ATTRACTOR SETS AND MULTIPLE EQUILIBRIA IN THE RESONANCE OF TOPOGRAPHICALLY FORCED WAVES

*Li Xianlang** (李贤琅)

Center for Meteorology and Physical Oceanography, Massachusetts Institute of Technology

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ABSTRACT

The resonance of topographically forced waves is studied using a quasi-geostrophic spectral model on the rotating sphere. The use of complete spectral expansions without truncation leads to the exact solutions of the nonlinear coupling equations by means of the random phase approximation and the projection operator techniques under the dissipation-vanishing limit. The energy transfer process between topographically forced wave ensemble and zonal mean flow is described.

It is shown that the dynamical system loses stability and further bifurcation takes place when the topographic force has occurred. There are two sorts of equilibrium point in the resonance system. The unstable equilibrium is an isolated equilibrium point and, therefore, is hardly observed to occur. The stable equilibrium is an attractor set which is related to the phenomenon of blocking.

I. INTRODUCTION

Recent developments in the field of geophysical fluid dynamics have provided a new and stimulating point of view about the problem of the bifurcation. New insights in this field have been achieved primarily through computer studies of the dynamics of simple nonlinear models, including the Lorenz equations (Lorenz, 1963). The relatively simple systems of three or more coupled nonlinear first-order equations often have chaotic solutions. These solutions are sometimes called strange attractors.

Usually we may raplace the partial differential equations of geophysical fluid dynamical system by an infinite set of ordinary differential equations, with time as the independent variable. The system is regarded as a mechanical system with dissipation. Mathematically, its motion is viewed as governed by a first-order differential equation on a state space. Although the state space for the geophysical fluid dynamical system is infinite in dimension, the point of view to be developed is that the phenomena underlying the system are essentially finitely dimensional. The intuition is that there is some finite set of essential "modes" or degrees of freedom which effectively govern the behavior, while the remaining infinite degrees of freedom simply respond passively.

Since the motions will be of large scale, they will be quasi-geostrophic and therefore be governed by conservation of potential vorticity. Furthermore, we may expand the variations in orthonormal eigenfunctions of the Laplace operator, which are the double Fourier functions for β-plane problem or the spherical harmonics for sphere problem, etc., respectively. The

* On leave from Chengdu Institute of Meteorology, Chengdu, Sichuan, People's Republic of China.

low-order models (highly-truncated spectral models) have often been used in this case since the pioneering work by Lorenz (1960). On a certain point, the most serious problem is the effect of severe truncation. If the truncation is not so severe and sufficient spectral components are retained, the spectral form is a good approximation of the original equations. However, there has been no assurance that the low-order models can approximate the nature of the original equations. As a result, the domain of applicability of truncated model systems in explaining the large-scale atmospheric motions is not yet very clear.

The present study of bottom topographic forcing in the rotating barotropic atmosphere is a complet spectral model without truncation. Charney and DeVore (1979) considered a resonance mechanism involving interaction with bottom topography. Their work (hereafter referred to as CD) has stimulated a wide range of new theoretical works on the multiple flow equilibria. Compared with the model in CD, the main difference is that each mode in the present model consists of many wavevectors. Hence, the mode is regarded as a wave ensemble. We emphasize that, because of a finite bandwidth of the spectrum of each of the modes, as mentioned previously, we must sum up over many wavevectors that satisfy the constraint conditions. And therefore, in such a case, the use of the complex Fourier amplitude of a single wavevector does not make sense and only the phase independent quantity such as the energy spectral density of wave ensemble is useful.

II. MODEL

Taking a (earth's radius) and Ω^{-1} (earth's angular velocity)$^{-1}$ as length and time scales, the nondimensional equations defining our spherical equivalent barotropic model are

$$\frac{\partial}{\partial t}\nabla^2\psi+2\frac{\partial}{\partial\lambda}\psi+J(\psi\nabla^2\psi+\eta)=\frac{1}{\mathscr{R}}\nabla^4\psi\ , \tag{1}$$

where ψ is the streamfunction, ∇^2 the horizontal Laplacian, $J(a,b)$ the horizontal Jacobian, $\mathscr{R}$ the Reynolds number, η the function of the bottom topography.

In order to write (1) in spectral form, both the streamfunction and the orographic function are expanded in spherical harmonics:

$$\psi(\theta,\lambda,t)=\sum_{l=0}^{\infty}\sum_{m=-l}^{+l}\Psi_l^m(t)Y_l^m(\theta,\lambda) \tag{2a}$$

$$\eta(\theta,\lambda)=\sum_{l=0}^{\infty}\sum_{m=-l}^{+l}H_l^mY_l^m(\theta,\lambda). \tag{2b}$$

The variety of wavevectors with order l and m can be labeled as $\boldsymbol{K}=(l_{\boldsymbol{K}},m_{\boldsymbol{K}})$ and Eq. (2) becomes

$$\psi(t)=\sum_{\boldsymbol{K}}\Psi_{\boldsymbol{K}}(t)Y_{\boldsymbol{K}} \tag{2a$'$}$$

$$\eta=\sum_{\boldsymbol{K}}H_{\boldsymbol{K}}Y_{\boldsymbol{K}}. \tag{2b$'$}$$

Note that $Y_{\boldsymbol{K}}$ is the eigenfunction of the sphere Laplace operator

$$\nabla^2Y_{\boldsymbol{K}}=-\Lambda_{\boldsymbol{K}}Y_{\boldsymbol{K}}, \tag{3}$$

where

$$\Lambda_{\boldsymbol{K}}=l_{\boldsymbol{K}}(l_{\boldsymbol{K}}+1). \tag{4}$$

On substituting Eqs. (2), (3) into Eq. (1) and integrating by operation, we obtain

$$\int_0^{2\pi} d\lambda \int_0^{\pi} \sin\theta d\theta Y_{\mathbf{K}}^{*} \left(\quad \cdot \quad \right)$$

