THE WIND IN THE BAROCLINIC BOUNDARY LAYER WITH THREE SUBLAYERS INCORPORATING THE WEAK NON-LINEAR EFFECT

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ABSTRACT

In considering the weak non-linear effect, and using the small parameter expansion method, the analytical expressions of the wind distribution within PBL (planetary boundary layer) and the vertical velocity at the top of the PBL are obtained when the PBL is divided into three layers and different eddy transfer coefficients K are adopted for the three layers. The conditions of barotropy and neutrality for the PBL are extended to that of baroclinity and non-neutral stratification. An example of a steady circular vortex is used to display the characteristics of the horizontal wind within the PBL and the vertical velocity at the top of the PBL. Some new results have been obtained, indicating that the magnitude of the speed in the lower height calculated by the present model is larger than that by the model in which k is a constant within the whole boundary layer, for example, in the classical Ekman boundary layer model and the model by Wu (1984). The angle between the wind at the top of the PBL and the wind near the surface calculated by the present model is less than that calculated by the single K model. These results are in agreement with the observations.

I. INTRODUCTION

In the theory of the classical Ekman boundary layer, air within the PBL moves under the balance of three forces: the pressure gradient force, the Coriolis force and the frictional force. Wu and Blumen (1982) introduced the geostrophic momentum approximation suggested by Hoskins (1975) into the PBL, and improved the theory of the Ekman boundary layer. Near the center of a circular vortex, however, the wind approaches the gradient wind and deviates largely from the geostrophic wind. Introducing non-linear effect, Wu further improved the application of the geostrophic momentum approximation within the PBL by means of a small parameter expansion method, which is helpful to improve the parameterization of the PBL and to research the feedback of the PBL to the free atmosphere.

Wu's work is the basis of this paper. Here, the boundary layer is divided into three layers and the different eddy transfer coefficients K are adopted for the three layers instead of the single K model. The first layer is the surface layer where the atmospheric motion may be described as follows: the turbulent viscous stress is a constant and therefore, the well-known logarithmic distribution of wind under the neutral condition can be obtained. The second and third layers form the Ekman layer where the motion equations include the terms of non-linear advection. To get the approximational solutions, a small parameter expansion method will be adopted in the present study. In the three layers, the equations describing the motions are different and so are the numerical values of k. These cause some difficulties for solving the equations. But by using the connecting conditions that both the wind and the viscous stress are continuous on the interfaces between each two layers, the wind in the three layers may be obtained easily and uniquely. Furthermore, by means of the continuity equation, the vertical velocity at the top of the PBL can be gained.

In the Ekman layer, the first-order approximation solution is similar to the classical Ekman solution which is obtained under the condition that the geostrophic wind is steady and homogeneous. In the second-order approximation solution, the effects of the local and advective change terms are included, that is, the influences of nonsteady and inhomogeneous of geostrophic wind are incorporated. Furthermore, in adopting the three sections K model, i. e. considering the variation of turbulent viscosity with height, it can be avoided that the magnitude of the speed calculated by the single K model is so small for the surface layer that the results of the present model are more in agreement with the observations

There are six sections in this paper. The governing equations are given and the horizontal velocity is derived in Section 2; the height of the top of the PBL and the vertical velocity there are derived and several factors affecting the vertical velocity are simply interpreted in Section 3; an example of steady circular vortex is introduced and the results of the present solutions are diagramatically analyzed in Section 4. Above-mentioned results under the conditions of barotropy and neutrality are extended into that of baroclinity and non-neutral stratification in Section 5; some brief conclusions are given.

II. BASIC EQUATIONS

According to Wu, the set of the atmospheric motion equations are:

$$\begin{aligned}
&\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}=-\frac{\partial \phi}{\partial x}+fv-\frac{\partial}{\partial z}\left(k\frac{\partial u}{\partial z}\right),\\
&\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}=-\frac{\partial \phi}{\partial y}-fu+\frac{\partial}{\partial z}\left(k\frac{\partial v}{\partial z}\right),\\
&\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0,
\end{aligned}\tag{1}$$

where (u,v,w) are the velocity components along the coordinate axes (x,y,z), ϕ refers to the geopotential deviation, f, the constant Coriolis parameter, and k the eddy transfer coefficient.

In order to get the non-dimensional equations, the following expressions are commonly used:

$$\begin{aligned}
&(x,y)=L(x',\ y'),\ \ z=H_0z',\\
&(u,v)=V(u',\ v'),\ \ w=Ww',\\
&\phi=\Phi\phi',\ \ k=Kk',\\
&\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=D\left(\frac{\partial u'}{\partial x'}+\frac{\partial v'}{\partial y'}\right),\ \ \Phi=fVL,\\
&D=\frac{W}{H}+R_0\frac{V}{L},\qquad R_0=\frac{V}{fL},
\end{aligned}\tag{2}$$

where R_0 is the Rossby number. For the PBL, $R_0<1$, for example, $R_0\leqslant 0.3$. Substituting (2) into (1), (1) is reduced to:

$$R_0\left(\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+R_0 w\frac{\partial u}{\partial z}\right)=-\frac{\partial \phi}{\partial x}+v+\frac{1}{2}E\frac{\partial}{\partial z}\left(k\frac{\partial u}{\partial z}\right),$$

$$R_0\left(\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+R_0 w\frac{\partial v}{\partial z}\right)=-\frac{\partial \phi}{\partial y}-u+\frac{1}{2}E\frac{\partial}{\partial z}\left(k\frac{\partial v}{\partial z}\right), \quad (3)$$

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0,$$

where primes have been omitted. E is Ekman number:

$$E=\frac{2K}{fH_0^2}. \quad (4)$$

The PBL is divided into the surface layer and the Ekman layer, the latter is divided into two layers again. There are three layers on the whole. k in the three sections model may be written as follows:

$$k=\begin{cases} k_1 z, & \text{(First layer: } z_0\leqslant z<h_1) \\ k_2=k_1h_1, & \text{(Second layer: } h_1\leqslant z<h_2) \\ k_3=\gamma k_2, & \text{(Third layer: } h_2\leqslant z) \end{cases} \quad (5)$$

where γ is a positive number less than 1.

