Variational Principle of Instability of Atmospheric Motions ①

Zeng Qingcun (曾庆存)
Laboratory of Numerical Modelling for Atmospheric Sciences and GFD,
Institute of Atmospheric Physics, Academia Sinica, Beijing, China
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ABSTRACT

Problems of instability of rotating atmospheric motions are investigated by using nonlinear governing equations and the variational principle. The method suggested in this paper is universal for obtaining criteria of instability in all models with all possible basic flows. For example, the model can be barotropic or baroclinic, layer or continuous, quasi-geostrophic or primitive equations; the basic flow can be zonal or nonzonal, steady or unsteady.

Although the basic flows possess a great deal of variety, they all are the stationary points in the functional space determined by an appropriate invariant functional. The basic flow is an unsteady one if the conservation of angular momentum is included in the associated functional.

The second variation, linear or nonlinear, gives the criteria of instability. Especially, the general criteria of instability for unsteady basic flow, orographically disturbed flow as well as nongeostrophic flow are first obtained by the method described in this paper.

It is also shown that the difference between the criteria of instability obtained by the linear theory and our variational principle clearly indicates the importance of using nonlinear governing equations.

In the appendix the theory is extended to cases such as in a β-plane where the fluid does not possess finite total energy, hence the variational principle can not be directly applied. However, a generalized Liapounoff norm can still be obtained on the basis of variational consideration.

I. INTRODUCTION

Hydrodynamic instability including the instability of atmospheric motions as one of its parts is a classical but difficult problem. A well developed method for the study of this problem is to solve the generalized eigenvalue problem of the corresponding linearized equation or equations. However, this method usually is suitable for zonal (parallel) and steady basic flow (see Lin, 1955). In the case of nonzonal but steady basic flow Arnold (1965) suggested a powerful variational method for obtaining criteria of instability. However, the general criteria of instability which have been found so far in the literature are only for the curvilinear and steady basic flows in the two-dimensional nondivergent or geostrophic model (Arnold, 1965; Dikii, 1965), or three-dimensional geostrophic model with isentropic bottom boundary (Blumen, 1968). No general criterion of instability has been found for the three-dimensional primitive equations neither by the variational method, nor by the linearization. Moreover, no

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general method has been developed, hence no general criterion of instability has been found for unsteady basic flow of any atmospheric model.

In order to fill this gap we have developed a generalized variational method based on the Arnold's one. Our variational principle is universal for obtaining criteria of instability in all models with all possible basic flows. i.e., the model can be barotropic or baroclinic, quasi-geostrophic or nongeostrophic; and the basic flow can be zonal or nonzonal, steady or unsteady.

Note that the linear theory of instability is referred to the linearized equation (or equations), while the variational method is based on the nonlinear governing equation (or equations), and therefore, the theory developed by the variational method is referred as nonlinear one, which can be applied to the studies of stability properties of small perturbations if the Liapounoff's definition of stability is adopted. Besides, the variational method can be also applied to the studies of stability properties of large amplitude disturbances in some paticular cases.

II. GENERAL THEOREMS FOR THE TWO-DIMENSIONAL QUASI-GEOSTROPHIC MODEL

The governing equaiton is the conservation of potential vorticity (or vorticity in the nondivergent model):

$$\frac{\partial q}{\partial t}+\vec{v}\cdot\nabla q=0, \tag{2.1}$$

where

$$q=\Delta\psi-\kappa\frac{f_0^2}{\varphi_0}\psi+2\omega\cos\theta, \tag{2.2}$$

$$\vec{v}=\vec{\theta}^0\left(-\frac{\partial\psi}{a\sin\theta\partial\lambda}\right)+\vec{\lambda}^0\left(\frac{\partial\psi}{a\partial\theta}\right), \tag{2.3}$$

ψ is the stream function, φ_0 the mean geopotential of the equivalent free surface, f_0 the mean Coriolis parameter, ω the Earth's angular velocity. ∇ and Δ are the gradient operator and Laplacian in the spherical surface with radius a respectively. $\kappa=0.1$. $\kappa=0$ is the case of nondivergent model. Orography is omitted first for simplicity.

From (2.1) we have conservations of total energy, "generalized enstrophy" and angular momentum. Therefore, we have an invariant functional (function of ψ, depending on an arbitrary function Q and some parameters r_0, r_1 and r_2):

$$I(\psi)=\iint_S\left\{r_0\left[|\vec{v}|^2+\kappa\frac{f_0^2}{\varphi^2}\psi^2\right]+r_1Q(q)+r_2\left[\sin\theta\frac{\partial\psi}{\partial\theta}-\kappa\frac{a^2f_0^2}{\varphi_0}\psi\cos\theta\right]\right\}dS$$
$$=\text{invariant}, \tag{2.4}$$

where the integration is taken over the whole spherical surface. The functional I is the extension of Arnold–Dikii's one which in turn is the degeneration of our I by taking $r_2=0$. Later, we can see the importance of this extension. In other words, the conservation of total angular momentum plays a particular role, although it is a functional of first order unlike the conservations of total energy and generalized enstrophy which are of second or higher order.

Giving a perturbation $\delta\psi$, the first and second variations, δI and $\delta^2 I$, and the difference $I(\psi+\delta\psi)-I(\psi)$ are obtained after elementary calculations as follows:

$$\delta I = \iint_S \left\{ -2r_0\psi + r_1 Q'(q) + r_2 a^2 \sin\theta \right\} \delta q dS. \tag{2.5}$$

$$\delta^2 I = \iint_S \left\{ r_0 \left[|\delta \vec{v}|^2 + \kappa \frac{f_0^2}{\varphi_0^2} (\delta\psi)^2 \right] + r_1 \frac{1}{2} Q''(q)(\delta q)^2 \right\} dS. \tag{2.6}$$

$$I(\psi + \delta\psi) - I(\psi) = \delta I + \delta^2 I + \cdots, \tag{2.7}$$

where $Q'(q) = \frac{dQ}{dq}$, and $Q'' = \frac{d^2 Q}{dq^2}$, and

$$I(\psi + \delta\psi) - I(\psi) = \delta I + \Delta^2 I \tag{2.8}$$

by the use of Lagrange formulation of truncated Taylor series, where

$$\Delta^2 I = \iint_S \left\{ r_0 \left[|\delta \vec{v}|^2 + \kappa \frac{f_0^2}{\varphi_0^2} (\delta\psi)^2 \right] + \frac{r_1}{2} Q''(q^*)(\delta q)^2 \right\} dS. \tag{2.9}$$

$$q^* = q + r^* \delta q, \qquad 0 \leqslant r^* \leqslant 1.$$

$\Delta^2 I$ differs from $\delta^2 I$ in that. $Q''(q)$ is replaced by $Q''(q^*)$. We assume that the properties of the flow $(\psi + \delta\psi, \vec{v} + \delta\vec{v}$, and $q + \delta q)$ all $\in L_2$, (see Zeng, 1979), and both I and $\Delta^2 I$ exist.

We have the following theorems:

Theorem 2.1 Every function $\psi(\theta, \lambda - \dot{\lambda}_0 t)$ which is a solution to (2.1)–(2.3) (propagating wave with a constant phase angular velocity $\dot{\lambda}_0$) is a stationary point of I in the functional space ψ, i.e. $\delta I = 0$, and the phase angular velocity $\dot{\lambda}_0 = -r_2 / 2r_0$. The inverse theorem is also true.

Proof If $\psi(\theta, \lambda - \dot{\lambda}_0 t)$ is a solution to (2.1), we have $\partial q / \partial t = -\dot{\lambda}_0 \partial q / \partial \lambda = J(\dot{\lambda}_0 a^2 \cos\theta,\ q)$, where $J(\ ,\ )$ denotes the Jacobian on the spherical surface with radius a. Hence from (2.1) we have

$$J(\psi + \dot{\lambda}_0 a^2 \cos\theta, q) = 0. \tag{2.10}$$

This means that $\psi + \dot{\lambda}_0 a^2 \cos\theta$ is an arbitrary function of argument q and vice versa, say.

$$\psi + \dot{\lambda}_0 a^2 \cos\theta = \tilde{Q}(q). \tag{2.11}$$

Let $r_1 Q'(q) = \tilde{Q}(q)$, i. e.

$$r_1 Q = \int_{q_0}^{q} \tilde{Q}(x) dx. \tag{2.12}$$

and $\dot{\lambda}_0 = -r_2/2r_0$, where $r_0 \neq 0$, from (2.11) we obtain

$$-2r_0\psi + r_1 Q'(q) + r_2 a^2 \cos\theta = 0. \tag{2.13}$$

Therefore, $\delta I = 0$ is satisfied. The theorem is proved.