$$\frac{d}{dt}\Psi_{\mathbf{K}}(t) - \left(i\frac{2m_{\mathbf{K}}}{\Lambda_{\mathbf{K}}} - \frac{\Lambda_{\mathbf{K}}}{\mathscr{R}}\right)\Psi_{\mathbf{K}}(t)$$

$$= i\sum_{\mathbf{K}'}\sum_{\mathbf{K}''}[B_{\mathbf{K}}(\mathbf{K}',\mathbf{K}'')\Psi_{\mathbf{K}'}(t)\Psi_{\mathbf{K}''}(t) + S_{\mathbf{K}}(\mathbf{K}',\mathbf{K}'')\Psi_{\mathbf{K}'}(t)H_{\mathbf{K}''}], \tag{5}$$

where

$$B_{\mathbf{K}}(\mathbf{K}',\mathbf{K}'') = \frac{1}{i}\frac{\Lambda_{\mathbf{K}'} - \Lambda_{\mathbf{K}''}}{2\Lambda_{\mathbf{K}}}\int_0^{2\pi} d\lambda \int_0^{\pi}\sin\theta d\theta Y_{\mathbf{K}}^{*} J(Y_{\mathbf{K}'}, Y_{\mathbf{K}''}) \tag{6a}$$

$$S_{\mathbf{K}}(\mathbf{K}',\mathbf{K}'') = \frac{1}{i}\frac{1}{\Lambda_{\mathbf{K}}}\int_0^{2\pi} d\lambda \int_0^{\pi}\sin\theta d\theta Y_{\mathbf{K}}^{*} J(Y_{\mathbf{K}'}, Y_{\mathbf{K}''}). \tag{6b}$$

We may write out the constrained condition by dummy-variable $\mathbf{K}$:

$$m_{\mathbf{K}} = m_{\mathbf{K}'} + m_{\mathbf{K}''}. \tag{7}$$

By means of the multiple-time-scale analysis (see Li, 1985), using the first-order equation of ε, we obtain the dispersion relation

$$\sigma_{\mathbf{K}} = -\frac{2m_{\mathbf{K}}}{l_{\mathbf{K}}(l_{\mathbf{K}}+1)} - i\frac{l_{\mathbf{K}}(l_{\mathbf{K}}+1)}{\mathscr{R}} \mathrel{\hat=} \sigma_{r\mathbf{K}} + i\sigma_{i\mathbf{K}}. \tag{8}$$

Consider the dissipation-vanishing limit and using the second-order equation of ε, we obtain the nonlinear coupling equations for the normalized complex amplitude

$$\frac{d}{dT}A_{\mathbf{K}}(T) = i\sum_{m_{\mathbf{K}} = m_{\mathbf{K}'} + m_{\mathbf{K}''}}[G_{\mathbf{K}}(\mathbf{K}',\mathbf{K}'')A_{\mathbf{K}'}(T)A_{\mathbf{K}''}(T)$$
$$+ F_{\mathbf{K}}(\mathbf{K}',\mathbf{K}'')A_{\mathbf{K}'}(T)\tilde{H}_{\mathbf{K}''}]e^{+i(\Delta\sigma_{r\mathbf{K}} - io^{+})t}, \tag{9}$$

where

$$G_{\mathbf{K}}(\mathbf{K}',\mathbf{K}'') = |B_{\mathbf{K}}(\mathbf{K}',\mathbf{K}'')B_{\mathbf{K}'}(\mathbf{K}'',\mathbf{K})B_{\mathbf{K}''}(\mathbf{K},\mathbf{K}')|^{\frac{1}{2}} \tag{10a}$$

$$F_{\mathbf{K}}(\mathbf{K}',\mathbf{K}'') = [S_{\mathbf{K}}(\mathbf{K}',\mathbf{K}'')/B_{\mathbf{K}}(\mathbf{K}',\mathbf{K}'')]G_{\mathbf{K}}(\mathbf{K}',\mathbf{K}'') \tag{10b}$$

$$\Delta\sigma_{r\mathbf{K}} \mathrel{\hat=} \sigma_{r\mathbf{K}} - (\sigma_{r\mathbf{K}'} + \sigma_{r\mathbf{K}''}), \quad |\Delta\sigma_{r\mathbf{K}}| \ll |\sigma_{r\mathbf{K}}|. \tag{10c}$$

Note that the $\mathbf{K}$ is the dummy-variable. Mathematically, in that case, it is an infinite set of first-order ordinary differential equations. Physically, in large-scale atmospheric motion, it can be considered as the case that there are some planetary wave modes which effectively govern the behavior as a wave ensemble and interfere with each other through the nonlinear resonance interaction.

Compared with the model in CD, we consider the topographic function is included only in the mode $\mathbf{K}''$ with a normalized complex amplitude $\tilde{H}_{\mathbf{K}''}$, and thus we have

$$\frac{\partial}{\partial T}A_{\mathbf{K}_0} = i\sum_{m_{\mathbf{K}_0} = m_{\mathbf{K}'} + m_{\mathbf{K}''}}[G_{\mathbf{K}_0}(\mathbf{K}',\mathbf{K}'')A_{\mathbf{K}'}A_{\mathbf{K}''} - F_{\mathbf{K}_0}(\mathbf{K}',\mathbf{K}'')A_{\mathbf{K}'}\tilde{H}_{\mathbf{K}''}]$$
$$\times e^{+i(\Delta\sigma_{r\mathbf{K}_0} - io^{+})t} \tag{11a}$$

$$\frac{\partial}{\partial T}A_{K_1}=i\sum_{m_{K_1}=m_K-m_{K''}}[G_{K_1}(K''^*,K)A^*_{K''}A_K-F_{K_1}(K''^*,\ K)\tilde{H}^*_{K''}A_K]$$
$$\times e^{-i(\Delta\sigma_{rK_1}+i0^+)t} \tag{11b}$$

$$\frac{\partial}{\partial T}A_{K_2}=i\sum_{m_{K_2}=m_K-m_{K'}}[G_{K_2}(K,K'^*)A_KA^*_{K'}]e^{-i(\Delta\sigma_{rK_2}+i0^+)t}. \tag{11c}$$