Physically, the frictional force is very important within the PBL and can not be omitted; mathematically, although E is a small parameter, it precedes the highest order derivative and the equations are singular. If all terms containing E are neglected, the solutions can not satisfy the full boundary conditions that at the top of the PBL, the wind equals the wind of the free atmosphere and at the bottom of the PBL, the wind equals the wind at the earth's surface.

In the first layer, following the folding model (Gandin, 1958), we can find that the wind (u_1, v_1) will satisfy following equations:

$$k\,\partial u_1/\partial z=\text{constant},\quad k\,\partial v_1/\partial z=\text{constant}. \quad (6)$$

That is, the viscous stresses do not change with height, in other words, the transfer fluxes of turbulent momentum in vertical direction are constants. Therefore, this layer also refers to the layer of constant flux.

In the Ekman layer, the Coriolis and frictional forces are approximately the same order of magnitude. Introducing the stretch transformation:

$$\eta=E^{-\frac{1}{2}}z, \quad (7)$$

and setting:

$$\tilde{w}=E^{-\frac{1}{2}}w, \quad (8)$$

we obtain the motion equations of the second and third layer

$$\frac{1}{2}k_n\frac{\partial^2 u_n}{\partial \eta^2}+v_n=v_g+R_0\left(\frac{\partial u_n}{\partial t}+u_n\frac{\partial u_n}{\partial x}+v_n\frac{\partial v_n}{\partial y}+R_0\tilde{w}_n\frac{\partial u_n}{\partial \eta}\right),$$

$$\frac{1}{2}k_n\frac{\partial^2 v_n}{\partial \eta^2}-u_n=-u_g+R_0\left(\frac{\partial v_n}{\partial t}+u_n\frac{\partial v_n}{\partial x}+v_n\frac{\partial v_n}{\partial y}+R_0\tilde{w}_n\frac{\partial v_n}{\partial \eta}\right),$$

$$\frac{\partial u_n}{\partial x}+\frac{\partial v_n}{\partial y}+\frac{\partial \tilde{w}_n}{\partial \eta}=0, \quad (9)$$

$$(n=2,\quad E^{-\frac{1}{2}}h_1=\eta_1<\eta<\eta_2=E^{-\frac{1}{2}}h_2;\ n=3,\ \eta_2\leqslant\eta).\ ,$$

where u_g, v_g are geostrophic winds. The upper and lower boundary conditions of the PBL are:

$$\eta\longrightarrow\infty,\ \text{u, v are limited},$$
$$z=z_0,\qquad u=0,\qquad v=0, \tag{10}$$

where z_0 is roughness.

The aim of the paper is to solve the basic equations (6) and (9) under the boundary conditions (10). Since the motion equations in the three layers are different, at the inner boundaries ($z=h_1$, h_2), the connecting conditions must be satisfied. Physically, at $z=h_1$, h_2, the wind and the viscous stress should be and must be continuous mathematically; if these two connecting conditions are satisfied, the solutions (u_n, v_n) $n=1,2,3$ are in existence and unique. This can be seen obviously from the following solving procedure.

In order to avoid reducing two second-order differential equations into a fourth-order differential equation, the complex velocity F_n and the complex geostrophic wind F_g are employed as

$$F_n=u_n+iv_n,\quad (n=1,2,3)$$
$$F_g=u_g+iv_g. \tag{11}$$

First, suppose that the PBL is barotropic, thus the geostrophic wind is independent of the height. Eqs. (9) can be reduced to

$$k_1\eta\frac{\partial F_1}{\partial\eta}=\alpha_1$$

$$\frac{1}{2}k_n\frac{\partial^2 F_n}{\partial\eta^2}-iF_n=-iF_g-R_0\left(\frac{\partial F_n}{\partial t}+\frac{F_n+\bar{F}_n}{2}\frac{\partial F_n}{\partial x}+\frac{F_n-\bar{F}_n}{2i}\frac{\partial F_n}{\partial y}-R_0\bar{w}_n\frac{\partial F_n}{\partial\eta}\right),$$
$$(n=2,\ 3), \tag{12}$$

where the bar above the letters expresses the conjugate complex. α_1 is a pending complex constant. F_n is expanded with the small parameter R_0. For simplicity, only the first-order and the second-order approximation are taken in the paper.

Setting

$$F_n=F_{0n}+R_0F_{1n},\quad (n=1,2,3),$$
$$\alpha_1=\alpha_{01}+R_0\alpha_{11}, \tag{13}$$

and substituting (13) into (12) and classified with for the same power of R_0, we obtain

$$R_0^0:\ k_1\eta\frac{\partial F_{01}}{\partial\eta}=\alpha_{01},$$

$$\frac{1}{2}k_n\frac{\partial^2 F_{0n}}{\partial\eta^2}-iF_{0n}=-iF_g,\ (n=2,3); \tag{14}$$

$$R_0^1:\ k_1\eta\frac{\partial F_{11}}{\partial\eta}=\alpha_{11},$$

$$\frac{1}{2}k_n\frac{\partial^2 F_{1n}}{\partial\eta^2}-iF_{1n}=\frac{\partial F_{0n}}{\partial t}+\frac{F_{0n}+\bar{F}_{0n}}{2}\frac{\partial F_{0n}}{\partial x}+\frac{F_{0n}-\bar{F}_{0n}}{2i}\frac{\partial F_{0n}}{\partial y},$$
$$(n=2,3). \tag{15}$$

The boundary conditions (10) are correspondingly reduced to

$$R_0^0: \eta\to\infty,\ F_{03}\ \text{is limited},$$
$$\eta=\eta_0,\ F_{01}=0; \tag{16}$$
$$R_0^1: \eta\to\infty,\ F_{13}\ \text{is limited},$$
$$\eta=\eta_0,\ F_{11}=0, \tag{17}$$

where $\eta_0=E^{-1/2}z_0$. The formal solutions of (14) under (16) are

$$F_{01}=\frac{\alpha_{01}}{k_1}\ln\frac{\eta}{\eta_1},$$
$$F_{02}=F_g+\alpha_{02}e^{-r_2\eta}+\alpha'_{02}e^{-r_2\eta},$$
$$F_{03}=F_g+\alpha'_{03}e^{-r_3\eta}, \tag{18}$$

where

$$r_n=\frac{1+i}{\sqrt{k_n}},\quad (n=2,3). \tag{19}$$

$\alpha_{01},\alpha_{02},\alpha'_{02},\alpha'_{03}$ are the pending complex constants.

