\frac{\partial}{\partial X}, \frac{\partial}{\partial Y}, \frac{\partial}{\partial Z}\right), \tag{27'}$$

and

$$E \equiv \frac{1}{2}\rho_{00}(uu^* + vv^* + \phi_z\phi_z^*/N^2). \tag{28'}$$

Similar definitions for $\boldsymbol{F}_u, \boldsymbol{F}_v$ and $\boldsymbol{F}_\phi$ can be obtained by replacing u', v', w', ϕ' and ρ_0 by u, v, w, ϕ and ρ_{00} in (29), respectively.

By using the polarization relations (22) we may show that

$$E = \rho_{00}\frac{K^2\hat{\omega}^2\phi\phi^*}{(\hat{\omega}^2 - f^2)^2}, \tag{32}$$

$$\begin{aligned}\boldsymbol{F}_\phi &\equiv (\rho_{00}\mathrm{Re}\{u\phi\}, \rho_{00}\mathrm{Re}\{v\phi\}, \rho_{00}\mathrm{Re}\{w\phi\}) \\ &= (\hat{c}_{gX}E, \hat{c}_{gY}E, \hat{c}_{gZ}E),\end{aligned} \tag{33a}$$

$$\rho_{00}uu^* = \frac{k}{\hat{\omega}}\hat{c}_{gX}E + \frac{f^2}{\hat{\omega}^2}E, \tag{33b}$$

$$\rho_{00}vv^* = \frac{l}{\hat{\omega}}\hat{c}_{gY}E + \frac{f^2}{\hat{\omega}^2}E, \tag{33c}$$

$$\rho_{00}\mathrm{Re}\{uv\} = \frac{k}{\hat{\omega}}\hat{c}_{gY}E = \frac{l}{\hat{\omega}}\hat{c}_{gX}E, \tag{33d,e}$$

$$\rho_{00}\mathrm{Re}\left\{uw - \frac{f}{N^2}v\phi_z\right\} = \frac{k}{\hat{\omega}}\hat{c}_{gZ}E, \tag{33f}$$

$$\rho_{00}\mathrm{Re}\left\{uw + \frac{f}{N^2}z\phi_z\right\} = \frac{l}{\hat{\omega}}\hat{c}_{gZ}E, \tag{33g}$$

where

$$(\hat{c}_{gX}, \hat{c}_{gY}, \hat{c}_{gZ}) = (c_{gX} - U, c_{gY} - V, c_{gZ}) \tag{34}$$

is Doppler shifted group velocity.

It can be shown that, unlike the two dimensional case (Coy, 1983), kinetic energy

(E_K) and potential energy (E_P) are not equally partitioned because of the influence of the rotation. Their ratio is given by, from (33b, c),

$$\frac{E_K}{E_P}=\frac{\hat{\omega}^2+f^2}{\hat{\omega}^2-f^2}>1. \tag{35}$$

Substituting (33) into (26') gives

$$\frac{D_g E}{DT}+E\nabla\cdot\boldsymbol{c}_g+\frac{E}{\hat{\omega}}k(\hat{\boldsymbol{c}}_g\cdot\nabla U)+\frac{E}{\omega}l(\hat{\boldsymbol{c}}_g\cdot\nabla V)=d'/\epsilon. \tag{36}$$

In deriving (36) we have used nondivergence condition since the mean flow is quasi-geostrophic. We see that wave energy is not conserved as the wave packets propagate in a nonhomogeneous medium even if dissipation and damping vanish. The wave kinematic relation (25a, b, d) gives

$$\begin{aligned}
\hat{\omega}E\frac{D_g(\hat{\omega}^{-1})}{DT}&=-\frac{E}{\hat{\omega}}\left\{\frac{D_g\omega}{DT}-\frac{D_g}{DT}(kU+lV)\right\}\\
&=-\frac{E}{\hat{\omega}}\left\{k\frac{\partial U}{\partial T}+l\frac{\partial V}{\partial T}-\frac{\partial}{\partial T}(kU+lV)-U\boldsymbol{c}_g\cdot\nabla k-V\boldsymbol{c}_g\cdot\nabla l-k\boldsymbol{c}_g\cdot\nabla U-l\boldsymbol{c}_g\cdot\nabla V\right\}\\
&=\frac{E}{\hat{\omega}}\left\{U\frac{D_g k}{DT}+V\frac{D_g l}{DT}+k\boldsymbol{c}_g\cdot\nabla U+l\boldsymbol{c}_g\cdot\nabla V\right\}\\
&=\frac{E}{\hat{\omega}}\left\{-U\left(k\frac{\partial U}{\partial X}+l\frac{\partial V}{\partial X}\right)-V\left(k\frac{\partial U}{\partial Y}+l\frac{\partial V}{\partial Y}\right)+k\boldsymbol{c}_g\cdot\nabla U+l\boldsymbol{c}_g\cdot\nabla V\right\}\\
&=\frac{E}{\hat{\omega}}\{k\hat{\boldsymbol{c}}_g\cdot\nabla U+l\hat{\boldsymbol{c}}_g\cdot\nabla V\}\\
&=\text{Last two terms on the LHS of (36)}.
\end{aligned} \tag{37}$$