Now we prove the inverse theorem. Giving a ψ, we have (2.13) satisfied if $\delta I = 0$. Taking Jacobian upon (2.13) and q, we obtain

$$J(-2r_0\psi + r_2 a^2 \cos\theta,\ q) = 0. \tag{2.14}$$

This equation is transformed into (2.10) by taking $\dot{\lambda}_0 = -r_2/2r_0$ if $r_0 \neq 0$, hence $\psi = \psi(\theta, \lambda - \dot{\lambda}_0 t)$ is a solution to (2.1). If $r_0 = 0$, from (2.14) we obtain that q and ψ are functions of θ only, hence ψ is also a solution to (2.1) with phase velocity equal to every arbitrary constant $\dot{\lambda}_0$.

Note 2.1 If either r_0 or r_1 is equal to zero, the flow determined by the stationary point of I is a zonal one, and even a solid rotation as $r_1 = 0$, that is $\psi = -a^2 \dot{\lambda}_2 \cos\theta$, where $\dot{\lambda}_2$ is a constant (the angular velocity). The case $Q'(q) =$ constant is equivalent to $r_1 = 0$. However, if and only if $r_2 \neq 0$, the flow with $\delta I = 0$ is unsteady one (with phase angular velocity $\dot{\lambda}_0 \neq 0$) according to Theorem 2.1. This means that our theory can be applied to steady as well as some unsteady basic flows by taking the conservation of total angular momentum into account in the functional I, but Arnold's and Dikii's theory can be applied only to the steady basic flow.

Theorem 2.2 A (basic) flow $\psi(\theta, \lambda - \dot{\lambda}_0 t)$ determined by $\delta I = 0$ with a function $Q(q)$ and parameters r_0, r_1 and r_2 is always stable with respect to every of those small perturbations $\delta\psi$ whose $r_1 Q''(q^*)/2$ and r_0 both are either non-negative or non-positive everywhere in the fluid: and the stability holds with respect to every perturbation if $r_1 Q''(x)$ as a function of its argument x always has the same sign as r_0 or is equal to zero.

Proof As the condition mentioned in Theorem 2.2 is satisfied, we can take $|\Delta^2 I|$ or a simpler one,

$$\|\delta\psi\|_W^2 = \left| r_0 \kappa \frac{f_0^2}{\varphi_0} \right| \cdot \|\delta\psi\|^2 + |r_0| \cdot \|\delta\vec{v}\|^2 + \left| r_1 \frac{Q''_m}{2} \right| \cdot \|\delta q\|^2, \tag{2.15}$$

as a Liapounoff's functional, where the norm $\|\cdot\|$ is simply taken in the L_2 space, $\|\cdot\|_W$ is a norm in a Sobolev's space, and $|Q''_m|$ is the lower bound of $|Q''(q^*)|$, that is

$$|Q''(q^*)|_{\delta\psi \in S_c} \geq Q''_m, \tag{2.16}$$

and $\delta\psi \in S_c$ denotes the sub-space consisting of those perturbations which satisfy the condition mentioned in Theorem 2.2. As $\delta I = 0$, and $I(\psi + \delta\psi)$ and $I(\psi)$ are two conservative, $\Delta^2 I$ is also conserved all the time t, $0 < t < \infty$. Moreover, for a given perturbation $\delta\psi \in S_c$, we have $|\Delta^2 I| \geq \|\delta\psi\|_W^2$. Therefore, if the initial perturbation $\delta\psi^{(0)} \in S_c$ is small enough that its $|\Delta^2 I^{(0)}| < \delta$, i. e.

$$|\Delta^2 I^{(0)}| = \left| r_0 \kappa \frac{f_0^2}{\varphi_0} \right| \cdot \| \delta\psi^{(0)} \|^2 + |r_0| \cdot \| \delta \vec{v}^{(0)} \|^2 + \left| \frac{r_1}{2} \iint_S Q''(q^{*(0)}) (\delta q^{(0)})^2 ds \right| < \delta, \tag{2.17}$$

we have

$$\| \delta\psi \|_W^2 \leqslant |\Delta^2 I^{(0)}| < \delta, \quad (0 \leqslant t < \infty, \ \delta\psi \in S_c), \tag{2.18}$$

and the basic flow ψ satisfying $\delta I = 0$ is stable with respect to perturbations $\delta\psi \in S_c$.

If $r_1 Q''(x)$ as a function of its argument x always has the same sign as r_0 or is equal to zero, the sub–space S_c coincides with the whole space, therefore, (2.15) and (2.18) hold for every $\delta\psi$, and the basic flow is always stable with respect to all perturbations no matter whether their amplitude is small or finite.

The geometric situation of (2.18) is illustrated in Fig.1.

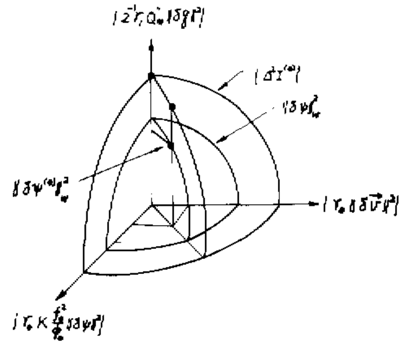

Fig. 1. Geometric representation of $\|\delta\psi\|_W^2$ and $|\Delta^2 I^{(0)}|$.

Note 2.2 The smallness of $\Delta^2 I^{(0)}$ means that the L_2 norms of $\delta\psi$ and its derivatives of first and second orders, $\delta\vec{v}$ and δq, all are necessarily small if $r_1 Q''(q^*) \not\equiv 0$. This indicates the important role of vorticity in the instability properties except $r_1 Q''(q) \equiv 0$ (the basic flow is a solid rotation). As $r_1 Q''(q) \equiv 0$, δq is not included in the norm $\|\delta\psi\|_W$, but

$$\|\delta\psi\|_W^2 = |\Delta^2 I| = |r_0| \left[\kappa \frac{f_0^2}{\varphi_0} \|\delta\psi\|^2 + \|\delta\vec{v}\|^2 \right] \tag{2.19}$$

is also a norm in the whole space $\delta\psi \in W^1$, and conserved all the time $0 < t < \infty$. Even in this case we can also prove the conservation of $\| \delta q \|$, although the boundness of $\| \delta q \|$ for all $t < \infty$ is not necessarily required in the definition of stability. In fact, we have the conservation of angular momentum of the perturbation,

$$\delta M \equiv \iint_S \left[\sin\theta \frac{\partial \delta\psi}{\partial\theta} - \kappa \frac{a^2 f_0^2}{\varphi_0} \cos\theta \cdot \delta\psi \right] dS = \delta M^{(0)}, \tag{2.20}$$

due to the linearity of angular momentum, and the conservation of total potential enstrophy

$$\| q + \delta q \|^2 = \|q\|^2 + 2\iint_S q\delta q dS + \|\delta q\|^2. \tag{2.21}$$

Now, the first term on the right hand side is conserved, and for a basic flow represented by a solid rotation we have

$$q = \left\{ 2\dot{\lambda}_z \left[1 + \frac{\kappa f_0^2}{2\varphi_0} a^2 \right] + 2\omega \right\} \cos\theta, \tag{2.22}$$

hence, from the conservation of δM and (2.22) we obtain the conservation of the second term on the right hand side of (2.21), (see Zeng, 1979). Finally, we obtain the conservation of $\| \delta q \|$, i.e. the last term on the right hand side of (2.21).