The expressions for the continuity of the viscous stress and the wind at $\eta=\eta_1$ are

$$\alpha_{01}=E^{-\frac{1}{2}}k_2\frac{\partial F_{02}}{\partial\eta}, \tag{20}$$
$$F_{01}=F_{02};$$

at $\eta=\eta_2$ are

$$E^{-\frac{1}{2}}k_2\frac{\partial F_{02}}{\partial\eta}=E^{-\frac{1}{2}}k_3\frac{\partial F_{03}}{\partial\eta},$$
$$F_{02}=F_{03}. \tag{21}$$

Substituting (18) into (20) and (21) yields a closed set of linear algebraic equations for these four complex constants. It is easy to solve the set. The results of the first-order approximation solutions are

$$F_{01}=\frac{A_1}{k_1}F_g\ln\frac{\eta}{\eta_1},$$
$$F_{02}=F_g+A_2F_ge^{r_2\eta}+A'_2F_ge^{-r_2\eta},$$
$$F_{03}=F_g+A'_3F_ge^{-r_3\eta}, \tag{22}$$

where A_1, A_2, A'_2, A'_3 are given in Appendix 1.

The method for solving (15) under (17) is similar to the preceding method, but the special solutions of the inhomogeneous ordinary differential equations are more complex than the aforesaid. Substituting (22) into (15) yields the special solutions of F_{12} and F_{13} Y_2, Y_3 as follows

$$Y_2=Y_2^I+Y_2^B,\quad Y_3=Y_3^I+Y_3^B,$$
$$Y_2^I=Y_3^I=i\frac{d_gF_g}{dt},\quad \frac{d_g}{dt}=\frac{\partial}{\partial t}+u_g\frac{\partial}{\partial x}+v_g\frac{\partial}{\partial y}, \tag{23}$$

Y_2^B, Y_3^B are given in Appendix 2.

(23) shows that Y_2 can be divided into two terms: the non-viscosity term Y_2^I and the viscosity term Y_2^B. The former is only dependent on the pressure field but independent of the viscous friction since it does not include the viscous coefficient. The latter is dependent on both the geostrophic wind of the free atmosphere and the friction of the PBL, which can be seen to include F_g and k_2 For Y_3 the same analysis can be made.

The formal solutions of (15) under (17) may be assumed

$$F_{11}=\frac{\alpha_{11}}{k_1}\ln\frac{\eta}{\eta_0},$$
$$F_{12}=Y_2+\alpha_{12}e^{r_2\eta}+\alpha'_{12}e^{-r_2\eta},$$
$$F_{13}=Y_3+\alpha'_{13}e^{-r_3\eta}, \tag{24}$$

where α_{11}, α_{12}, α'_{12}, α'_{13} are the pending complex constants which can be obtained through solving the closed algebraic equations, i.e. the connecting conditions analogous to (20) and (21). They are given in Appendix 3.

Up to now, the second-order approximation solutions have been obtained. The winds of three layers are

$$F_1=F_{01}+R_0F_{11},$$
$$F_2=F_{02}+R_0F_{12},$$
$$F_3=F_{03}+R_0F_{13}, \tag{25}$$

where F_{01}, F_{02}, F_{03} are given by (22), F_{11}, F_{12}, F_{13} by (24).

Combining the well-known formula

$$k_1=\kappa V_*\tag{26}$$

and

$$V_*^2=\left|k\frac{\partial F_1}{\partial z}\right|,\tag{27}$$

yields

$$k_1^2=\kappa_2|\alpha_1|,\tag{28}$$

where V_* is the frictional velocity, κ is the Von-Karman constant, generally it is taken as 0.4. Eq. (28) is a implicit equation which connects the geostrophic wind and k_1, in dimensionless form, it is reduced to

$$k_1=\kappa\left(\frac{2V}{EfH_0}\right)^{\frac{1}{2}}|A_1F_g+R_0\alpha_{11}|^{\frac{1}{2}},\tag{29}$$

where A_1 and α_{11} have been given in Appendix 1, 3. By use of the iteration method. it is very easy to get k_1 from known pressure field.

Cancelling the surface layer and taking $r=1$, we have

$$\eta_0=\eta_1=0,$$
$$k_2=k_3=1,\tag{30}$$

the solutions (22) and (24) are reduced to

$$F_{02}=F_{03}=F_g-F_ge^{-(1+i)\eta},$$
$$F_{12}=F_{13}=Y+D_{11}e^{-(1+i)\eta},\tag{31}$$

where

$$D_{11}=-A_i+\frac{Ci}{3}+\left(\frac{2}{5}-\frac{3i}{10}\right)D,$$

$$Y=A_i+\frac{A+C}{2}(1-i)\eta e^{-(1+i)\eta}-\frac{Ci}{3}e^{-2(1+i)\eta}+\frac{1}{2}Die^{-(1-i)\eta}-\frac{2+i}{5}De^{-2\eta}.\tag{32}$$

The first expressions of (31) is the same expression as (16) of Wu, the second expression of (31) as (18) of Wu's, showing that the single K model is a special case of the three sections K model.

The present solutions are of many differences from the solutions of the single K model within the first layer, e.g. the former is of the logarithmic distribution of wind which is in agreement with the observations, but the latter is representative of small wind. In single K model, k is taken as an average value in the PBL larger than real k within the first layer; moreover, viscous stress $k\,\partial v/\partial z$ should be a constant, consequently; the shear value $\partial v/\partial z$ of wind is to be very small. Since the velocity at the earth's surface is zero, the wind calculated by the single K model in the first layer is obviously too small.