Combining (36) with (37) would result in the following wave action equation:

$$\frac{\partial A}{\partial T}+\nabla\cdot(\boldsymbol{c}_g A)=\frac{d'}{\epsilon\hat{\omega}}, \tag{38}$$

where the wave action density

$$A=\frac{E}{\hat{\omega}} \tag{39}$$

is the ratio of wave energy density to the Doppler shifted frequency.

Equation (38) gives the wave action density conservation law in the order of ϵ if $d'=0$. The above derivation shows that we require the mean state satisfying (13c) for the action conservation to hold. Wave action conservation will not hold in this special model if N^2 depends on time and/or horizontal variables.

In the real atmosphere the right-hand sides of (1)—(3) may include eddy flux terms (Holton, 1975), eddy diffusion and/or Rayleigh friction, and Newtonian cooling. It is easy to show that eddy flux terms do not change wave action density, which implies that wave-wave nonlinear interaction for the packet vanishes at the first-order approximation. We let damping terms be $\left(\frac{\partial}{\partial z}D\frac{\partial}{\partial z}-a\right)(u',v')$ on the right-hand sides of the momentum equations (1) and (2), and $\left(\frac{\partial}{\partial z}\kappa\frac{\partial}{\partial z}-\alpha\right)\phi'_z$ in (3). Here D is kinematic diffusion coefficient, a is Rayleigh friction coefficient, κ is thermal diffusion coefficient and α is Newtonian cooling coefficient. If we assume that all these damping coefficients are slowly varying functions of space, we get, for the dissipation and damping terms in the first-order approximation

$$\frac{d}{\epsilon\hat{\omega}}=-\left\{D\left(m^2-\frac{1}{4H^2}\right)\left[(1+P_r)+(1-P_r)\frac{f^2}{\hat{\omega}^2}\right]+a\left(1+\frac{f^2}{\hat{\omega}^2}\right)+\alpha\left(1-\frac{f^2}{\hat{\omega}^2}\right\}\frac{A}{\epsilon},\tag{40}$$

where $P_r=D/\kappa$ it Prandtl number.

We now complete the derivation of wave action equation (38). The equation is a general form in the sense that zonal averaging given by (7) may change in a large scope. Since mean flow evolution equations are different for different averaging regimes, we may result in different kinds of wave-mean flow interaction dynamics in different regimes though we have the same wave action equation. Here we will consider two different averaging regimes in the following two sections.

III. EQUATIONS FOR THE MEAN FLOW EVOLUTION. 1: GLOBAL WAVE FORCING

We first consider the case that the averaging of the mean states is over the whole latitude. This case should correspond to the inertio-gravity waves of horizontal wavelength longer than hundreds kilometers. Such kind of forcing in a large extent will result in the mean fields changing in large scales and rotation should be included.

When rotation is included, inertio-gravity wave induced meridional mean flow also becomes important in inducing zonal mean flow acceleration through Coriolis torque. We thus have the following Eulerian mean governing equations for the zonal mean flow on β-plane (Holton, 1975)

$$\frac{\partial\bar{u}}{\partial t}-f\bar{v}=-\frac{\partial}{\partial y}(\overline{u'v'})-\frac{1}{\rho_0}\frac{\partial}{\partial z}(\rho_0\overline{u'w'}),\tag{41}$$

$$f\bar{u}+\frac{\partial\bar{\Phi}}{\partial y}=0,\tag{42}$$

$$\frac{\partial^2\bar{\Phi}}{\partial t\partial z}+N^2\bar{w}=-\frac{\partial}{\partial y}(\overline{v'\phi'_z})-\frac{1}{\rho_0}\frac{\partial}{\partial z}(\rho_0\overline{w'\phi'_z}),\tag{43}$$

$$\frac{\partial\bar{v}}{\partial y}+\frac{1}{\rho_0}\frac{\partial}{\partial z}(\rho_0\bar{w})=0.\tag{44}$$

The model shows that the zonal mean flow is in quasi-geostrophic balance. Mean meridional and vertical velocity are just required to maintain the quasi-geostrophic flow.