If $Q''_m = 0$ but $r_1 Q''(q) \neq 0$, the boundness of $\| \delta q \|^2$ can also be inferred from the conservation of total potential enstrophy (2.21) by using the inequality

$$\|\delta q\| = \|(q + \delta q) - q\| \leqslant \|q + \delta q\| + \|q\|, \tag{2.23}$$

although the smallness of $\| \delta q \|$ for all $t < \infty$ can not be guaranteed.

Theorem 2.3 A flow $\psi(\theta, \lambda - \dot{\lambda}_0 t)$ determined by $\delta I = 0$ with a function $Q(q)$ and parameters r_0, r_1 and r_2 might be unstable if either the sign of r_0 is opposite to that of $r_1 Q''(q^*)$, or Q'' is a sign–nondefinitive function of its argument in the fluid.

Proof The conditions mentioned in Theorem 2.3 are necessary for the instability, otherwise the flow is stable according to Theorem 2.2.

Note 2.3 For a given set of r_0, r_1, r_2 and $Q(x)$, there might be several solutions to equation (2.13) if $Q'(x)$ is not a linear function of its argument x, and therefore, we might have several basic flows determined by the same set (r_0, r_1, r_2, and $Q(x)$) . Suppose that one of them, say, ψ_1, (θ, λ, t), satisfies the following conditions, (i) its $\delta^2 I$ is a sign definitive functional, and (ii) $Q''(q_1 + \delta q)$ is also a sign definitive function if $|\delta q| < \varepsilon$, but a sign–alternative if $|\delta q| < \varepsilon$ is not satisfied, where q_l is the potential vorticity of flow ψ_1. The functional $I(\psi, \vec{v}, q)$ (i.e., complicated functional $I(\psi)$) reaches its local minimum I_1 (if $\delta^2 I > 0$) or maximum I_1 (if $\delta^2 I < 0$) at the point ($\psi_1, \vec{v}_1, q_1$) in the dense subspace ($\psi_1 + \delta\psi, \vec{v}_1 + \delta\vec{v}, q_1 + \delta q$), where ($\delta\psi, \delta\vec{v}, \delta q$) all $\in C^\infty$. Perhaps, ψ_1 is stable with respect to all small enough perturbations whose initial $\delta q^{(0)}$ satisfies $|\delta q^{(0)}| < \varepsilon' < \varepsilon$, and a transition from the vicinity of ψ_1 to the vicinity of another basic flow, say ψ_2, might occur by the action of perturbation whose $|\delta q^{(0)}|$ is not bounded by ε'.

III. INSTABILITY OF LINEAR AND NONLINEAR HAURWITZ WAVES

A whole family of Haurwitz waves can be obtained by $\delta I = 0$, i.e. determined by equation (2.13).

Theorem 3.1 Linear (classical) Haurwitz waves,

$$\psi - \psi_0 = -a^2 \dot{\lambda}_z \cos\theta + \sum_{m=0}^{n} A_m P_n^m (\cos\theta) e^{im(\lambda - \dot{\lambda}_0 t)}, \tag{3.1}$$

are determined by $\delta I=0$ with linear function $r_1Q'(q)=2b_2q+b_1$, where $P_n^m(\cos\theta)e^{im\lambda}$ are normalized spherical harmonics, A_m $(m=0,1,...,n)$ are some arbitrary constants, and

$$\begin{cases}\psi_0=b_1/[2(r_0+b_2\kappa f_0^2/\varphi_0)],\\ \dot{\lambda}_z=[2\omega+a^2r_2/2b_1]/[n(n+1)-2],\\ \dot{\lambda}_0=\{-2\omega+\dot{\lambda}_z[n(n+1)-2]\}\ /\ \{n(n+1)+\kappa a^2f_0^2/\varphi_0\},\end{cases}\tag{3.2}$$

$$(n=2,3,...).$$

The proof of the Theorem is directly made by substituting of (3.1) and (3.2) into (2.13).

Note 3.1 For a given linear function $r_lQ'(q)$, there exists a solution to (2.13) only if r_0 and b_2 satisfy the following conditions

$$\begin{cases}\dfrac{r_0}{b_2}=-\left[\dfrac{n(n+1)}{a^2}+\kappa\dfrac{f_0^2}{\varphi_0}\right],\\ n=2,3,4,....\end{cases}\tag{3.3}$$

This means that for an arbitrarily given function r_1Q and parameters r_0 and r_2, there is not necessarily a solution to the equation $\delta I=0$, i. e. functional I might not have stationary point.

Note 3.2 For a linear Haurwitz wave determined by (3.1), we have

$$\Delta^2I=\delta^2I=r_0\iint_S\left\{\left[|\delta\overline{v}|^2+\kappa\frac{f_0^2}{\varphi_0}(\delta\psi)^2\right]-\frac{(\delta q)^2}{\dfrac{n(n+1)}{a^2}+\kappa\dfrac{f_0^2}{\varphi_0}}\right\}dS=\text{invariant},\tag{3.4}$$

where $\delta\psi$, $|\delta\overline{v}|$ and δq can be not small, i.e., the disturbance $\delta\psi$ can be a large amplitude one. Therefore, according to Theorem 2.3, we have:

Linear Haurwitz wave might be unstable, or, at least metastable.

Hoskines (1973) and many others have shown that linear Haurwitz wave

$$\psi=-a^2\dot{\lambda}_z\cos\theta+A_mP_n^m(\cos\theta)e^{im(\lambda-\dot{\lambda}_0t)}\tag{3.1}$$

can be stable or unstable with respect to the perturbations represented by maximumly truncated series of spherical harmonics, depending on the amplitude A_m, the wavenumber m of the Haurwitz wave and the wavenumber of the perturbations. However, the question on the stability of linear Haurwitz wave in the general case, i.e. when the perturbation posses infinitive degrees of freedom, is still open.

Now, denoting the energy and potential enstrophy of the perturbation by E' and P', i.e.

$$\begin{cases}E'\equiv\dfrac{1}{2}\ (\|\delta\overline{v}\|^2+\kappa\dfrac{f_0^2}{\varphi_0}\|\delta\psi\|^2)\ ,\\ P'\equiv\dfrac{1}{2}\|\delta q\|^2,\end{cases}\tag{3.5}$$

we can represent P' by E' as follows (see, Zeng, 1979),

$$P' = \frac{1}{a^2} N_p E', \tag{3.6}$$

and, consequently, rewrite (3.4) into the following

$$(1 - \frac{N_p}{N_h})E' = \frac{\delta^2 I}{2r_0}, \tag{3.7}$$

where

$$N_p \equiv n_p(n_p + 1) + a^2 \kappa f_0^2 / \varphi_0, \tag{3.8}$$

$$N_h \equiv n(n+1) + a^2 \kappa f_0^2 / \varphi_0, \tag{3.9}$$

and n_p is the weighted mean two–dimensional wavenumber on spherical surface of the perturbation. (3.7) tell us, that (i), if the initial mean scale is larger than that of basic flow, i. e. $n_p^{(0)} < n$, we have $\delta^2 I / 2r_0 > 0$ and $n_p < n$ all the time. Therefore, when energy descade, i.e. $n_p(t) < n_p^{(0)}$ takes place with the perturbation, its energy $E'(t)$ and potential enstrophy $P'(t)$ both decrease, i.e. $E'(t) < E'(0)$, and $P'(t) < P'(0)$; while when energy cascade $n_p(t) > n_p^{(0)}$ takes place, both $E'(t)$ and $P'(t)$ increase. (ii), if $n_p^{(0)} > n$ we have $\delta^2 I / 2r_0 < 0$ and $n_p(t) >$ n all the time. Therefore, $E'(t) < E'(0)$ and $P'(t) < P'(0)$ as $n_p(t) > n_p^{(0)}$, but $E'(t) > E'(0)$ and $P'(t) > P'(0)$ as $n_p(t) < n_p^{(0)}$. These mean, that energy and potential enstrophy of a perturbation always synchronously increase or decrease. This is quite different from that situation when the basic flow satisfies the sufficient condition of stability. In fact, according to the linear theory (see, Zeng, 1983) or the conservation of $\Delta^2 I$ represented by (2.9), a mutual compensation between energy and weighted potential enstrophy of the perturbation takes place as the basic flow satisfies the sufficient condition of stability.