III. VERTICAL VELOCITY

At the bottom of the free atmosphere, i. e. the top of the PBL, the friction force is negligible, where the weak non-linear wind F_T can be obtained by omitting the terms containing the second-order derivative in (14) and (15) as follows

$$F_T = F_g + R_0 i \frac{d_g F_g}{dt}. \tag{33}$$

It it conventional to designate the height η_T at which the wind for the first time, in the PBL is parallel to F_T at the top of the PBL. Although the wind in the PBL is still variable above the height, its variety in magnitude and direction is very small, only oscillates slightly near F_T, which is only of mathematical significance. In fact, η_T may approximately be regarded as the top of the PBL. Following this traditional method, η_T can be gained from the equation

$$\arg F_3 = \arg F_T. \tag{34}$$

Substituting (25) into the continuity equation (9) and integrating this equation with respect to η, we obtain the vertical velocity at the top of the PBL as follows

$$\begin{aligned}
\tilde{w}_T &= w_T^{(1)} + w_T^{(2)} + w_T^{(3)} + w_T^{(4)}, \\
w_T^{(1)} &= \frac{E^{1/2}}{k_1}\left(\eta_1 - \eta_0 - \eta_1 \ln \frac{\eta_1}{\eta_0}\right)\nabla\cdot(A_1 F_g + R_0 \alpha_{11}), \\
w_1^{(2)} &= R_0 E^{1/2}(\eta_T - \eta_1)\frac{d_g \zeta_g}{dt}, \\
w_T^{(3)} &= -\nabla\cdot\left\{\frac{F_g}{r_2}[A_2(B_{22} - B_{21}) - A_2'(B_{22}^{-1} - B_{21}^{-1})] - \frac{F_g}{r_3}A_3'(B_{33}^{-1} - B_{32}^{-1})\right\} \\
w_T^{(4)} &= -R_0\{G\}, \quad (G \text{ is given in Appendix 4.})
\end{aligned} \tag{35}$$

where

$$B_{33} = e^{r_3 \eta_T}. \tag{36}$$

The vertical velocity at z_0 has been assumed to be zero.

$\nabla\cdot$ indicates the divergence operation, it is defined as

$$\nabla\cdot X = \frac{\partial X_r}{\partial x} + \frac{\partial X_i}{\partial y}. \tag{37}$$

X is any complex variable; X_r, X_i are its real and imaginary part respectively.

Eq. (35) shows that the vertical velocity at the top of the PBL is dependent on four terms. The first term $W_T^{(1)}$ is the contribution of the first layer. The second term $W_T^{(2)}$ is a non-viscous and nonlinear term. It has no viscous coefficient. It let R_0 equal zero, this term will disappear; therefore there is no such term in the classical theory of the Ekman boundary layer, which is caused by non-linearity because it includes R_0 which is connected with the term of the non-linear inertial force. In addition, $W_T^{(2)}$ also includes two factors $\eta_T - \eta_t$ and $d_g \zeta_g / dt$, showing that this term is directly proportional to the depth of the Ekman layer and the rate of change of geostrophic vorticity following motion. For example, when the local rate of change of geostrophic vorticity is not equal to zero, the vertical motion at the top of the PBL will occur. The third term $W_T^{(3)}$ is a viscous term but is independent of non-linearity since it contains k, not R_0. This term is just corresponding to the vertical velocity at the top of the PBL in classical Ekman theory. Introducing (30), and setting $R_0=0$, we obtain

$$\bar{w}_T = w_T^{(3)} = \frac{1}{2}\zeta_g. \tag{38}$$

It is Charney and Eliassen (1949) who obtained the vertical velocity at the top of the PBL, directly proportional to the geostrophic vorticity. The fourth term $W_T^{(4)}$ is a term of viscosity and non-linearity since it includes both k_2, k_3 and R_0 but not included in the Ekman theory.

IV. EXAMPLE

For simplicity, some properties of the boundary layer solutions developed above will be illustrated as a steady, axisymmetric circular vortex. A pressure field is assumed to be

$$\phi = \pm\left[1 - \frac{\alpha}{2}(x^2 + y^2)\right] e^{-\frac{\alpha}{2}(x^2+y^2)}, \tag{39}$$

where ϕ is the non-dimensional geopotential deviation. The same expression as (39) is used for an anticyclonic (+) and cyclonic (−) vortex. α is a constant, e.g. 0.5.

Figure 1 is a hodograph of horizontal wind in the PBL at $x=10^6$ (m), $y=0$. E line represents the Ekman solution, GM, the semi-geostrophic solution of the single K model which corresponds to Eqs. (23a) and (23b) of Wu and Blumen (1982), NL the weak non-linear solution of the three sections K model which is (25) of the present paper. The subscripts 1 and 2 denote a cyclonic and anticyclonic vortex respectively.

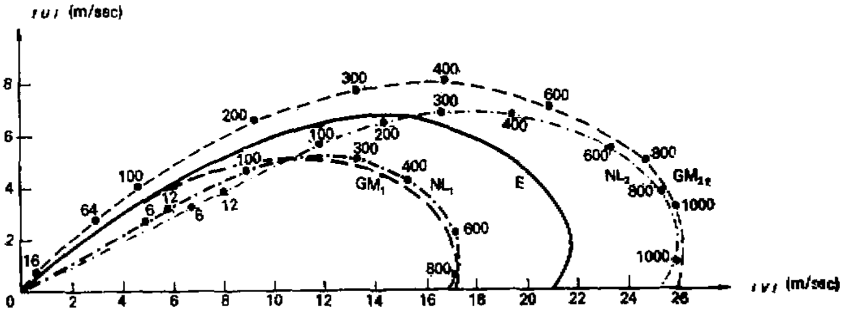

Fig. 1. Hodograph for horizontal wind in the PBL (The legend refers to the text). ($R_1=0.3$, $E=0.001$, $F=0.0001$)

It can be easily seen from Fig. 1 that near the top of the PBL NL line approaches GM line and both of them are reasonable, that is, the wind is larger than the Ekman solution for an anticyclonic vortex and is smaller for a cyclonic vortex. The reason is that after incorporating geostrophic momentum approximation or weak non-linear effect, the inertial centrifugal force makes the wind increase an anticyclonic vortex and decrease a cyclonic vortex. The Ekman solution can not reflect this difference of wind between the cyclonic and the anticyclonic vortexes so it is not in agreement with the fact.