Similarly, WKB approximation expansion of eddy terms with the form of (14) gives the following set of equations

$$\rho_0\left(\frac{\partial U}{\partial T}-\frac{fV}{\epsilon}\right)+\rho_{00}\left(\frac{\partial}{\partial Y}\mathrm{Re}\{uv\}+\frac{\partial}{\partial Z}\mathrm{Re}\{uw\}\right)=0,\tag{45}$$

$$\frac{fU}{\epsilon}+\frac{\partial\bar{\Phi}}{\partial Y}=0,\tag{46}$$

$$\epsilon^2\rho_0\frac{\partial^2\bar{\Phi}}{\partial T\partial Z}+\rho_0 N^2\bar{w}=\epsilon\rho_{00}\mathrm{Re}\left\{-\frac{\partial}{\partial Y}(v\phi_z)-\frac{\partial}{\partial Z}(w\phi_z)\right\}, \tag{47}$$

$$\rho_0\frac{\partial\bar{V}}{\partial Y}+\frac{\partial}{\partial Z}(\rho_0\bar{w})=0. \tag{48}$$

According to the polarization relation (22c) we have

$$\mathrm{Re}\{w\phi_z\}=0, \tag{49}$$

which allows us to reduce (46)—(48) into

$$\rho_0\frac{\partial^2 V}{\partial Y^2}=\epsilon\frac{\partial}{\partial Z}\left(-\frac{\rho_0}{N^2}\frac{\partial^2 U}{\partial T\partial Z}+\frac{\rho_{00}}{N^2}\frac{\partial^2}{\partial Y^2}\mathrm{Re}\left\{\frac{v\phi_z}{N^2}\right\}\right). \tag{50}$$

Combining (45) and (50) gives

$$\frac{\partial}{\partial T}\left[\frac{\partial^2 U}{\partial Y^2}+\frac{1}{\rho_0}\frac{\partial}{\partial Z}\left(\rho_0\frac{f^2}{N^2}\frac{\partial U}{\partial Z}\right)\right]$$

$$+\frac{\rho_{00}}{\rho_0}\frac{\partial^2}{\partial Y^2}\left(\frac{\partial}{\partial Z}\mathrm{Re}\{-fv\phi_z/N^2+uw\}+\frac{\partial}{\partial Y}\mathrm{Re}\{uv\}\right)=0. \tag{51}$$

However we can express eddy terms by wave action flux along the group velocity. Therefore (51) results in, by (33e, f) and (39),

$$\frac{\partial}{\partial T}\left[\frac{\partial^2 U}{\partial Y^2}+\frac{1}{\rho_0}\frac{\partial}{\partial Z}\left(\rho_0\frac{f^2}{N^2}\frac{\partial U}{\partial Z}\right)\right]+\frac{1}{\rho_0}\frac{\partial^2}{\partial Y^2}\left(\frac{\partial}{\partial Y},\frac{\partial}{\partial Z}\right)(c_{gY},c_{gZ})(Ak)=0, \tag{52}$$

where zonal wave number k is a constant. Equation (52) can also be written as

$$\frac{\partial}{\partial T}Q_Y=\frac{\epsilon^2}{\rho_0}\frac{\partial^2}{\partial Y^2}\left(\frac{\partial}{\partial Y},\frac{\partial}{\partial Z}\right)(c_{gY},c_{gZ})(Ak), \tag{53}$$

where Q_Y is meridional gradient of mean flow potential vorticity:

$$Q_Y=\beta-\epsilon^2\left[\frac{\partial^2 U}{\partial Y^2}+\frac{1}{\rho_0}\frac{\partial}{\partial Z}\left(\rho_0\frac{f^2}{N^2}\frac{\partial U}{\partial Z}\right)\right]. \tag{54}$$

Equation (53) implies that when rotation is included wave-mean flow interactions will change the global mean field by geostrophic adjustment processes.