Note that the $\tilde{H}_{K''}$ is not time-dependent, and we have

$$\frac{\partial}{\partial T}\tilde{H}_{K''}=0. \tag{11d}$$

III. RANDOM PHASE APPROXIMATION AND PROJECTION OPERATOR TECHNIQUES

We consider the phase of the three wave modes changes rapidly at random times during the process of the nonlinear interaction. In such a case, the phases of wavevectors are distributed randomly on all the components of a wave ensemble. As a consequence, the wave ensemble statistical average can be superseded in favour of the time-average. Note that the time-averaged amplitude $\langle A_K\rangle$ vanishes, when averaged over a time τ which is much larger than the average period of the phase change but smaller than the nonlinear interaction time scale T. This occurs because

$$\langle A_K\rangle \triangleq \frac{1}{\tau}\int_0^\tau |A_K(T)|e^{i\varphi_K(t)}dt, \tag{12}$$

where $\varphi_K(t)$ represents the phase of A_K which is rapidly varying compared with the time scale of T, hence

$$\langle A_K\rangle=|A_K(T)|\langle\cos\varphi_K(t)+i\sin\varphi_K(t)\rangle=0. \tag{13}$$

When the nonlinear interaction starts to take plase, however, such a randomly changing phase of the three wave modes will begin to be correlated. If we write A_K as a sum of the terms with a completely random phase $A_K^{(0)}$ and a small term but with a correlated phase $A_K^{(1)}$,

$$A_K=A_K^{(0)}+A_K^{(1)}, \tag{14}$$

where

$$\langle A_K^{(0)}\rangle=0.$$

To the average of the product of the two wave modes amplitudes, we have

$$\langle A_{K'}A_{K''}\rangle=\langle A_{K'}^{(0)}A_{K''}^{(0)}\rangle+\langle A_{K'}^{(1)}A_{K''}^{(0)}\rangle+\langle A_{K'}^{(0)}A_{K''}^{(1)}\rangle$$
$$=\langle A_{K'}A_{K''}^{(0)}\rangle+\langle A_{K'}^{(0)}A_{K''}\rangle. \tag{15}$$

And the average of the product of the three wave modes amplitudes becomes

$$\langle A_KA_{K'}A_{K''}\rangle=\langle A_KA_{K'}^{(0)}A_{K''}^{(0)}\rangle+\langle A_K^{(0)}A_{K'}A_{K''}^{(0)}\rangle+\langle A_K^{(0)}A_{K'}^{(0)}A_{K''}\rangle. \tag{16}$$

In addition, the average of the products of the four amplitudes can be reduced to sums of products of two amplitudes because of the assumed statistical independence. That is

$$\langle ABCD\rangle=\langle AB\rangle\langle CD\rangle+\langle AC\rangle\langle BD\rangle+\langle AD\rangle\langle BC\rangle. \tag{17}$$

The above equation is the same as the Eq. (6.7) of Salmon (1982).

Now, the average of the products of the two amplitudes which are statistically independent, such as $\langle A_{K'}A^*_{K''}\rangle$, has a non-vanishing value only when $K'=K''$, hence

$$\langle A_{K'}A^*_{K''}\rangle=\langle |A_{K'}|^2\delta_{K',K''} \quad . \tag{18}$$

The use of this relation is known as RANDOM PHASE APPROXIMATION.

Furthermore, note that there is a quite difference between the two time scales t and T, the time variation of the amplitude A_K due to the nonlinear interaction is slow compared to that associated with the phase which fluctuates rapidly. It can be regarded that the slowly relaxing variables as a "slow subspace" and the random phase as an element of "fast subspace" which is orthogonal of this "slow subspace". Thus there will be a large separation in the time scales, then A_K in the lowest order can be obtained by integrating the coupled Eqs. (11 a–11 c) from $t=-\infty$ to t by means of the PROJECTION OPERATOR TECHNIQUES (see Berne, 1977), Hence,

$$A_{K_0}\approx\sum_{m_{K_0}=m_{K'}+m_{K''}}\frac{1}{\Delta\sigma_{rK_0}-i0^+}[G_{K_0}(K',K'')A^{(0)}_{K'}A^{(0)}_{K''}$$

$$+F_{K_0}(K',K'')A^{(0)}_{K'}\bar{H}_{K''}]e^{+i(\Delta\sigma_{rK_0}-i0^+)t} \tag{19a}$$

$$A_{K_1}\approx-\sum_{m_{K_1}=m_K-m_{K''}}\frac{1}{\Delta\sigma_{rK_1}+i0^+}[C_{K_1}(K''^*,K)A^{(0)*}_{K''}A^{(0)}_{K}$$

$$-F_{K_1}(K''^*,K)\bar{H}^*_{K''}A^{(0)}_{K}]e^{-i(\Delta\sigma_{rK_1}+i0^+)t} \tag{19b}$$

$$A_{K_2}\approx-\sum_{m_{K_2}=m_K-m_{K'}}\frac{1}{\Delta\sigma_{rK_2}+i0^+}[G_{K_2}(K,K'^*)A^{(0)}_{K}A^{(0)*}_{K'}]$$

$$\times e^{-i(\Delta\sigma_{rK_2}+i0^+)t}, \tag{19c}$$

where the items $\pm i0^+$ make the integral convergent. This causality is very important to show the characters of dissipation (see Prigoging 1975).