From the above analysis we conclude that Haurwitz wave (3.1) is stable with respect to those perturbations whose $n_p(t) < n_p^{(0)} < n$ all the time as $\delta^2 I(0) / 2r_0 > 0$ or $n_p(t) > n_p^{(0)} > n$ as $\delta^2 I(0) / 2r_0 < 0$, but unstable with respect to all other perturbations. In this sence a Haurwitz wave might be referred to as metastable one.

Note 3.3 We can find out upper and lower boundaries of E'. Suppose that a basic flow is a Haurwitz wave given by (3.1) but rewritten as $\bar{\psi}$, and a perturbation is given by ψ' whose initial $\psi'^{(0)}$ is orthogonal to $\bar{\psi}^{(0)}$ (denoted by $\psi'^{(0)} \perp \bar{\psi}^{(0)}$) and angular momentum $M'^{(0)} = 0$. Otherwise, if there is any component "paralled" to $\bar{\psi}^{(0)}$ (denoted by $\psi'^{(0)}_{\parallel}$), we can subtract $\psi'^{(0)}_{\parallel}$ from $\psi'^{(0)}$ and add it to $\bar{\psi}^{(0)}$. For convenience we write

$$\psi' = \psi'_{\perp} + \psi'_{\parallel}, \tag{3.10}$$

and $\psi'^{(0)}_{\parallel} = 0$. Hereafter, we will denote an integral of function $F(\theta, \lambda)$ on the spherical surface as $<\mathrm{F}>$, for example, $\psi'^{(0)}_{\parallel} = 0$ means

$$\langle \bar{\psi}^{(0)} \psi'^{(0)} \rangle = 0. \tag{3.11}$$

The conservation of energy of disturbed flow $\psi = \bar{\psi} + \psi'$ yields

$$E(\overline{\psi}+\psi')=\overline{E}+E'+\langle\nabla\overline{\psi}\cdot\nabla\psi'+\kappa\frac{f_0^2}{\varphi_0}\overline{\psi}\psi'\rangle=E^{(0)},$$

and

$$E'+\langle\nabla\overline{\psi}\cdot\nabla\psi'+\kappa\frac{f_0^2}{\varphi_0}\overline{\psi}\psi'\rangle=E^{(0)}-\overline{E}=E'^{(0)}, \tag{3.12}$$

where $\overline{E}$ is the energy of the basic flow. The second equality in (3.12) is obtained by the use of (3.11). The conservation of $\delta^2 I$ and that of M' yield

$$E'-\frac{a^2}{N_b}P'=\frac{\delta^2 I(0)}{2r_0}=(1+\frac{N_p^{(0)}}{N_b})E'^{(0)}, \tag{3.13}$$

$$M'\equiv\langle\frac{\partial\psi'}{\partial\theta}\sin\theta-\kappa\frac{a^2 f_0^2}{\varphi_0}\psi'\cos\theta\rangle=M'^{(0)}=0. \tag{3.14}$$

Now, possible maxima or minima of E' under constraints (3.12), (3.13) and (3.14) can be determined by the Lagrange's method, i.e. by $\delta J=0$, where

$$J=E'+\lambda_1\left\{E'+\langle\nabla\overline{\psi}\cdot\nabla\psi'+\kappa\frac{f_0^2}{\varphi_0}\overline{\psi}\psi'\rangle\right\}$$
$$\lambda_2\left\{E'-(a^2/N_b)P'\right\}+\lambda_3 M', \tag{3.15}$$

and λ_1, λ_2 and λ_3 are some constants under determination. After elaborated calculations the upper and lower bounds of E', denoted by E'_u and E'_l respectively, are determined by the following

$$E'_u=\left(\frac{\lambda_1}{1+\lambda_1}\right)_u^2 A^2+D_u^2\equiv(E'_{\parallel}+E'_{\perp})_u, \tag{3.16}$$

$$E'_l=\left(\frac{\lambda_1}{1+\lambda_1}\right)_l^2 A^2+D_l^2\equiv(E'_{\parallel}+E'_{\perp})_l, \tag{3.17}$$

$$E'_{u,l}-E'^{(0)}=2\left(\frac{\lambda_1}{1+\lambda_1}\right)_{u,l}A^2, \tag{3.18}$$

where $E'_{\perp}$ and $E'_{\parallel}$ are the energy of components orthogonal and parallel to the whole family of "pure" Haurwitz wave (3.1) with $\lambda_z=0$ respectively,

$$A^2=\frac{1}{2}N_b\sum_{m=0}^{n}|A_m|^2, \tag{3.19}$$

$$\left[1-\frac{N'}{N_b}\right]_{u,p}D_{u,e}^2=\left[1-\frac{N_p^{(0)}}{N_b}\right]E'^{(0)}, \tag{3.20}$$

$$\left(\frac{\lambda_1}{1+\lambda_1}\right)_u=1+\left(1+\frac{E'^{(0)}-D_u^2}{A^2}\right)^{\frac{1}{2}}, \tag{3.21}$$

$$\left(\frac{\lambda_1}{1+\lambda_1}\right)_l=1-\left(1+\frac{E'^{(0)}-D_u^2}{A^2}\right)^{\frac{1}{2}}, \tag{3.22}$$

$$N' = n'(n'+1) + \kappa a^2 f_0^2 / \varphi_0 . \tag{3.23}$$

and n' is an integer. Depanding on $n_p^{(0)} > n, = n$ or $< n$, we have $n' \to \infty, = n$ or $= 1$ respectively.

If $E'^{(0)} / A^2 \ll 1$, we have

$$E'_u - E'^{(0)} \approx E'_u \approx 4A^2, \tag{3.24}$$

and

$$E'^{(0)} - E'_l \approx \begin{cases} E'^{(0)}\left[\left(\dfrac{n_p^{(0)}(n_p^{(0)}+1)-2}{n(n+1)-2}\right)+\varepsilon\right], & (n_p^{(0)} < n), \\ \varepsilon E'^{(0)}, & (n_p^{(0)} > n) \end{cases} \tag{3.25}$$

where $\varepsilon > 0$, and $O(\varepsilon) = O(E'^{(0)} / A^2)$.

(3.24) shows that

$$\psi' \approx -2(\bar{\psi} + a^2 \lambda_z \cos\theta).$$

This means that the basic wavy flow might completely be distorted, and ψ just has a phase angle opposite to that of $\bar{\psi}$. While, (3.25) shows that by using the nonlinear equation there is always some energy kept by the perturbation.

The results obtained above are illustrated by Fig.2.

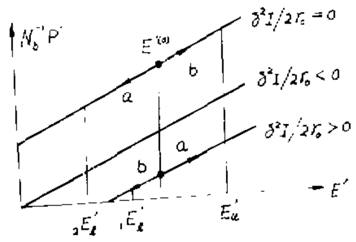

Fig. 2. Evolution of energy, E', and potential enstrophy, P', of the perturbations superimposed on linear Hourwitz wave. $_1E'_l$ and $_2E'_l$ are the lower bounds of the perturbation energy corresponding to $\delta^2 I / 2r_0 > 0$ and $\delta^2 I / 2r_0 < 0$, respectively; and $E'_u - E'^{(0)} \approx 4A^2$. a and b denote the directions corresponding to $n_p < n_p^{(0)}$ and $n_p > n_p^{(0)}$ respectively.

At the end we point out that nonlinear (generalized) Haurwitz waves can be determined by (3.1), where $Q'(q)$ is a nonlinear function of q. They might be stable or unstable depending on the conditions stated in Theorems 2.2 and 2.3.