It can be seen from Fig. 1 that near the origin, the angle between E line or GM line

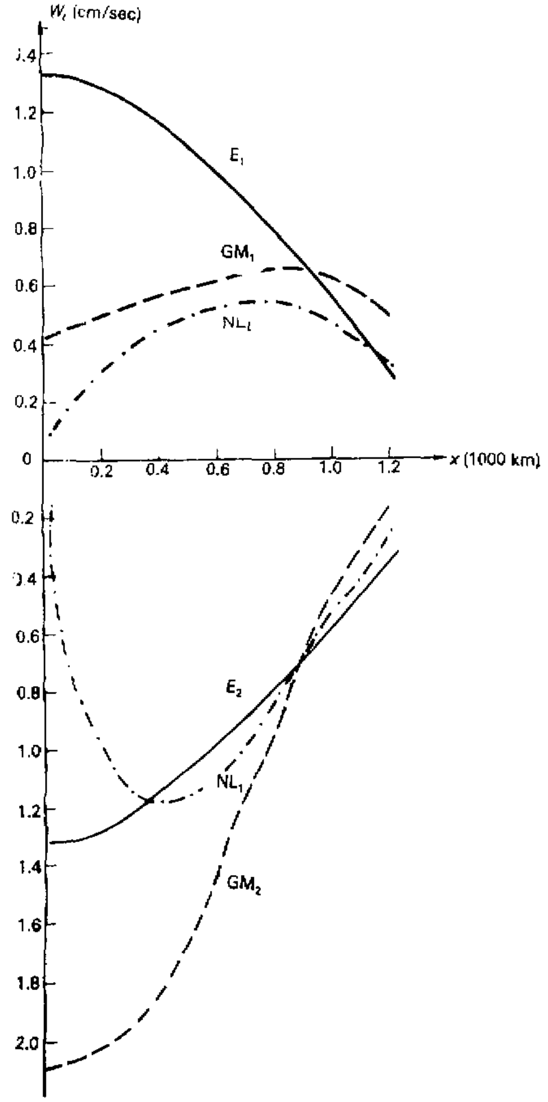

Fig. 2. A distribution for the vertical velocity at the top of the PBL (The explanation refers to Fig. 1).

and the horizontal axis is about 45°, but NL line is less than 30°, suggesting that for these two models, the angles between the wind near surface and the wind at the top of PBL are different. Fig. 2 shows the distribution of the vertical velocity at the top of PBL with (x, y). E_1 line and E_2 line are symmetric, but GM_1 line and GM_2 line are not symmetric, neither are NL_1 line and NL_2 line. This can be explained simply by: 1) E_1 line and E_2 line are from (38) and the signs of ζ_g of a cyclonic and an anticyclonic vortex are opposite but their absolute values are equal; 2) E_1 line and E_2 line are symmetric. Both GM line and NL line contain the effect of inertial centrifugal force, i. e. the terms involve R_0-$W_T^{(1)}$, $W_T^{(2)}$, $W_T^{(4)}$, all of which contribute to $\bar{W}_T$. $W_T^{(2)}$ is taken as a sample to interpret simply. The rate of advective changes is represented by $u_g\frac{\partial\zeta_g}{\partial x}+v_g\frac{\partial\zeta_g}{\partial y}$. For a cyclonic and an anticyclonic vortex, the signs of u_g, v_g are opposite, so is ζ_g, but their products are the same, the total absolute value of $W_T^{(2)}$ and $W_T^{(3)}$ which only contains F_g are not equal. Therefore, the curves are not symmetric, which shows that non-linear inertial centrifugal force causes the difference of vertical velocity at the top of the PBL between a cyclonic and an anticyclonic vortex.

The most evident difference between the single K model and the three sections K model appears near the center of the steady circular vortex. The $\bar{W}_T$, calculated by the former approaches the maximum, but the latter tends to zero due to the values of different k. It is known that the friction stress is caused by the vertical shear between the wind at the top of the PBL and the wind at the earth's surface. Near the center, both of them approach zero, so does k; thus the convergent rising current and the divergent sinking current caused by the frictional effect are certainly very weak.

V. THE EFFECTS OF STRATIFICATION AND BAROCLINITY

In the surface layer, if the atmosphere is non-neutral, the logarithm distribution of wind does not hold again. In this case, the atmosphere is stably or unstably stratified, the Businger commonly used function

$$\phi_m=\begin{cases}\left(1-15\frac{z}{L}\right)^{-1/4}, & \text{(for unstable stratification } \frac{z}{L}<0.)\\ 1+4.7\frac{z}{L} & \text{(for stable stratification } \frac{z}{L}>0.)\end{cases} \tag{40}$$

may be adopted, where L is the length of Monin-Obukhov. Consequently, the wind in the surface layer satisfies the equation as follows

$$k_1\eta\frac{\partial F_1}{\partial\eta}=\partial_1\phi_m\left(\frac{\eta E^{\frac{1}{2}}}{L}\right). \tag{41}$$

The first equation of formal solutions of (18) and (24) is reduced to

$$F_{01}=\frac{a_{01}}{k_1}\left[\ln\frac{\eta}{\eta_0}-\psi_m\left(\frac{\eta E^{\frac{1}{2}}}{L}\right)\right],$$

$$F_{11}=\frac{a_{11}}{k_1}\left[\ln\frac{\eta}{\eta_0}-\psi_m\left(\frac{\eta E^{\frac{1}{2}}}{L}\right)\right], \tag{42}$$

where

$$\psi_m\left(\frac{\eta E^{\frac{1}{2}}}{L}\right)=\begin{cases} 2\ln\frac{1+\xi}{2}+\ln\frac{1+\xi^2}{2}-2\mathrm{arctg}\xi+\frac{\pi}{2}, & \left(\frac{\eta E^{\frac{1}{2}}}{L}<0\right) \\ & \left(\xi=\left(1-15\frac{\eta E^{\frac{1}{2}}}{L}\right)^{\frac{1}{4}}\right), \\ -4.7\frac{\eta E^{\frac{1}{2}}}{L} & \left(\frac{\eta E^{\frac{1}{2}}}{L}>0\right). \end{cases} \tag{43}$$