IV. ATTRACTOR SET AND MULTIPLE EQUILIBRIA SOLUTIONS

With the above preparations, we now construct the time differential equation for the $\langle|A_{K_0}|^2\rangle$s. First consider $\langle|A_{K_0}|^2\rangle$, multiply Eq. (11 a) by $A^*_{K_0}$ and add to it the product of A_{K_0} and the complex conjugate of Eq. (11 a); then operate wave ensemble average. Thus we have

$$\frac{\partial}{\partial T}\langle|A_{K_0}|^2\rangle=i\sum_{m_{K_0}=m_{K_1}+m_{K_2}}[G_{K_0}(K_1,K_2)\langle A^*_{K_0}A_{K_1}A_{K_2}e^{+i(\Delta\sigma_{rK_0}-i0^+)t}\rangle$$

$$+F_{K_0}(K_1,K_2)\bar{H}_{K_2}\langle A^*_{K_0}A_{K_1}e^{+i(\Delta\sigma_{rK_0}-i0^+)t}\rangle]+C.C. \tag{20}$$

In this expression, we have switched the dummy-variables K' 'K'' to K_1 and K_2 since the wave ensemble average is used. In case $m_{K_2}=0$, i.e. the mode A_{K_2} represents the zonal mean flow, we obtain (see Appendix)

$$\frac{\partial}{\partial T}\langle|A_{K_0}|^2\rangle=\iint dm_K\, d\sigma_r \delta(m_{K_0}-m_{K_1}-m_{K_2})\,\delta(\sigma_{rK_0}-\sigma_{rK_1}-\sigma_{rK_2})$$
$$\times[-4\pi G^2\langle|A_{K_0}|^2\rangle^2+4\pi F^2\bar{H}^2\langle|A_{K_0}|^2\rangle], \tag{21a}$$

where

$$G^2\hat{=}|G_{K_0}(K_1,K_2)|^2=|G_{K_1}(K_2^*,K_0)|^2=|G_{K_2}(K_0,K^*)|^2 \tag{22}$$

$$F^2\hat{=}[S_{K_0}(K_1,K_2)/B_{K_0}(K_1,K_2)]^2G^2=\frac{4}{|\Lambda_{K_0}-\Lambda_{K_2}|^2}G^2. \tag{23}$$

similarly, we have

$$\frac{\partial}{\partial T}\langle|A_{K_1}|^2\rangle=\iint dm_K\, d\sigma_r \delta(m_{K_1}-m_{K_1}-m_{K_2})\,\delta(\sigma_{rK_0}-\sigma_{rK_1}-\sigma_{rK_2})$$
$$\times[-4\pi G^2\langle|A_{K_1}|^2\rangle^2+4\pi F^2\bar{H}^2\langle|A_{K_1}|^2\rangle]. \tag{24}$$

We can find out that Eq. (24) is equivalent to Eq. (21 a) as a result of the case $m_{K_2}=0$ *i.e.* $K_0=K_1$.

From Eq. (11 c and d), we can write formally

$$\frac{\partial}{\partial T}\left(A_{K_2}+\frac{F}{G}\bar{H}\right)=\frac{\partial}{\partial T}A_{K_2}=i\sum_{m_{K_2}=m_K-m_{K'}}G_{K_2}(K,K'^*)A_K A^*_{K'}e^{-i(\Delta\sigma_{rK_2}+i0^+)t}. \tag{25}$$

Finally, we obtain

$$\frac{\partial}{\partial T}\left\langle\left|A_{K_2}+\frac{F}{G}\bar{H}\right|^2\right\rangle=\frac{\partial}{\partial T}\left[\langle|A_{K_2}|^2\rangle+\frac{F^2}{G^2}\bar{H}\right]$$
$$=\frac{\partial}{\partial T}\langle|A_{K_2}|^2\rangle=\iint dm_K\, d\sigma_r \delta(m_{K_0}-m_{K_1}-m_{K_2})\,\delta(\sigma_{rK_0}-\sigma_{rK_1}-\sigma_{rK_2})$$
$$\times[4\pi G^2\langle|A_{K_0}|^2\rangle^2-4\pi F^2\bar{H}^2\langle|A_{K_0}|^2\rangle]. \tag{21b}$$

Becuse of the δ -function in the right side of Eq. (21), the nontrivial solution exists only when

$$m_{K_0}=m_{K_1}+m_{K_2} \tag{26a}$$

$$\sigma_{rK_0}=\sigma_{rK_1}+\sigma_{rK_2} \tag{26b}$$

Eq. (26) is just the resonance constraints. By means of the resonance constraints and the character of δ-function, we can obtain the integration on the right side of Eq. (21). Note that $|A_K|^2$ is in the direct ratio of the wave energy and $\langle|A_K|^2\rangle$ is related to the energy spectral density, consequently $\langle|A_K|^2\rangle$ may represents the mean power which is carried by a certain mode of wave ensemble. Let

$$W_K\hat{=}\langle|A_K|^2\rangle, \tag{27}$$

it can be regarded as the mean energy in a certain mode of wave ensemble since the ergodicity of phase exists due to random phase approximation. Hence,

$$\frac{\partial}{\partial T}W_{K_0}=-4\pi G^2W_{K_0}\left[W_{K_0}-\frac{4\bar{H}^2}{|\Lambda_{K_0}-\Lambda_{K_2}|^2}\right] \tag{28a}$$

$$\frac{\partial}{\partial T}W_{K_2}=4\pi G^2W_{K_0}\left[W_{K_0}-\frac{4\bar{H}^2}{|\Lambda_{K_0}-\Lambda_{K_2}|^2}\right]. \tag{28b}$$

On adding Eq. (28 a) to Eq. (28 b), we obtain conservation law

$$\frac{\partial}{\partial T}(W\boldsymbol{K}_0+W\boldsymbol{K}_2)=0. \tag{29}$$

We have

$$W\boldsymbol{K}_0+W\boldsymbol{K}_2=C_W. \tag{3c}$$

It means that the total energy of the resonant system is conservative through the course of nonlinear resonance of topographically forced waves.

According to the dynamical systems theory, Eq. (28) can be regarded as a two-dimensional autonomous system. The topographic function $\tilde{H}^2$ and the order $|\Lambda\boldsymbol{K}_0-\Lambda\boldsymbol{K}_2|^2$ are the parameters. Before giving the exact solution, let us analyze the system itself. There are two sorts of equilibrium point in the system (28):

$$(W\boldsymbol{K}_0=0,\quad W\boldsymbol{K}_2=C_W); \tag{31a}$$

and

$$\left(W\boldsymbol{K}_0=\frac{4\tilde{H}^2}{|\Lambda\boldsymbol{K}_0-\Lambda\boldsymbol{K}_2|^2},\ W\boldsymbol{K}_2=C_W-\frac{4\tilde{H}^2}{|\Lambda\boldsymbol{K}_0-\Lambda\boldsymbol{K}_2|^2}.\right. \tag{31b}$$

They are also the singularities in Eq. (28).