IV. THE INFLUENCE OF OROGRAPHY ON THE STEADY FLOW AND ITS INSTABILITY

If orographic influence is taken into account we have the same equations (2.1) and (2.3), but instead of (2.2), now we have

$$q = \Delta\psi - \kappa\frac{f_0^2}{\varphi_0}\psi + (2\omega\cos\theta + \frac{f_0\varphi_s}{\varphi_0}), \tag{4.1}$$

where φ_s is the geopotential of the orography. The total energy and generalized enstrophy are still conserved, but in general case the total angular momentum is no longer a conservative integral. Therefore, we can take $I(\psi)$ defined by (2.4) as an invariant functional but with $r_2=0$, and, consequently, (2.5)–(2.19) still are all valid but with $r_2=0$ and q defined by (4.1). We have

Theorem 4.1 Every possible steady flow over the topography $h_s=\varphi_s/g$ is determined by the stationary point of I, i.e. by the equations

$$\begin{cases} -2r_0\psi + r_1 Q'(q) = 0, \\ q = \Delta\psi - \kappa\dfrac{f_0^2}{\varphi_0}\psi + (2\omega\cos\theta + \dfrac{f_0\varphi_s}{\varphi_0}). \end{cases} \tag{4.2}$$

The stability or instability of such flow with orographic influence can be determined according to Theorems 2.2 and 2.3.

Theorem 4.2 The steady disturbance generated by the orography and superimposed on a zonal flow with constant angular velocity $\dot{\lambda}_z$ (solid rotation) exists and is unique if $Q=q^2$ and $r_2=0$ are taken in I, and $(\kappa f_0^2/\varphi_0 + r_0/r_1)a^2 \neq -n(n+1)$, $n=1,2,\ldots$, where r_0/r_1 is determined by

$$-\dot{\lambda}_z = 2\omega\cdot\left[2+\left(\kappa\frac{f_0^2}{\varphi_0}+\frac{r_0}{r_1}\right)a^2\right]^{-1}. \tag{4.3}$$

The flow is stable if $r_0/r_1>0$, i. e.

$$-\left(1+\kappa\frac{f_0^2}{2\varphi_0}a^2\right)^{-1} \leqslant \frac{\dot{\lambda}_z}{\omega} < 0; \tag{4.4}$$

and it might be unstable if condition (4.4) is not satisfied.

Proof Taking $Q=q^2$, the governing equation for the steady flow is determined by $\delta I=0$ as follows:

$$\Delta\psi - \left(\kappa\frac{f_0^2}{\varphi_0}+\frac{r_0}{r_1}\right)\psi = -2\omega\cos\theta - \frac{f_0^2}{\varphi_0}\left(\frac{\varphi_s}{f_0}\right), \tag{4.5}$$

whose solution consisting of zonal flow with constant angular velocity $\dot{\lambda}_z$ and disturbance F generated by the orography is

$$\psi = -a^2\dot{\lambda}_z\cos\theta + F, \tag{4.6}$$

where $\dot{\lambda}_z$ is given by (4.3), and F satisfies the following equation

$$\Delta F - \left(\kappa\frac{f_0^2}{\varphi_0}+\frac{r_0}{r_1}\right)F = -\frac{f_0^2}{\varphi_0}\left(\frac{\varphi_s}{f_0}\right) \tag{4.7}$$

Therefore, F is uniquely determiend by φ_s, provided $(\kappa f_0^2/\varphi_0 + r_0/r_1)\cdot a^2 \neq -n(n+1)$, $n=1,2,\cdots$.

Next, according to Theorem 2.2, sufficient condition for stability is satisfied, provided $(r_0/r_1)a^2 \geqslant 0$. In our case,

$$\frac{r_0}{r_1}a^2 = -\left(2+\frac{f_0^2 a^2}{\varphi_0}+\frac{\omega}{\dot{\lambda}_z}\right),$$

and $(r_0/r_1)a^2>0$ results in (4.4). Theorem is proved.

Note 4.1 The conditions

$$\left(\kappa\frac{f_0^2}{\varphi_0}+\frac{r_0}{r_1}\right)a^2 = -n(n+1),\ n=1,2,\cdots, \tag{4.8}$$

correspond to the resonant cases, which can occur only if $\dot{\lambda}_z>0$ (westerlies) in accordance with (4.3). For a given n, which satisfies (4.8), there exist solutions to (4.5) only if

$$\iint_S \left[2\omega\cos\theta+\left(\frac{f_0^2}{\varphi_0}\right)\frac{\varphi_s}{f_0}\right]P_n^m(\cos\theta)e^{im\lambda}dS=0. \tag{4.9}$$

$$m=0,1,2,\cdots,n,$$

and the solutions consist of forced and free wave parts represented by arbitrary combinations of spherical harmonics with the same n.

Note 4.2 It seems that the orography might not influence the instability because φ_s does not include in the criterion (4.4) for stability. In fact, this is not true because in the absence of orography the solid rotation can be obtained by taking $r_1=0$ but $r_0\neq 0$ and $r_2\neq 0$, hence it is always stable in accordance with Theorem 2.2. However, similarly to Haurwitz wave, the wave-like basic flow induced by the orography embedded in westerlies might be unstable.

Note 4.3 In the real atmosphere $1+(f_0^2/2\varphi^0)a^2\approx 3$. Therefore, the orographically induced steady motion superimposed on uniform westerlies or easterlies with angular velocity $0>\dot{\lambda}_z>-\omega/3$ is stable, and might be unstable as $\dot{\lambda}_z>0$ or $\dot{\lambda}_z<-\omega/3$.

V. GENERAL THEOREMS FOR THE THREE-DIMENSIONAL QUASI-GEOSTROPHIC MODEL

The governing equation is the conservation of potential vorticity written in the same form as (2.1), but q is defined as follows:

$$q=\Delta\psi+\frac{\partial}{\partial\zeta}\left(\frac{f_0^2\zeta^2}{c^2}\frac{\partial\psi}{\partial\zeta}\right)+2\omega\cos\theta, \tag{5.1}$$

where $0\leqslant\zeta\leqslant 1$, $c^2\equiv\alpha R\tilde{T}$, $\alpha=R(\gamma_a-\tilde{\gamma})/g$, $\gamma_a=g/c_p$. $\tilde{T}(z)$ and $\tilde{\gamma}$ are the mean temperature and its vertical gradient respectively. ψ should also satisfy two boundary conditions (see Zeng, 1979):

$$E<\infty, \tag{5.2}$$

$$\left(\frac{\partial}{\partial t}+\kappa'\vec{v}_s\cdot\nabla\right)b=0,\quad \left(b\equiv\left(\frac{\partial\psi}{\partial\zeta}\right)_s+\kappa\alpha_s\psi_s\right), \tag{5.3}$$

where E is total energy (see below), the subscript s denotes the function given at the bottom boundary $\zeta=1$, the orographic influence is omitted, κ and $\kappa'=0.1$, $\kappa=0$ corresponds to the approximation of vertically integrated nondivergency, and $\kappa'=0$ corresponds to isentropic bottom boundary.

From the conservation of potential vorticity and the boundary conditions (5.2) and (5.3), we have conservations of total energy E, "generalized enstrophy" F, angular momentum M and the "generalized boundary energy" B derived from (5.3) (see below). The invariant functional $I(\psi)$ suitable for our purpose is a linear combination of all the conservations mentioned above with some parameters r_n, $n=0,1,2,3$,

$$I(\psi) = 2r_0 E + r_1 F + r_2 M + 2r_3 B = \text{invariant}, \tag{5.4}$$

where

$$E \equiv \frac{1}{2}\iint_S \left\{\kappa \frac{f_0^2 \alpha_s}{c_s^2}\psi_s^2 + \int_0^1 \left[|\nabla\psi|^2 + \left(\frac{f_0\zeta}{c}\frac{\partial\psi}{\partial\zeta}\right)^2\right]d\zeta\right\}dS, \tag{5.5}$$

$$F \equiv \iint_S \int_0^1 Q(q)d\zeta dS, \tag{5.6}$$

$$M \equiv \iint_S \left\{-\kappa\frac{f_0^2\alpha_s a^2}{c_s^2}\psi_s + \int_0^1 v_\lambda a\sin\theta d\zeta\right\}dS, \tag{5.7}$$

$$B \equiv \iint_S G(b)dS, \tag{5.8}$$

and G is an arbitrary function of argument b.