We can use the above-mentioned method to solve the wind in the non-neutral stratification atmosphere. Rewriting the left hand side of the connecting conditions (20), we have

$$\alpha_{01}\to\alpha_{01}\phi_m\left(\frac{\eta_1 E^{\frac{1}{2}}}{L}\right),$$

$$F_1=\alpha_{01}H,\quad \left(H=\frac{1}{k_1}\ln\frac{\eta_1}{\eta_0}\right)\to F_1=\alpha_{01}H',\quad \left(H'=\frac{1}{k_1}\left[\ln\frac{\eta_1}{\eta_0}-\psi_m\left(\frac{\eta_1 E^{\frac{1}{2}}}{L}\right)\right]\right). \tag{44}$$

We obtain the first order approximations of the wind in the non-neutral stratified PBL as

$$F_{01}=\frac{A_1'}{k_1\phi_m\left(\frac{\eta_1 E^{1/2}}{L}\right)}\left[\ln\frac{\eta}{\eta_0}-\psi_m\left(\frac{\eta E^{\frac{1}{2}}}{L}\right)\right]. \tag{45}$$

A_1' in (45) and A_1 in Appendix 1 are of the same form, but their values are different, and so the physical process is also different, because the H in A has come to H' now. This shows that the stratified effect has been considered. The procedure solving F_{11} by means of the same method is not described here. The treatment of the stratification in the Ekman layer is not described in this paper either.

The baroclinity in the PBL must be considered when the temperature field T is inhomogeneous. Supposing that the pressure and temperature field at the bottom of the free atmosphere are given, we have

$$\begin{aligned} &F_g(\eta)=F_{gT}-\tilde{F}(\eta)=F_{gT}-\tilde{T}(\eta_T-\eta), \\ &\tilde{F}(\eta)=\tilde{u}(\eta)+i\tilde{v}(\eta), \\ &\tilde{u}(\eta)=-\frac{g}{fT}\frac{\partial T}{\partial y}(\eta_T-\eta),\quad \tilde{v}(\eta)=\frac{g}{fT}\frac{\partial T}{\partial x}(\eta_T-\eta), \\ &\tilde{T}=\frac{g}{fT}\left(-\frac{\partial T}{\partial y}+i\frac{\partial T}{\partial x}\right), \end{aligned} \tag{46}$$

where F_{gT} is the complex geostrophic wind at the top of the PBL, $\tilde{F}(\eta)$ is the complex thermal wind. The special solutions of the second equation of (14) are still F_g, but the special solutions Y_2, Y_3 of the second equation of (15) are much complex than that in barotropic atmosphere because the second power terms appear on the right-hand side of (15) under consideration of the non-linear advective effect.

The first-order approximate formal solutions in the baroclinic atmosphere can be written as

$$\begin{aligned} &F_{01}=\frac{\alpha_{01}}{k_1}\ln\frac{\eta}{\eta_1}, \\ &F_{02}=F_g(\eta)+\alpha_{02}e^{r_2\eta}+\alpha_{02}'e^{-r_2\eta}, \\ &F_{03}=F_g(\eta)+\alpha_{03}'e^{-r_3\eta}. \end{aligned} \tag{47}$$

The expressions of the complex constants α_{01}, α_{02}, α'_{02}, α'_{03} are given in Appendix 5. The formal solutions of F_{11}, F_{12}, F_{13} can still be written as (24); $\alpha_{11}, \alpha_{12}, \alpha'_{12}, \alpha'_{13}$ are still given in Appendix 3, but now the viscous parts of special solutions Y_2, Y_3 are different from those in Appendix 2; necessary is a reference to Appendix 6. Up to now, the second order approximate solutions have been solved which still are (25) formally, but the value of the solutions and the physical process are different from those in the barotropic atmosphere, because the thermal factor has been considered. For example, the geostrophic wind is no longer independent of height but is the function of the height. If the temperature is higher in the south than in the north, the thermal wind blows east. If the geostrophic wind near the earth's surface blows south (north), it will turns counterclockwise (clockwise) with height. If the geostrophic wind near the earth's surface blows east (west), it will increase (decrease) with height. It can be seen immediately from (47) that the variations in direction and magnitude of the geostrophic wind directly affect F_{02}, F_{03}, i. e. the wind within the Ekman layer. In the surface layer, the wind is derived from (47), Appendix 5 and (46) as follows

$$F_{01} = -\tilde{F}(\eta)\frac{E^{\frac{1}{2}}h_1}{\eta_T - \eta}\ln\frac{\eta_1}{\eta} + \frac{1}{k_1}(S_2 B_{21}\alpha_{02} - S_2 B_{21}^{-1}\alpha'_{02})\ln\frac{\eta}{\eta_1}, \quad (\eta_0 < \eta < \eta_1). \tag{48}$$

It can be seen from (22) and Appendix 1 that the second term of (48) and F_{01} in barotropic atmosphere are formally identical. Thus it is the first term of (48) that revise the wind in the surface layer under the baroclinic condition. If temperature in the south is higher than that in the north, the complex thermal wind $\tilde{F}(\eta)$ blows east, whereas $-\tilde{F}(\eta)$, blows west. In this case there exists a westward revision of the wind in the surface layer. It is helpful to predict exactly the direction and magnitude of the wind near the earth's or ocean's surface.

VI. CONCLUSION

Considering the non-linear advective effect and the variation of the viscous coefficient k with either height or geostrophic wind, and using a small parameter expansion method, we have derived the analytical expressions (25) of wind in the PBL and (35) of the vertical velocity at the top of the PBL in this paper. By means of Businger's universal function the condition of the neutral atmosphere in the surface layer has been extended to the stratified atmosphere, the first order approximation solution is (45). Finally, by introducing thermal wind, the barotropic atmosphere has been extended to the baroclinic atmosphere, the wind is expressed by (25) and Appendix 6.

The present results are used to obtain exactly the wind in the PBL. The improving of the calculation of vertical velocity at the top of the PBL contributes to the research of the feedback of the PBL to free atmosphere. But there exists a solving condition $R_0 < 1$ in this paper, and is necessary to have a further research in regard to the application of lower latitude. In addition, it is, after all, so crude that k takes three sections model, which also needs improving.