Moreover, we can also introduce a diffusion term in the zonal mean flow equation (41), so that (52) becomes

$$\frac{\partial}{\partial T}\left[\frac{\partial^2 U}{\partial Y^2}+\frac{1}{\rho_0}\frac{\partial}{\partial Z}\left(\rho_0\frac{f^2}{N^2}\frac{\partial U}{\partial Z}\right)\right]$$

$$+\frac{1}{\rho_0}\frac{\partial^2}{\partial Y^2}\left(\frac{\partial}{\partial Y},\frac{\partial}{\partial Z}\right)(c_{gY},c_{gZ})(Ak)=\epsilon\frac{\partial^2}{\partial Y^2}\left(\frac{1}{\rho_0}\frac{\partial}{\partial Z}\rho_0 D\frac{\partial U}{\partial Z}\right). \tag{55}$$

The coupled model consisting of (38), (40) and (55) is a 3-dimensional generalization of the models of Dunkerton (1981) and Coy (1983). One interesting feature of the wave equation arising from rotation is that the damping effects by Rayleigh friction and Newtonian cooling are unequal even if we set the two coefficients to be equal. The ratio of Rayleigh friction damping to Newtonian cooling damping is just the ratio of kinetic energy and potential energy (given by (35)). As $|\hat{\omega}|\longrightarrow f$ Newtonian cooling damping is reduced to zero while Rayleigh friction damps the wave exponentially. Note that if $a=\alpha$ the resulting damping rate $[2a]$ is independent on wave frequency. Eddy diffusion damps the wave most efficiently as $|\hat{\omega}|\to f$ because the vertical wavenumber approaches infinity.

IV. EQUATIONS FOR THE MEAN FLOW EVOLUTION. 2: LOCAL WAVE FORCING

In this section we consider the case that zonal averaging of the mean state is much less than the whole latitude. Since most meso-scale gravity waves in troposphere produced mainly by orography (lee waves) and wind shear are of horizontal wavelengths less than 100 km (Atkinson, 1981) interactions between such kinds of meso-scale gravity waves and the mean flow averaged over hundreds km may be important in the lower stratosphere. It is well known that mean flow in the low stratosphere could depart largely from symmetric polar vortex. This fact implies that the meridional mean velocity could have approximate value of zonal mean velocity. Now the mean flow equations are

$$\frac{\partial\bar{u}}{\partial t}+\bar{u}\frac{\partial\bar{u}}{\partial x}+\bar{v}\frac{\partial\bar{u}}{\partial y}-f\bar{v}=-\frac{\partial\bar{\Phi}}{\partial x}-\frac{\partial\overline{u'^2}}{\partial x}-\frac{\partial\overline{u'v'}}{\partial y}-\frac{1}{\rho_0}\frac{\partial(\rho_0\overline{u'w'})}{\partial z}, \tag{56a}$$

$$\frac{\partial\bar{v}}{\partial t}+\bar{u}\frac{\partial\bar{v}}{\partial x}+\bar{v}\frac{\partial\bar{v}}{\partial y}+f\bar{u}=-\frac{\partial\bar{\Phi}}{\partial y}-\frac{\partial\overline{u'v'}}{\partial x}-\frac{\partial\overline{v'^2}}{\partial y}-\frac{1}{\rho_0}\frac{\partial(\rho_0\overline{v'w'})}{\partial z}, \tag{56b}$$

$$\frac{\partial\bar{\Phi}_z}{\partial t}+\bar{u}\frac{\partial\bar{\Phi}_z}{\partial x}+\bar{v}\frac{\partial\bar{\Phi}_z}{\partial y}+N^2\bar{w}=-\frac{\partial}{\partial x}(\overline{u'\phi'_z})-\frac{\partial}{\partial y}(\overline{v'\phi'_z})-\frac{1}{\rho_0}\frac{\partial}{\partial z}(\rho_0\overline{w'\phi'_z}), \tag{56c}$$

$$\frac{\partial\bar{u}}{\partial x}+\frac{\partial\bar{v}}{\partial y}+\frac{1}{\rho_0}\frac{\partial}{\partial z}(\rho_0\bar{w})=0. \tag{56d}$$