Since Eq. (28 a) is a special case of Riccati Equation, the exact solution of Eq. (28) can be obtained as follows

$$\begin{cases} W\boldsymbol{K}_0(T)=\dfrac{\dfrac{4\tilde{H}^2}{|\Lambda\boldsymbol{K}_0-\Lambda\boldsymbol{K}_2|^2}}{\left(1+\dfrac{4\tilde{H}^2}{|\Lambda\boldsymbol{K}_0-\Lambda\boldsymbol{K}_2|^2W\boldsymbol{K}_0(0)}-1\right)e^{-4\pi G^2\frac{4\tilde{H}^2}{|\Lambda\boldsymbol{K}_0-\Lambda\boldsymbol{K}_2|^2}T}} \\ W\boldsymbol{K}_2(T)=C_W-W\boldsymbol{K}_0(T) \end{cases} \tag{32}$$

For all $W\boldsymbol{K}_0(0)>0$, we always have

$$\begin{cases} W\boldsymbol{K}_0(T\to\infty)=\dfrac{4\tilde{H}^2}{|\Lambda\boldsymbol{K}_0-\Lambda\boldsymbol{K}_2|^2} \\ W\boldsymbol{K}_2(T\to\infty)=C_W=\dfrac{4\tilde{H}^2}{|\Lambda\boldsymbol{K}_0-\Lambda\boldsymbol{K}_2|^2}. \end{cases} \tag{33}$$

Hence, Eq. (31b) is the asymptotically stable equilibrium point and, consequently, it can be regarded as the Attractors (see Lorenz, 1980).

When $W\boldsymbol{K}_0(0)=0$, we have

$$\begin{cases} W\boldsymbol{K}_0(T)=0 \\ W\boldsymbol{K}_2(T)=C_W. \end{cases} \tag{34}$$

Hence, Eq. (30a) is an isolated unstable equilibrium point.

In order to have a better understanding of the character of Eq. (31 a), we assume that $\tilde{H}^2=0$, i.e. there is no topographic force in the system. Then Eq. (28) reduces to

$$\frac{\partial}{\partial T}W\boldsymbol{K}_0=-4\pi G^2W\boldsymbol{K}_0^2 \tag{35a}$$

$$\frac{\partial}{\partial T}W\boldsymbol{K}_2=4\pi G^2W\boldsymbol{K}_0^2. \tag{35b}$$

In such a case, the exact solution of Eq. (34) can readily be given (Li, 1985)

$$\begin{cases} W\boldsymbol{K}_0(T)=\dfrac{W\boldsymbol{K}_0(0)}{1+4\pi G^2 W\boldsymbol{K}_0(0)T} \\ W\boldsymbol{K}_2(T)=C_W-\dfrac{W\boldsymbol{K}_0(0)}{1-4\pi G^2 W\boldsymbol{K}_0(0)T}. \end{cases} \tag{36}$$

For all $W\boldsymbol{K}_0(0)$, we always have

$$\begin{cases} W\boldsymbol{K}_0(T\to\infty)=0 \\ W\boldsymbol{K}_2(T\to\infty)=C_W. \end{cases} \tag{37}$$

Eq. (31 a) therefore becomes an asymptotically stable equilibrium point, i.e. an attractor. Since the total energy C_W is an arbitrary constant, in this case, every point on the positive $W\boldsymbol{K}_2$ axis is an equilibrium point for system (35) Therefore, it is an attractor set.

V. DISCUSSION AND CONCLUSIONS

Utilizing a complete spectral model without truncation in the barotropic atmosphere with bottom topographic forcing, we have shown that the fixed point ($W\boldsymbol{K}_0=0$, $W\boldsymbol{K}_2=C_W$) is a bifurcation point of the dynamical system (28). When the parameter $\tilde{H}^2$ passes the critical value 0 and increases, the dynamical system loses stability and further bifurcation takes place.

In a preliminary work (Li, 1985), we have shown that the energy of finite-amplitude disturbance wave ensemble can be completely transferred into the zonal mean flow by nonlinear resonance interaction when $\tilde{H}^2=0$. The energy transfer process would continue until the motion becomes purely zonal flow. Since the equilibrium point ($W\boldsymbol{K}_0=0, W\boldsymbol{K}_2=C_W$) is a stable point in the case of $\tilde{H}^2=0$, the initial energy of finite-amplitude disturbance wave ensemble $W\boldsymbol{K}_0(0)$ should be given by the external thermal source which is outside the barotropic system. We emphasize that our barotropic model neglects diabatic heating. Nevertheless, it is hoped that this kind of initial condition also may lead to valuable insights into the baroclinic effects.

We now discuss the case $\tilde{H}^2>0$ as follows:

(1) $W\boldsymbol{K}_0(0)=0$

We will have unstable equilibrium mathematically as mentioned in Section IV, which, in practice, is hardly observed to occur.

(2) $W\boldsymbol{K}_2(0)>\dfrac{4\tilde{H}^2}{|A\boldsymbol{K}_0-A\boldsymbol{K}_2|^2}\gg W\boldsymbol{K}_0(0)$

At an early stage of the instability, $W\boldsymbol{K}_0$ will grow exponentially since the linear item of Eq. (28). When $W\boldsymbol{K}_0$ becomes larger, the nonlinear damping will gradually stabilize the linear instability through the nonlinear item $-4\pi G^2 W\boldsymbol{K}_0^2$. It is clear that the only way the topographically forced waves can grow is by extracting energy from the zonal mean flow. The behavior of the system (28) is shown in Figs. 1 and 2.