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APPENDICES

1. $$A_1 = s_2 B_{21} A_2 - s_2 B_{21}^{-1} A_2', \quad A_2 = \frac{1}{B_{21}(s_2 H - 1)} + C_1 A_2'.$$

$$A_2' = C_2 A_3', \qquad A_3' = \frac{-B_{22}}{B_{21}(s_2 H - 1)[C_2(B_{22}^{-1} + B_{22} C_1) - B_{32}^{-1}]},$$

$$C_1 = \frac{s_2 H + 1}{B_{21}^{2}(s_2 H - 1)}, \qquad C_2 = \frac{B_{22}}{2B_{32}}\left(\frac{s_3}{s_2} + 1\right),$$

$$s_n = E^{-1/2} k_n r_n, \quad (n = 2, 3), \quad B_{21} = e^{r_2 \eta_1}, \quad B_{22} = e^{r_2 \eta_2},$$

$$B_{32} = e^{r_3 \eta_2}, \quad H = \frac{1}{k_1} \ln \frac{\eta_1}{\eta_0}.$$

2. $$Y_2^B = \frac{A+C}{k_2 r_2}(A_2 e^{r_2 \eta} - A_2' e^{-r_2 \eta})\,\eta - \frac{C_i}{3}(A_2^2 e^{2r_2 \eta} + A_2'^2 e^{-2r_2 \eta})$$

$$-D\left[\frac{i}{2}(\bar{A}_2 e^{\bar{r}_2 \eta} + \bar{A}_2' e^{-\bar{r}_2 \eta}) + \frac{2+i}{5}(A_2 \bar{A}_2 e^{2\eta/\sqrt{k_2}} + A_2' \bar{A}_2' e^{-2\eta/\sqrt{k_2}})\right.$$

$$\left. + \frac{-2+i}{5}\left(A_2 \bar{A}_2' e^{2i\eta/\sqrt{k_2}} + A_2' \bar{A}_2 e^{-2i\eta/\sqrt{k_2}}\right)\right] + 2C_i A_2 A_2',$$

$$Y_3^B = -\frac{A+C}{k_3 r_3} A_3' e^{-r_3 \eta} - \frac{C_i}{3} A_3'^2 e^{-2r_3 \eta} - D\left(\frac{i}{2}\bar{A}_3' e^{-\bar{r}_3 \eta} + \frac{2+i}{5} A_3' \bar{A}_3' e^{-2\eta/\sqrt{k_3}}\right)$$

$$A = \frac{d_g F_g}{dt}, \quad C = \frac{i}{2} F_g \zeta_g, \quad D = -\frac{1}{2}\bar{F}_g\left[\frac{\partial u_g}{\partial x} - \frac{\partial v_g}{\partial y} + i\left(\frac{\partial v_g}{\partial x} + \frac{\partial u_g}{\partial y}\right)\right],$$

$$\zeta_g = \frac{\partial v_g}{\partial x} - \frac{\partial u_g}{\partial y}.$$

3. $$\alpha_{11} = D_{11}' + s_2 B_{21} \alpha_{12} - s_2 B_{21}^{-1} \alpha_{12}', \qquad \alpha_{12} = D_{11} + C_1 \alpha_{12}',$$

$$\alpha_{12}' = D_{12}' + C_2 D_{12}, \qquad \alpha_{13}' = D_{12},$$

$$D_{11} = \frac{Y_2(\eta_1) - H D_{11}'}{B_{21}(s_2 H - 1)}, \qquad D_{11}' = E^{-1/2} k_2 \frac{\partial Y_2(\eta_1)}{\partial \eta},$$

$$D_{12} = \frac{Y_3(\eta_2) - Y_2(\eta_2) - B_{22} D_{11} - (B_{22}^{-1} + B_{22} C_1) D_{11}'}{C_2(B_{22}^{-1} + B_{22} C_1) - B_{32}^{-1}},$$

$$D_{12}' = \frac{B_{22}}{2}\left[Y_3(\eta_2) - Y_2(\eta_2) - \frac{E^{-1/2}}{s_2}\left(k_3 \frac{\partial Y_3(\eta_2)}{\partial \eta} - k_2 \frac{\partial Y_2(\eta_2)}{\partial \eta}\right)\right],$$

$$\frac{\partial Y_2(\eta)}{\partial \eta} = \frac{A+C}{k_2}\left[A_2 e^{r_2 \eta}\left(\eta + \frac{1}{r_2}\right) + A_2' e^{-r_2 \eta}\left(\eta - \frac{1}{r_2}\right)\right]$$

$$-\frac{2C_i}{3} r_2 (A_2^2 e^{2r_2 \eta} - A_2'^2 e^{-2r_2 \eta}) - D\left[\frac{i}{2}\bar{r}_2(\bar{A}_2 e^{\bar{r}_2 \eta} - \bar{A}_2' e^{-\bar{r}_2 \eta})\right.$$

$$+\frac{4+2i}{5\sqrt{k_2}}\left(A_2\bar{A}_2e^{2\eta/\sqrt{k_2}}-A_2'\bar{A}_2'e^{-2\eta/\sqrt{k_2}}\right)$$

$$-\frac{2+4i}{5\sqrt{k_2}}\left(A_2\bar{A}_2'e^{2i\eta/\sqrt{k_2}}-A_2'\bar{A}_2e^{-2i\eta/\sqrt{k_2}}\right)\Bigg],$$

$$\frac{\partial Y_3(\eta)}{\partial\eta}=\frac{A+C}{k_3}A_3'e^{-r_3\eta}\left(\eta-\frac{1}{r_3}\right)+\frac{2Ci}{3}r_3A_3'^2e^{-2r_3\eta}$$

$$+D\left(\frac{i}{2}\bar{A}_3'\bar{r}_3e^{-\bar{r}_3\eta}+\frac{4+2i}{5\sqrt{k_3}}A_3'\bar{A}_3'e^{-2\eta/\sqrt{k_3}}\right).$$