Now, the first and second variations are as follows

$$\begin{aligned}\delta I = &\iint_S \int_0^1 \left[-2r_0\psi + r_1 Q'(q) + r_2 a^2\cos\theta\right]\delta q d\zeta dS \\ &+ \iint_S \left\{r_3 G'(b) + \frac{f_0^2}{c_s^2}(2r_0\psi_s - r_2 a^2\cos\theta)\right\}\left[\left(\frac{\partial\delta\psi}{\partial\zeta}\right)_s + \kappa\alpha_s\delta\psi_s\right]dS,\end{aligned} \tag{5.9}$$

$$\begin{aligned}\delta^2 I = &\iint_S \int_0^1 \left\{r_0\left[|\nabla\delta\psi|^2 + \left(\frac{f_0\zeta}{c}\frac{\partial\delta\psi}{\partial\zeta}\right)^2\right] + \frac{r_1}{2}Q''(q)(\delta q)^2\right\}d\zeta dS \\ &+ \iint_S \left\{r_0\kappa\frac{f_0^2\alpha_s}{c_s^2}(\delta\psi_s)^2 + \frac{1}{2}r_3 G''(b)(\delta b)^2\right\}dS.\end{aligned} \tag{5.10}$$

And

$$I(\psi + \delta\psi) - I(\psi) = \delta I + \Delta^2 I, \tag{5.11}$$

where

$$\begin{aligned}\Delta^2 I = &\iint_S \int_0^1 \left\{r_0\left[|\nabla\delta\psi|^2 + \left(\frac{f_0\zeta}{c}\frac{\partial\delta\psi}{\partial\zeta}\right)^2\right] + \frac{r_1}{2}Q''(q^*)(\delta q)^2\right\}d\zeta dS \\ &+ \iint_S \left\{r_0\kappa\frac{f_0^2\alpha_s}{c_s^2}(\delta\psi_s)^2 + \frac{r_3}{2}G''(b^*)(\delta b)^2\right\}dS.\end{aligned} \tag{5.12}$$

$$q^* = q + r^* \delta q, \qquad 0 \leq r^* \leq 1,$$
$$h^* = h + r^{**} \delta h, \qquad 0 \leq r^{**} \leq 1.$$

Theorem 5.1 Every function $\psi(\theta, \lambda - \dot{\lambda}_0 t, \zeta)$ which is a solution of the three–dimensional quasi–geostropic model (5.1) and satisfies the boundary conditions (5.2) and (5.3) is a stationary point of $I(\psi)$ and satisfies the following equation and boundary conditions:

$$-2r_0\psi + r_1 Q'(q) + r_2 a^2 \cos\theta = 0, \tag{5.13}$$
$$E < \infty, \tag{5.14}$$
$$r_3 G'(h) + \frac{f_0^2}{c_s^2}(2r_0\psi_\zeta - r_2 a^2 \cos\theta) = 0, \tag{5.15}$$

where Q and G are two given functions and r_n are some parameters, $n=0,1,2,3$. The phase angular velocity $\dot{\lambda}_0 = -r_2 / 2r_0$. The inverse theorem is also true.

The proof of Theorem 5.1 is essentially the same as that of Theorem 2.1.

Theorem 5.2 A three–dimensional basic flow $\psi(\theta, \lambda - \dot{\lambda}_0 t, \zeta)$ which is determined by $\delta I = 0$ with functions $Q(q)$ and $G(h)$ and parameters r_n $(n=0,1,2,3,)$ is always stable with respect to every of those small perturbations $\delta\psi$, if r_0, $r_1 Q''(q^*)$, $r_0 \kappa \alpha_s / c_s^2$ and $r_3 G''(h^*)$ all are either non–negative or non–positive, and the stability holds with respect to every large amplitude perturbation if $Q''(x)$ and $G''(x)$ as functions of their argument x are also sign–definitive or equal to zero.

Theorem 5.3 A flow $\psi(\theta, \lambda - \dot{\lambda}_0 t, \zeta)$ might be unstable if either r_0, $r_1 Q''$, $r_0 \kappa \alpha_s / c_s^2$ and $r_3 G''$ do not have the same sign or one of Q'' and G'' is a sign–nondefinitive function of $(\theta, \lambda, \zeta, t)$.

The proof of Theorems 5.2 and 5.3 is also essentially the same as that of Theorems 2.2 and 2.3, but with three–dimensional norms, $\| \cdot \|_3$ in L_2–space and $\| \cdot \|_{3w}$ in Sobolev space. Similar to (2.15), now we take

$$\|\delta\psi\|_{3w}^2 = \left| r_0 \kappa \frac{f_0^2}{\varphi_0} \right| \cdot \|\delta\psi_\zeta\|^2 + \left| \frac{1}{2} r_3 G''_m \right| \cdot \|\delta h\|^2 + |r_0| \cdot \|\nabla_3 \delta\psi\|_3^2$$
$$+ \left| \frac{1}{2} r_1 Q''_m \right| \cdot \|\delta q\|_3^2, \tag{5.16}$$

where $\| \cdot \|$ is the same as defined in (2.15), i.e. defined on a spherical surface with radius a, and $|Q''_m|$ and $|G''_m|$ are the lower bounds of $|Q''(q^*)|$ and $|G''(h^*)|$ respectively, i.e.

$$\begin{cases} |Q''(q^*)|_{\delta\psi \in S_\varepsilon} \geq Q_m, \\ |G''(h^*)|_{\delta\psi \in S_\varepsilon} \geq G_m. \end{cases} \tag{5.17}$$

$\nabla_3 \delta\psi$ is a pseudo–three dimensional operator,

$$\nabla_3 \delta\psi \equiv \nabla\delta\psi + \overline{k^2} \left(\frac{f_0 \zeta}{c}\right) \frac{\partial \delta\psi}{\partial \zeta}. \tag{5.18}$$

and its norm $\|\nabla_3 \delta\psi\|_3$ is determined as follows.

$$\|\nabla_3 \delta\psi\|_3^2 = \|\nabla \delta\psi\|_3^2 + \|\frac{f_0\zeta}{c}\frac{\partial \delta\psi}{\partial \zeta}\|_3^2 = \|\delta\vec{v}\|_3^2 + \|\frac{f_0\zeta}{c}\frac{\partial\delta\psi}{\partial\zeta}\|_3^2. \tag{5.19}$$

We have

$$\|\delta\psi\|_{3w}^2 \leq |\Delta^2 I^{(0)}|, \qquad (0 \leq t < \infty). \tag{5.20}$$

Therefore, $\|\delta\psi\|_{3w}^2 < \delta$ all the time as $|\Delta^2 I^{(0)}| < \delta$.

The geometric representation of Theorem 5.2 is similar to Fig.1.

Note 5.1 A whole family of Haurwitz waves in the three–dimensional baroclinic atmosphere can be determined by $\delta I = 0$, and we have the similar results as in Section III. Indeed, such baroclinic Haurwitz waves have given in the literature (see Zeng. 1979).

Note 5.2 The criterion of instability obtained by Blumen (1968) for the steady basic flow and the quasi–geostrophic model with isentropic bottom surface is obviously a special case of our general criterion. That criterion of instability obtained by Blumen (1978) and Zeng (1983) for the steady and zonal basic flow and the linearized model without the assumption on the isentropicity of bottom surface is also a special case of our general criterion. However it is necessary to point out that the general criterion of instability for an unsteady basic flow can be obtained only by our generalized variational method with $r_2 \neq 0$.

Note 5.3 When orographic influence is taken into account, the conservation of potential vorticity as well as equations (5.1)–(5.3) are all valid but

$$b \equiv (\frac{\partial\psi}{\partial\zeta})_s + \kappa\alpha_s\psi_s + \alpha_s f_0^{-1}\varphi_s. \tag{5.21}$$

where $Z_s(\theta, \lambda) = \varphi_s(\theta, \lambda)/g$ is the elevation of the orography. The conservation of angular momentum is no longer valid, and we have results similar to Section IV. Theorems 5.1, 5.2 and 5.3 are all valid but with $r_2 = 0$, hence steady flows generated by orographic influence can be obtained by $\delta I = 0$, and the orographic influence on the conditions for stability are included in $G''(b)$ explicitly and in such a manner as described in Note 4.2.