(3) $W\boldsymbol{K}_0(0)=\dfrac{4\tilde{H}^2}{|A\boldsymbol{K}_0-A\boldsymbol{K}_2|^2}$

The stable equilibrium will continue. (see Fig. 3). As a specific case, we finally cosider

$$C_W\approx W\boldsymbol{K}_2(0)\approx\frac{4\tilde{H}^2}{|A\boldsymbol{K}_0-A\boldsymbol{K}_2|^2}\gg W\boldsymbol{K}_0(0)\approx 0,$$

i.e., the topographic function is extremely strong. It happened that the topograph was large and, relatively, the zonal flow was weaker.

In such a case, the zonal mean flow would be broken down since most of the energy have been extracted by the strongly growing up resonant topographically forced waves. This result agrees with CD, which suggested that the phenomenon of blocking is a metastable equilibrium state of the low-index, near-resonant character. We can sketch families of trajectories in the W_{K_0}——W_{K_3} plane as Fig. 4, in which, the arrows signify increasing time T.

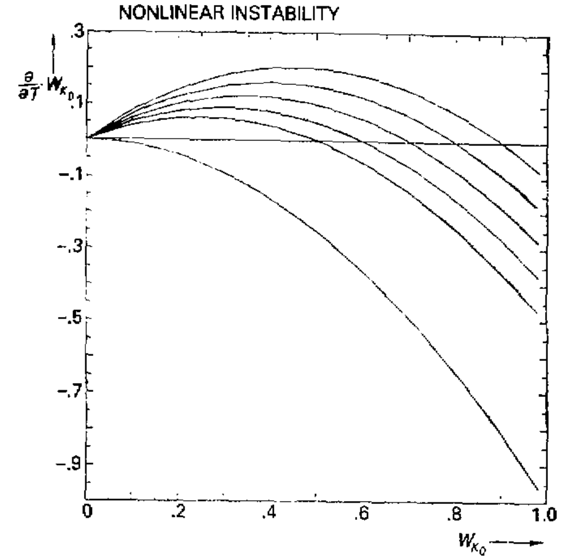

Fig. 1. The nonlinear instabilities for the topographically forced waves and the case of a purely dissipative system. Topographic functions are equal to zero, and 0.5; 0.6; 0.7; 0.8; 0.9, respectively.

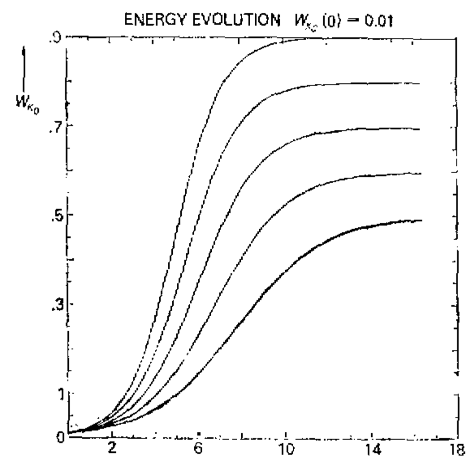

Fig. 2. Time variation of the energy of topographically forced waves. Topographical functions are equal to 0.5; 0.6; 0.7; 0.8; 0.9, respectively.

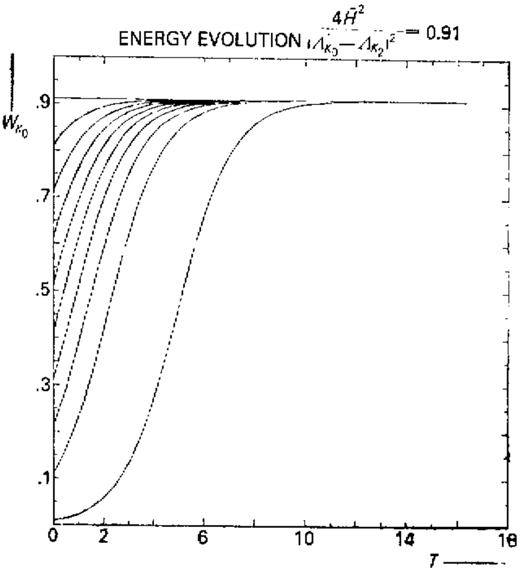

Fig. 3. An attractor for the topographically forced system. Initial values are equal to 0.01; 0.11; 0.21; 0.31; 0.41; 0.51; 0.61; 0.71; 0.81; 0.91; respectively.

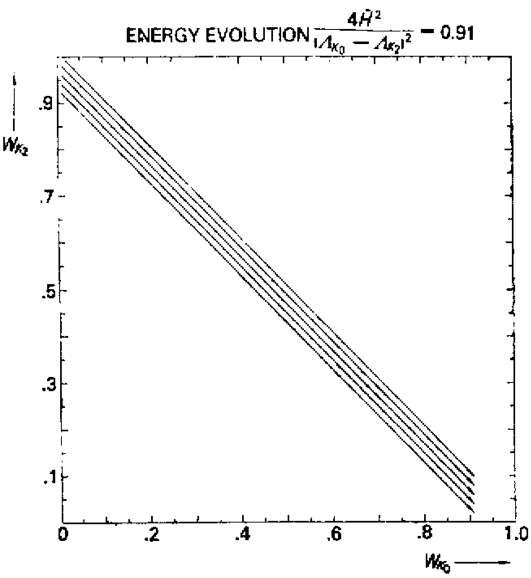

Fig. 4. The trajectories in the $W_{K_0}-W_{K_2}$ phase plane. Initial values of (W_{K_0}, W_{K_2}) are equal to (0.01, 0.92); (0.01, 0.94); (0.01, 0.96); (0.01, 0.98); (0.01, 1.00), respectively. The arrows signify increasing time T.

As indicated by the above discussion, the orographic forcing resonance is important in changing the pattern of the energy transfer on wave-wave interactions. It appears that the orographic forcing may be an important barotropic influence on the self-excited zonal mean flow. In the present study only the effect of topography was investigated. It seems that future efforts must be directed towards determining the behaviour of nonlinear baroclinic waves.

Finally, these results also reveal that some results in the low-order models have a validity to approximate the nature QUALITATIVELY (Lorenz, 1982). However, it is always necessary to assess the effect of truncation in the low-order models.