4. $$G=-\frac{i(A+C)}{2}\Bigg[A_2\left(B_{22}\eta_2-B_{21}\eta_1-\frac{B_{22}-B_{21}}{r_2}\right)$$

$$+A_2'\left(B_{22}^{-1}\eta_2-B_{21}^{-1}\eta_1+\frac{B_{22}^{-1}-B_{21}^{-1}}{r_2}\right)+A_3'\left(B_{33}^{-1}\eta_T-B_{32}^{-1}\eta_2+\frac{B_{33}^{-1}-B_{32}^{-1}}{r_3}\right)\Bigg]$$

$$-\frac{Ci}{6}\left[\frac{A_2^2}{r_2}(B_{22}^2-B_{21}^2)-\frac{A_2'^2}{r_2}(B_{22}^{-2}-B_{21}^{-2})-\frac{A_3'^2}{r_3}(B_{33}^{-2}-B_{32}^{-2})\right]$$

$$-D\Bigg[\frac{i}{2}\left(\frac{\bar{A}_2}{\bar{r}_2}(\bar{B}_{22}-\bar{B}_{21})-\frac{\bar{A}_2'}{\bar{r}_2}(\bar{B}_{22}^{-1}-\bar{B}_{21}^{-1})-\frac{\bar{A}_3'}{\bar{r}_3}(\bar{B}_{33}^{-1}-\bar{B}_{32}^{-1})\right)$$

$$+\frac{2+i}{10}\Big(\sqrt{k_2}A_2\bar{A}_2\left(e^{2\eta_2/\sqrt{k_2}}-e^{2\eta_1/\sqrt{k_2}}\right)-\sqrt{k_2}A_2'\bar{A}_2'\left(e^{-2\eta_2/\sqrt{k_2}}\right)-e^{-2\eta_2/\sqrt{k_2}}$$

$$-\sqrt{k_3}A_3'\bar{A}_3'\left(e^{-2\eta_T/\sqrt{k_3}}-e^{-2\eta_2/\sqrt{k_3}}\right)\Big)$$

$$+\frac{1+2i}{10}\sqrt{k_2}\left(A_2\bar{A}_2'\left(e^{2i\eta_2/\sqrt{k_2}}-e^{2i\eta_1/\sqrt{k_2}}\right)-\bar{A}_2A_2'\left(e^{-2i\eta_2/\sqrt{k_2}}-e^{-2i\eta_1/\sqrt{k_2}}\right)\right)\Bigg]$$

$$+2C_iA_2A_2'(\eta_2-\eta_1)+\frac{1}{r_2}(B_{22}-B_{21})\alpha_{12}-\frac{1}{r_2}(B_{22}^{-1}-B_{21}^{-1})\alpha_{12}'$$

$$-\frac{1}{r_3}(B_{33}^{-1}-B_{32}^{-1})\alpha_{13}'.$$

5. $$\alpha_{01}=D_{01}'+s_2B_{21}\alpha_{02}-s_2B_{21}^{-1}\alpha_{02}',\qquad \alpha_{02}=D_{02}+C_1\alpha_{02}',$$

$$\alpha_{02}'=D_{02}'+C_2\alpha_{03}',\qquad \alpha_{03}'=D_{03},$$

$$D_{01}=\frac{F_{g1}-HD_{01}'}{B_{21}(s_2H-1)},\quad F_{g1}=F_g(\eta_1),\quad D_{01}'=E^{-1/2}k_2\bar{T},$$

$$D_{02}=\frac{-B_{22}D_{01}-(B_{22}^{-1}+B_{22}C_1)D_{02}'}{C_2(B_{22}^{-1}+B_{22}C_1)-B_{32}^{-1}},\qquad D_{02}'=\frac{E^{-1/2}\bar{T}(k_3-k_2)}{-2s_2B_{22}^{-1}}.$$

6. $$Y_2=Y_2^I+Y_2^{II}=i\frac{d_gF_g}{dt}+\frac{1}{k_2r_2}\left(\frac{d_g}{dt}+\frac{i}{2}\xi_g\right)(\alpha_{02}e^{r_2\eta}-\alpha_{02}'e^{-r_2\eta})$$

$$-\frac{i}{6}\left(\alpha_{02}e^{2r_2\eta}\bar{\nabla}\alpha_{02}+\alpha_{02}'e^{-2r_2\eta}\bar{\nabla}\alpha_{02}'\right)+\frac{i}{4}\left(\bar{\alpha}_{02}e^{\bar{r}_2\eta}+\bar{\alpha}_{02}'e^{-\bar{r}_2\eta}\right)\nabla F_g$$

$$+\frac{2+i}{10}\left(\bar{\alpha}_{02}e^{i\eta/\sqrt{k_2}}\nabla\alpha_{01}+\bar{\alpha}'_{01}e^{-i\eta/\sqrt{k_2}}\nabla\alpha'_{02}\right)$$

$$+\frac{-2+i}{10}\left(\bar{\alpha}'_{02}e^{2i\eta/\sqrt{k_2}}\nabla\alpha_{02}+\bar{\alpha}_{02}e^{-2i\eta/\sqrt{k_2}}\nabla\alpha'_{02}\right)+\frac{i}{2}\left(\alpha_{02}\overline{\nabla}\alpha'_{02}+\alpha'_{02}\overline{\nabla}\alpha_{02}\right)$$

$$Y_3=Y'_3+Y''_3=i\frac{d_gE_g}{dt}+\frac{1}{k_3r_3}\left(\frac{d_g}{dt}+\frac{i}{2}\xi_g\right)\left(-\alpha'_{03}e^{-r_3\eta}\right)-\frac{i}{6}\alpha'_{03}e^{-2r_3\eta}\overline{\overline{\nabla}}\alpha'_{01}$$

$$+\frac{i}{4}\bar{\alpha}'_{01}e^{-\bar{r}_3\eta}\nabla F_g+\frac{2+i}{10}\bar{\alpha}'_{03}e^{-2\eta/\sqrt{k_3}}\nabla\alpha'_{03}.$$

$$\nabla X=\left(\frac{\partial}{\partial x}+i\frac{\partial}{\partial y}\right)(X_r+iX_i),$$

$$\overline{\nabla}X=\left(\frac{\partial}{\partial x}-i\frac{\partial}{\partial y}\right)(X_r+iX_i).$$

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