WKB approximation expansions of the eddy terms give the following mean flow equations

$$\frac{\partial U}{\partial T}+U\frac{\partial U}{\partial X}+V\frac{\partial U}{\partial Y}-\frac{f}{\epsilon}V=-\frac{\partial\bar{\Phi}}{\partial X}-\frac{1}{\rho_0}\nabla\cdot\boldsymbol{F}_u, \tag{57a}$$

$$\frac{\partial V}{\partial T}+U\frac{\partial V}{\partial X}+V\frac{\partial V}{\partial Y}+\frac{f}{\epsilon}U=-\frac{\partial\bar{\Phi}}{\partial Y}-\frac{1}{\rho_0}\nabla\cdot\boldsymbol{F}_v, \tag{57b}$$

$$\epsilon\left[\frac{\partial^2\bar{\Phi}}{\partial T\partial Z}+U\frac{\partial^2\bar{\Phi}}{\partial X\partial Z}+V\frac{\partial^2\bar{\Phi}}{\partial Y\partial Z}\right]+\frac{N^2}{\epsilon}\bar{w}=-\frac{1}{\rho_0}\nabla\cdot\boldsymbol{F}_{\phi_z}, \tag{57c}$$

$$\frac{\partial U}{\partial X}+\frac{\partial V}{\partial Y}+\frac{1}{\rho_0}\frac{\partial}{\partial Z}(\rho_0\bar{w})=0. \tag{57d}$$

We will see that it is convenient to divide the mean field (Ψ) into two parts

$$\Psi(X,Y,Z,T)=\Psi_1(\hat{X},\hat{Y},\hat{Z})+\Psi_2(X,Y,Z,T), \tag{58}$$

where new-introduced slowly varying variables $\hat{X},\hat{Y}$ and $\hat{Z}$ are defined by

$$\hat{X}=\epsilon X=\epsilon^2 x,\quad \hat{Y}=\epsilon Y=\epsilon^2 y,\quad \hat{Z}=\epsilon Z=\epsilon^2 z.$$

The steady part of the mean fields are more slowly varying functions of space and satisfy the following balance equation (Holton, 1979)

$$U_1\frac{\partial U_1}{\partial\hat{X}}+V_1\frac{\partial U_1}{\partial\hat{Y}}-\frac{f}{\epsilon^2}V_1=-\frac{\partial\bar{\Phi}_1}{\partial\hat{X}}, \tag{59a}$$

$$U_1\frac{\partial V_1}{\partial\hat{X}}+V_1\frac{\partial V_1}{\partial\hat{Y}}+\frac{f}{\epsilon^2}U_1=-\frac{\partial\bar{\Phi}_1}{\partial\hat{Y}}, \tag{59b}$$

$$\bar{w}_1=0, \tag{59c}$$

$$\frac{\partial U_1}{\partial \hat{X}}+\frac{\partial V_1}{\partial \hat{Y}}=0. \tag{59d}$$

This part of the mean field represents the mean state before the wave forcing occurs. The second part of the mean field represents the induced flow by the local wave forcing. Since we have introduced slowly varying variables through (58), we now should replace the derivatives in (56) by,

$$\frac{\partial}{\partial T}\to\frac{\partial}{\partial T},\ \frac{\partial}{\partial X}\to\frac{\partial}{\partial X}+\epsilon\frac{\partial}{\partial \hat{X}},\ \frac{\partial}{\partial Y}\to\frac{\partial}{\partial Y}+\epsilon\frac{\partial}{\partial \hat{Y}},\ \frac{\partial}{\partial Z}\to\frac{\partial}{\partial Z}+\epsilon\frac{\partial}{\partial \hat{Z}}. \tag{60}$$

Substituting (58) and (60) into (57) and using equations (59), we will arrive at the following equations for the local forcing induced mean flow changes:

$$\frac{\partial U_2}{\partial T}+U_1\frac{\partial U_2}{\partial X}+V_1\frac{\partial U_2}{\partial Y}-\frac{f}{\epsilon}V_2=-\frac{\partial\bar{\Phi}_2}{\partial X}-\frac{1}{\epsilon\rho_0}\nabla\cdot\boldsymbol{F}_u, \tag{61a}$$

$$\frac{\partial V_2}{\partial T}+U_1\frac{\partial V_2}{\partial X}+V_1\frac{\partial V_2}{\partial Y}+\frac{f}{\epsilon}U_2=-\frac{\partial\bar{\Phi}_2}{\partial Y}-\frac{1}{\epsilon\rho_0}\nabla\cdot\boldsymbol{F}_v, \tag{61b}$$

$$\epsilon\left[\frac{\partial^2\bar{\Phi}_2}{\partial T\partial Z}+U_1\frac{\partial^2\bar{\Phi}_2}{\partial X\partial Z}+V_1\frac{\partial^2\bar{\Phi}_2}{\partial Y\partial Z}\right]+\frac{N^2}{\epsilon}\varpi_2=-\frac{1}{\epsilon\rho_0}\nabla\cdot\boldsymbol{F}_{\phi_z}, \tag{61c}$$

$$\frac{\partial U_2}{\partial X}+\frac{\partial V_2}{\partial Y}+\frac{1}{\rho_0}\frac{\partial}{\partial Z}(\rho_0\varpi_2)=0. \tag{61d}$$