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APPENDIX

If we now make use of the expressions (15), (16) for the average of the products $A^*_{K_0}A_{K_1}A_{K_2}$, $A_{K_0}A^*_{K_1}A^*_{K_2}$, $A^*_{K_0}A_{K_1}$, and $A_{K_0}A^*_{K_1}$ in Eq. (20), there will appear ten terms inside the bracket. To evaluate the average of each term, we take the first conjugate pair out of the five pairs and designate it by the quantity a_1:

$$a_1 \hat{=} G_{K_0}(K_1,K_2) < A^*_{K_0}\, A^{(0)}_{K_1}\, A^{(0)}_{K_2} e^{+i(\Delta\sigma_{rK_0}-i0^+)t} \rangle + C.C. \qquad (A1)$$

On substituting Eq. (19 a) into Eq. (20), the exponential term in the wave ensemble average will be eliminated, and thus we have

$$a_1 = \sum_{m_{K_0}=m_{K'}+m_{K''}} \left[\frac{|G_{K_0}(K',K'')|^2}{\Delta\sigma_{rK_0}+i0^+} \langle A^{(0)*}_{K'}\, A^{(0)*}_{K''}\, A^{(0)}_{K_1}\, A^{(0)}_{K_2} \rangle \right.$$

$$\left. + \frac{G_{K_0}(K',K'')F^*_{K_0}(K',K'')\bar{H}^*_{K''}}{\Delta\sigma_{rK_0}+i0^+} \langle A^{(0)*}_{K'}\, A^{(0)}_{K_1}\, A^{(0)}_{K_2} \rangle \right] + C.C. \qquad (A2)$$

By means of Random Phase Approximation, the term $\langle A^{(0)*}_{K'}\, A^{(0)}_{K_1}\, A^{(0)}_{K_2}\rangle$ becomes zero and the term $\langle A^{(0)*}_{K'}\, A^{(0)*}_{K''}\, A^{(0)}_{K_1}\, A^{(0)}_{K_2}\rangle$ becomes

$$\langle A^{(0)*}_{K'}\, A^{(0)*}_{K''}\, A^{(0)}_{K_1}\, A^{(0)}_{K_2}\rangle = \langle |A^{(0)}_{K''}|^2\rangle \delta_{K',K''^*} \langle |A^{(0)}_{K_1}|^2\rangle \delta_{K_1,K_2^*}$$

$$+ \langle |A^{(0)}_{K_1}|^2\rangle \delta_{K_1,K'} \langle |A^{(0)}_{K_2}|^2\rangle \delta_{K_2,K''}$$

$$+ \langle |A^{(0)}_{K_2}|^2\rangle \delta_{K_2,K'} \langle |A^{(0)}_{K_1}|^2\rangle \delta_{K_1,K''}. \qquad (A3)$$

The first and the third terms on the right-hand side of this expression do not contribute. We can derive a similar expression for the $C.C.$ term. Substituting these expressions into a_1, we have

$$a_1 = \sum_{m_{K_0}=m_{K_1}+m_{K_2}} \left[\frac{|G_{K_0}(K_1,K_2)|^2}{\Delta\sigma_{rK_0}+i0^+} - \frac{|G_{K_0}(K_1,K_2)|^2}{\Delta\sigma_{rK_0}-i0^+} \right] \langle |A^{(0)}_{K_1}|^2\rangle \langle |A^{(0)}_{K_2}|^2\rangle$$

$$(A4)$$

Using the relations

$$|G_{K_0}(K_1,K_2)|^2=|G_{K_1}(K_2^*,K_0)|^2=|G_{K_2}(K_0,K_1^*)|^2\widehat{=}G^2$$

$$\Delta\sigma_{rK_0}=\Delta\sigma_{rK_1}=\Delta\sigma_{rK_2}\equiv\sigma_{rK_0}-\sigma_{rK_1}-\sigma_{rK_2}\widehat{=}\Delta\sigma_r\ ,$$

the second conjugate pairs inside the bracket in Eq. (20) can also be evaluated to give

$$a_2=-\sum_{m_{K_1}=m_{K_0}-m_{K_2}}\left[\frac{G^2}{\Delta\sigma_r+i0^+}-\frac{G^2}{\Delta\sigma_r-i0^+}\right]\langle|A_{K_2}^{(0)}|^2\rangle\ \langle|A_{K_0}^{(0)}|^2\rangle. \tag{A5}$$

In the third conjugate pair, however,

$$a_3=-\sum_{m_{K_2}=m_K-m_{K'}}\frac{G^2}{\Delta\sigma_r+i0^+}\langle A_{K_0}^{(0)*}A_{K_1}^{(0)*}A_K^{(0)}A_{K'}^{(0)*}\rangle+C.C. \tag{A6}$$

Note that

$$\langle A_{K_0}^{(0)*}A_{K_1}^{(0)}A_K^{(0)}A_{K'}^{(0)*}\rangle=\langle|A_{K_1}^{(0)}|^2\rangle\delta_{K_1,K_0}\langle|A_K^{(0)}|^2\rangle\delta_{K,K'}$$
$$+\langle|A_K^{(0)}|^2\rangle\delta_{K,K_0}\langle|A_{K_1}^{(0)}|^2\rangle\delta_{K_1,K'}$$
$$+\langle|A_{K_2}^{(0)}|^2\rangle\delta_{K_0,K'}^{*}\langle|A_{K_1}^{(0)}|^2\rangle\delta_{K_1,K^*}. \tag{A7}$$