VI. BAROTROPIC PRIMITIVE EQUATIONS

Governing equations are exactly those as for the shallow water but written on a rotating spherical surface. They can be transferred into the follows:

$$\frac{\partial v_\theta}{\partial t} - \varphi q v_\lambda = -\frac{\partial K}{a\partial\theta}, \tag{6.1}$$

$$\frac{\partial v_\lambda}{\partial t} + \varphi q v_\theta = -\frac{\partial K}{a\sin\theta\partial\lambda}, \tag{6.2}$$

$$\frac{\partial q}{\partial t} + \vec{v}\cdot\nabla q = 0, \tag{6.3}$$

where

$$K = \varphi + \frac{1}{2}(v_\theta^2 + v_\lambda^2). \tag{6.4}$$

$$q=\frac{1}{\varphi}\left\{\frac{1}{a\sin\theta}\left(\frac{\partial v_\lambda\sin\theta}{\partial\theta}-\frac{\partial v_\theta}{\partial\lambda}\right)+2\omega\cos\theta\right\},\tag{6.5}$$

and φ is the geopotential of the free surface. It is not difficult to prove that equations (6.1)–(6.3) are equivalent to those commonly used in fluid mechanics and dynamic meteorology. In fact the continuity equation

$$\frac{\partial\varphi}{\partial t}+\nabla\cdot\varphi\vec{v}=0\tag{6.6}$$

can be easily obtained from (6.1). (6.2) and the conservation of potential vorticity (6.3). $\nabla\cdot(\quad)$ in (6.6) is the two–dimensional divergency operator on a spherical surface with radius a.

Again we have the conservations of mass, angular momentum, energy and generalized enstrophy, and can construct an invariant functional as follows

$$2I(\vec{v},\ \varphi)=2r_0E+r_1F+2r_2M+r_3Ma,\tag{6.7}$$

where

$$E\equiv\frac{1}{2}\iint_S[\varphi|\vec{v}|^2+\varphi^2]dS,\tag{6.8}$$

$$F\equiv\iint_S\varphi Q(q)dS.\tag{6.9}$$

$$M\equiv\iint_S\varphi a(v_\lambda+a\omega\sin\theta)\sin\theta dS,\tag{6.10}$$

$$Ma\equiv\iint_S\varphi dS,\tag{6.11}$$

and $Q(q)$ is an arbitrary function of its argument q.

q is a function of $\vec{v}$ and φ. For convenience we denote the difference between $q(\vec{v}+\delta\vec{v},\varphi+\delta\varphi)$ and $q(\vec{v},\varphi)$ by δq , i. e.,

$$\begin{aligned}\delta q&\equiv q(\vec{v}+\delta\vec{v},\varphi+\delta\varphi)-q(\vec{v},\varphi)\\&=\frac{1}{\varphi+\delta\varphi}\left[\frac{1}{a\sin\theta}\left(\frac{\partial\delta v_\lambda\sin\theta}{\partial\theta}-\frac{\partial\delta v_\theta}{\partial\lambda}\right)\right]-q\frac{\delta\varphi}{\varphi+\delta\varphi}.\end{aligned}\tag{6.12}$$

We have

$$\delta q=\delta^1q+\delta^2q+\ldots,\tag{6.13}$$

and

$$\delta q=\delta^1q+\Delta^2q.\tag{6.14}$$

where

$$\delta^1q=\frac{1}{\varphi a\sin\theta}\left(\frac{\partial\delta v_\lambda\sin\theta}{\partial\theta}-\frac{\partial\delta v_\theta}{\partial\lambda}\right)-q\frac{\delta\varphi}{\varphi}.\tag{6.15}$$

$$\delta^2q=\frac{-1}{a\sin\theta}\left(\frac{\partial\delta v_\lambda\sin\theta}{\partial\theta}-\frac{\partial\delta v_\theta}{\partial\lambda}\right)\frac{\delta\varphi}{\varphi}+q\left(\frac{\delta\varphi}{\varphi}\right)^2=-\left(\frac{\delta\varphi}{\varphi}\right)\delta^1q.\tag{6.16}$$

$$\Delta^2 q = \delta q - \delta^1 q = \frac{\delta\varphi}{\varphi}\Bigg[\left(\frac{-1}{\varphi+\delta\varphi}\right) \frac{1}{a\sin\theta} \cdot \left(\frac{\partial \delta v_\lambda \sin\theta}{\partial\theta} - \frac{\partial \delta v_\theta}{\partial\lambda}\right) + q\frac{\delta\varphi}{\varphi+\delta\varphi}\Bigg] = -\left(\frac{\delta\varphi}{\varphi}\right)\delta q. \tag{6.17}$$

By using formulas (6.15)–(6.17) we obtain the first and second variations of functional I as follows,

$$2\delta I = \iint_S \Bigg\{\Bigg[r_0(2\varphi + |\vec{v}|^2) + 2r_2(v_\lambda + a\omega\sin\theta)a\sin\theta + r_1(Q - qQ') + r_3 \Bigg]\delta\varphi + \Bigg[r_0 2\varphi v_\theta + r_1 Q'' \frac{\partial q}{a\sin\theta\partial\lambda}\Bigg]\delta v_\theta + \Bigg[r_0 2\varphi v_\lambda + 2r_2\varphi a\sin\theta - r_1 Q''\frac{\partial q}{a\partial\theta}\Bigg]\delta v_\lambda\Bigg\} dS, \tag{6.18}$$

$$\begin{aligned} 2\delta^2 I = & \iint_S \Bigg\{\Bigg[r_0\Big[\varphi|\delta\vec{v}|^2 + (\delta\varphi)^2 + 2\delta\varphi\vec{v}\cdot\delta\vec{v}\Big] + r_1\Big[\frac{\varphi}{2}Q''(q)(\delta^1 q)^2 + Q'(q)(\varphi\delta^2 q \\ & + \delta\varphi\delta^1 q)\Big] + 2r_2 a\delta\varphi\delta v_\lambda\sin\theta\Bigg\} dS \\ = & \iint_S \Bigg\{ r_0\varphi\Big[\delta v_\lambda + (v_\lambda + a\frac{r_2}{r_0}\sin\theta)\frac{\delta\varphi}{\varphi}\Big]^2 + r_0\varphi\Big[\delta v_\theta + v_\theta\frac{\delta\varphi}{\varphi}\Big]^2 \\ & + r_0\left(1 - \frac{(v_\lambda + ar_0^{-1}r_2\sin\theta)^2 + v_\theta^2}{\varphi}\right)\Big[\delta\varphi\Big]^2 + r_1\frac{\varphi}{2}Q''(q)\Big[\delta^1 q\Big]^2\Bigg\} dS. \end{aligned} \tag{6.19}$$

The difference between $I(\vec{v}+\delta\vec{v},\ \varphi+\delta\varphi)$ and $I(\vec{v},\varphi)$ is given by

$$I(\vec{v}+\delta\vec{v},\ \varphi+\delta\varphi) - I(\vec{v},\ \varphi) = \delta I + \delta^2 I + \dots, \tag{6.20}$$

or

$$I(\vec{v}+\delta\vec{v},\varphi+\delta\varphi) - I(\vec{v},\ \varphi) = \delta I + \Delta^2 I, \tag{6.21}$$

where

$$\begin{aligned} 2\Delta^2 I = & \iint_S \Bigg\{ r_0\Big[\varphi^{**}|\delta\vec{v}|^2 + (\delta\varphi)^2 + 2\delta\varphi(\vec{v}\cdot\delta\vec{v})\Big] + r_1\Big[\frac{1}{2}\varphi^{**}Q''(q^*)(\delta q)^2 \\ & + \varphi Q'(q)\Delta^2 q + Q'(q)\delta\varphi\delta q\Big] + 2r_2 a\delta\varphi\delta v_\lambda\sin\theta\Bigg\} dS \\ = & \iint_S \Bigg\{ r_0\varphi^{**}\Big[\delta v_\lambda + (v_\lambda + ar_2r_0^{-1}\sin\theta)\frac{\delta\varphi}{\varphi^{**}}\Big]^2 \\ & + r_0\varphi^{**}\Big[\delta v_\theta + v_\theta\frac{\delta\varphi}{\varphi^{**}}\Big]^2 + r_1\frac{\varphi^{**}Q''(q^*)}{2}\Big[\delta q\Big]^2 \\ & + r_2\left(1 - \frac{(v_\lambda + ar_2r_0^{-1}\sin\theta)^2 + v_\theta^2}{\varphi^{**}}\right)\Big[\delta\varphi\Big]^2\Bigg\} dS, \end{aligned} \tag{6.22}$$

$$(\varphi^{**} \equiv \varphi + \delta\varphi,\ q^* = q + r^*\delta q,\ 0 \leqslant r^* \leqslant 1).$$