For the gravity waves of wavelength less than hundreds kilometers the rotation effect may be ignored. In that case the flux terms in (61) are, from (33) and (39),

$$\boldsymbol{F}_u=kA\hat{\boldsymbol{c}}_g,\ \ \boldsymbol{F}_v=lA\hat{\boldsymbol{c}}_g,\ \ F_{\phi_z}=0. \tag{62a,b,c}$$

We have thus completed the derivations of mean flow equations which connect the eddy terms with wave action equation. A simpler form of equations (61) may be obtained by noting that, in the real atmosphere, horizontal wavenumber of small upward propagating gravity waves does not show significant change as compared to vertical wavenumber. We can introduce the following time derivative along the "moving frame" with velocity $[U_1(\hat{X},\hat{Y},\hat{Z}),\ V_1(\hat{X},\hat{Y},\hat{Z})]$:

$$\frac{\partial}{\partial T'}=\frac{\partial}{\partial T}+U_1\frac{\partial}{\partial X}+V_1\frac{\partial}{\partial Y}. \tag{63}$$

Substituting (62) and (63) into (61) gives

$$\frac{\partial U_2}{\partial T'}-\frac{f}{\epsilon}V_2=-\frac{\partial\bar{\Phi}_2}{\partial X}+\frac{k}{\epsilon\rho_0}\left[\frac{\partial A}{\partial T'}-\frac{d}{\epsilon\hat{\omega}}\right], \tag{64a}$$

$$\frac{\partial V_2}{\partial T'}+\frac{f}{\epsilon}U_2=-\frac{\partial\bar{\Phi}_2}{\partial Y}+\frac{l}{\epsilon\rho_0}\left[\frac{\partial A}{\partial T'}-\frac{d}{\epsilon\hat{\omega}}\right], \tag{64b}$$

$$\frac{\partial^2\bar{\Phi}_2}{\partial T'\partial Z}+\frac{N^2}{\epsilon^2}\varpi_2=0, \tag{64c}$$

$$\frac{\partial U_2}{\partial X}+\frac{\partial V_2}{\partial Y}+\left(\frac{\partial}{\partial Z}-\frac{1}{\epsilon H}\right)\varpi_2=0, \tag{64d}$$

where we have used the wave action conservation equation (38) and the following approximation relation

$$\hat{c}_g \approx (c_{gX}-U_1,\ c_{gY}-V_1,\ c_{gZ}). \tag{65}$$

Equations (64) describe a geostrophic adjustment process in stratified atmosphere with local wave sources. It is well known that the complete solution of (64) may be represented as a superposition of elementary inertio-gravity waves and a steady geostrophic mode (Dickinson, 1966; Blumen, 1972; Zeng, 1981). Zhu and Holton (1986) showed that the characteristic wavelength of the forcing induced gravity waves is defferent from the original forcing gravity waves and is mainly dependent on the forcing distributions.

V. CONCLUDING REMARKS

Equations (21), (25), (38), and (55) or (64) are the complete equations of ray tracing and wave-mean flow interactions for gravity waves propagating in a nonuniform but slowly varying medium. The models formulated by these equations can be used to study the general circulations in middle atmosphere where momentum depositions by gravity waves originally generated at the troposphere may play an important role. We can rewrite those equations in physical variables straightforwardly, e. g.,

$$\frac{\partial A}{\partial t}+\nabla_3\cdot\boldsymbol{F}=-\left\{D\left(m^2-\frac{1}{4H^2}\right)\left[(1+P_r)+(1-P_r)\frac{f^2}{\hat{\omega}^2}\right]\right.$$

$$\left.+a\left(1+\frac{f^2}{\hat{\omega}^2}\right)+\alpha\left(1-\frac{f_2}{\hat{\omega}^2}\right)\right\}A, \tag{38'}$$

where

$$\boldsymbol{F}\equiv\boldsymbol{c}_g A \tag{66}$$

is defined as the three dimensional Eliassen-Palm flux for gravity waves. The relationship between the Eliassen-Palm flux and the wave action given by (66) is consistent with the same relationship for planetary waves (Palmer, 1982; Huang, 1986).

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