If $K=K'$, $m_{K_2}(=m_K-m_{K'})$ becomes zero, the mode A_{K_2} therefore is the zonal mean flow. Let us open discussion into that case. We have

$$a_3=-\sum_{m_{K_2}=m_{K_0}-m_{K_1}}\left[\frac{G^2}{\Delta\sigma_r+i0^+}-\frac{G^2}{\Delta\sigma_r-i0^+}\right]2\langle|A_{K_0}^{(0)}|^2\rangle^2 \tag{A8}$$

and

$$a_1+a_2=0\ . \tag{A9}$$

The fourth conjugate pair can be

$$a_4\widehat{=}F_{K_0}(K',K'')H_{K_2}\langle A_{K_0}^*A_{K_1}^{(0)}e^{-i(\sigma_{rK_0}-i0^+)t}\rangle+C.C.$$

$$=\sum_{m_{K_0}=m_{K'}+m_{K''}}\left[\frac{F_{K_0}(K',K'')\bar{H}_{K''}G_{K_0}(K',K'')}{\Delta\sigma_{rK_0}+i0^+}\langle A_{K'}^{(0)*}A_{K''}^{(0)*}A_{K_1}^{(0)}\rangle\right.$$
$$\left.+\frac{F^2\bar{H}^2}{\Delta\sigma_r+i0^+}\langle A_{K'}^{(0)*}A_{K_1}^{(0)}\rangle\right]-C.C.$$

$$=\sum_{m_{K_0}=m_{K_1}+m_{K_2}}\left[\frac{F^2\bar{H}^2}{\Delta\sigma_r+i0^+}-\frac{F^2\bar{H}^2}{\Delta\sigma_r-i0^+}\right]\langle|A_{K_1}^{(0)}|^2\rangle. \tag{A10}$$

Similarly, the last conjugate pair is

$$a_5=-\sum_{m_{K_1}=m_K-m_{K''}}\left[\frac{F_{K_1}(K',K'')\bar{H}_{K''}G_{K_1}(K''^*,K)}{\Delta\sigma_{rK_1}+i0^+}\langle A_{K_0}^{(0)*}A_{K''}^{(0)*}A_K^{(0)}\rangle\right.$$
$$\left.-\frac{F^2\bar{H}^2}{\Delta\sigma_r+i0^+}\langle A_{K_0}^{(0)*}A_K^{(0)}\rangle\right]+C.C.$$

$$= \sum_{m_{K_1} = m_{K_0} - m_{K_2}} \left[-\frac{F^2 \tilde{H}^2}{\Delta\sigma_r + i0^+} - \frac{F^2 \tilde{H}^2}{\Delta\sigma_r - i0^+} \right] \langle | A_{K_0}^{(0)} |^2 \rangle , \tag{A11}$$

Obviously, we have

$$a_4 = a_5 \ . \tag{A12}$$

Substitute these results into Eq. (20) and remove the superscript (0) which is no longer needed to consider the time development in the T scale. Furthermore, we assume a continuous zonal wavenumbers spectrum, the summation sign becomes integral. By means of the formulation of principal-valued integration

$$\frac{1}{\Delta\sigma_r \mp i0^+} = P.V. \frac{1}{\Delta\sigma_r} \pm i\pi\delta(\Delta\sigma_r) , \tag{A13}$$

we finally obtain:

$$\frac{\partial}{\partial T} \langle | A_{K_0} |^2 \rangle = \iint dm_K d\sigma_r \delta(m_{K_0} - m_{K_1} - m_{K_2}) \, \delta(\sigma_{rK_0} - \sigma_{rK_1} - \sigma_{rK_2}) \times [-4\pi G^2 \langle | A_{K_0} |^2 \rangle^2 + 4\pi F^2 \tilde{H}^2 \langle | A_{K_0} |^2 \rangle] \tag{21a}$$

Similarly, we have

$$\frac{\partial}{\partial T} \langle | A_{K_1} |^2 \rangle = \iint dm_K d\sigma_r \delta(m_{K_0} - m_{K_1} - m_{K_2}) \, \delta(\sigma_{rK_0} - \sigma_{rK_1} - \sigma_{rK_2}) \times [-4\pi G^2 \langle | A_{K_1} |^2 \rangle^2 + 4\pi F^2 \tilde{H}^2 \langle | A_{K_1} |^2 \rangle] . \tag{24}$$

From Eq. (11c and d), we can write formally

$$\frac{\partial}{\partial T} \left(A_{K_2} + \frac{F}{G} \tilde{H} \right) = \frac{\partial}{\partial T} A_{K_2} = i \sum_{m_{K_2} = m_K - m_{K'}} [G_{K_2}(K, K'^*) A_K A_{K'}^*] e^{-i(\Delta\sigma_{rK_2} + i0^+)t} . \tag{25}$$

Multiply Eq. (25) by $\left(A_{K_2} + \frac{F}{G} \tilde{H} \right)^*$ and add to it the product of $\left(A_{K_2} + \frac{F}{G} \tilde{H} \right)$ and the complex conjugate of Eq. (25), then operate on wave ensemble average, we have

$$\frac{\partial}{\partial T} \left\langle \left| A_{K_2} + \frac{F}{G} \tilde{H} \right|^2 \right\rangle = i \sum_{m_{K_2} = m_K - m_{K'}} [G \langle A_{K_2}^* A_{K_0} A_{K_1}^* e^{-i(\Delta\sigma_r + i0^+)t} \rangle + F^* \tilde{H}^* \langle A_{K_0} A_{K_1}^* e^{-i(\Delta\sigma_r + i0^+)t} \rangle] + C.C. \tag{A15}$$

Recall Eq. (13) and Eq. (11 d), we still have

$$\frac{\partial}{\partial T} \left\langle \left| A_{K_2} + \frac{F}{G} \tilde{H} \right|^2 \right\rangle = \frac{\partial}{\partial T} \left[\langle | A_{K_2} |^2 \rangle + \frac{F^2}{G^2} \tilde{H}^2 \right] = \frac{\partial}{\partial T} \langle | A_{K_2} |^2 \rangle . \tag{A16}$$

Finally, we obtain

$$\frac{\partial}{\partial T} \langle | A_{K_2} |^2 \rangle = \iint dm_K d\sigma_r \delta(m_{K_0} - m_{K_1} - m_{K_2}) \, \delta(\sigma_{rK_0} - \sigma_{rK_1} - \sigma_{rK_2}) \times [4\pi G^2 \langle | A_{K_0} |^2 \rangle^2 - 4\pi F^2 \tilde{H}^2 \langle | A_{K_0} |^2 \rangle] \tag{21b}$$

Eq. (21) is our topographic waves resonance coupling equations.

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