(6.22) differs from (6.19) in that the coefficients, φ and $Q''(q)$, in the integrand are replaced by φ^{**} and $Q''(q^*)$ respectively due to the fact that

$$(\varphi-\delta\varphi)|\vec{v}+\delta\vec{v}|^2=(\varphi+\delta\varphi)(|\vec{v}|^2+2\vec{v}\cdot\delta\vec{v}+|\delta\vec{v}|^2)=\varphi|\vec{v}|^2$$
$$+\left[|\vec{v}|^2\delta\varphi+2\varphi\vec{v}\cdot\delta\vec{v}\right]+\left[2\delta\varphi\vec{v}\cdot\delta\vec{v}+\varphi^{**}\delta\vec{v}|^2\right],$$
$$(\varphi+\delta\varphi)Q(q-\delta q)=(\varphi+\delta\varphi)\left[Q(q)+Q'(q)\delta q+\frac{1}{2}Q''(q^*)(\delta q)^2\right]=\varphi Q(q)$$
$$+\left[Q(q)\delta\varphi+\varphi Q'(q)\delta^1 q\right]+\left[\varphi Q'(q)\Delta^2 q+Q'(q)\delta\varphi\delta q+\frac{1}{2}\varphi^{**}Q''(q^*)(\delta q)^2\right].$$

Theorem 6.1 Every propagating wave solution set $(v_\theta(\theta,\lambda-\dot{\lambda}_0 t);\ v_\lambda(\theta,\lambda-\dot{\lambda}_0 t),\ \varphi(\theta,\lambda-\dot{\lambda}_0 t))$ to the primitive equations (6.1)–(6.3) corresponds to a stationary point of $I(\vec{v},\varphi)$ in the functional space $(\vec{v},\varphi)$, and is determined by the following equations:

$$2\Phi\equiv 2r_0\left[K+\frac{r_2}{r_0}a\sin\theta(v_\lambda+a\omega\sin\theta)\right]=-r_1(Q-qQ')-r_3. \tag{6.23}$$

$$\varphi q v_\theta=-\frac{\partial\Phi}{a\sin\theta\partial\lambda}. \tag{6.24}$$

$$\varphi q\left(v_\lambda+\frac{r_2}{r_0}a\sin\theta\right)=\frac{\partial\Phi}{a\partial\theta}. \tag{6.25}$$

The phase velocity $\dot{\lambda}_0=-r_2/r_0$, provided $r_0\neq 0$. The inverse theorem is also true.

Proof Let $(\vec{v}(\theta,\lambda-\dot{\lambda}_0 t),\ \varphi(\theta,\lambda-\dot{\lambda}_0 t))$ be a solution to the equations (6.1)–(6.3). Substituting them into (6.1)–(6.3), taking the relationship $\partial/\partial t=-\dot{\lambda}_0\partial/\partial\lambda$ into account, we have

$$-\dot{\lambda}_0\frac{\partial v_\theta}{\partial\lambda}-\varphi q v_\lambda=-\frac{\partial K}{a\partial\theta}. \tag{6.26}$$

$$-\dot{\lambda}_0\frac{\partial v_\lambda}{\partial\lambda}+\varphi q v_\theta=-\frac{\partial K}{a\sin\theta\partial\lambda}. \tag{6.27}$$

$$-\dot{\lambda}_0\frac{\partial q}{\partial\lambda}+\vec{v}\cdot\nabla q=0. \tag{6.28}$$

Introducing a function Φ by the first equality in (6.23) with $r_2/r_0=-\dot{\lambda}_0$ and $r_0=1$, we have

$$-\frac{\partial K}{a\partial\theta}=-\frac{\partial\Phi}{a\partial\theta}-\dot{\lambda}_0\varphi q a\sin\theta-\dot{\lambda}_0\frac{\partial v_\theta}{\partial\lambda},$$
$$-\frac{\partial K}{a\sin\theta\partial\lambda}+\dot{\lambda}_0\frac{\partial v_\lambda}{\partial\lambda}=-\frac{\partial\Phi}{a\sin\theta\partial\lambda}.$$

Substituting them into (6.26) and (6.27) results in (6.24) and (6.25). We have also

$$\vec{v}\cdot\nabla q=\left(\frac{\partial\Phi}{\varphi q a\partial\theta}+\dot{\lambda}_0 a\sin\theta\right)\frac{\partial q}{a\sin\theta\partial\lambda}-\left(\frac{\partial\Phi}{\varphi q a\sin\theta\partial\lambda}\right)\frac{\partial q}{a\partial\theta}$$
$$=\frac{1}{\varphi q}J(\Phi,q)+\dot{\lambda}_0\frac{\partial q}{\partial\lambda}. \tag{6.29}$$

Substituting it into (6.28) yields

$$\left(\frac{1}{\varphi q}\right)J(\Phi,\ q)=0. \tag{6.30}$$

Therefore, 2Φ is a function of argument q. Now, we can determine a function $r_1Q(q)$ by solving the following ordinary differential equation

$$r_1[Q(q)-qQ'(q)]-r_3=2\Phi(q), \tag{6.31}$$

where r_1 and r_3 are constants, and $r_1\neq 0$. Therefore, (6.23) is also satisfied. The theorem is proved.

Now we prove the inverse theorem. Suppose that functions $(\vec{v},\ \varphi)$ satisfy equations (6.23)–(6.25). Representing Φ by K and v_λ in accordance with (6.23), from (6.24) and (6.25) we obtain (6.26) and (6.27) with $\dot{\lambda}_0=-r_2/r_0$.

Next, from (6.24) and (6.25) we also obtain (6.29). In addition, the second equality of (6.23) means that 2Φ is a function of argument q, consequently $J(\Phi,\ q)=0$. Therefore, from (6.29) we obtain (6.28). All these show that functions $(\vec{v},\varphi)$ indeed construct a propagating wave solution set to equations (6.1)–(6.3), and $\dot{\lambda}_0=-r_2/r_0$.

Note 6.1 In order to seek some special solutions to the primitive equations Zeng in his book (1979) had already obtained equations (6.23)–(6.25) with $r_2=0$ for determining steady flows and developed a method similar to solving (6.23)–(6.25) with $r_2\neq 0$ for obtaining generalized Haurwitz waves which are solutions to the primitive equations but modifications of classical Haurwitz waves given in Section III, although the variational principle had not yet been indicated there. Some such solutions can be found in Zeng's book and Zeng, Zhang and Yuan's paper (1985).

Note 6.2 Suppose that for a given set (r_0, r_1, r_2, r_3 and $Q(q)$) equation $\delta I=0$ provides a solution $(\vec{v},\ \varphi)$ to equations (6.1)–(6.3). Repeating the procedure of proving Theorem 6.1 by using this solution $(\vec{v},\ \varphi)$ we can construct a new function $Q(q)$ by (6.31) but with $r_0\neq 0$. This means that we can always take $r_0\neq 0$ without any loss of generality. Moreover, we can also take $r_0=r_1=1$ without loss of generality, because r_1 can be considered as a coefficient in Q, and $r_1=0$ is equivalent to $Q\equiv 0$.

Theorem 6.2 A flow determined by $\delta I=0$ is stable with respect to those perturbations subspace whose $\Delta^2 I$ is either non–negative or non–positive functional of $(\delta\vec{v},\ \delta\varphi,\ \delta q)$, but might be unstable with respect to the supplementary subspace.

The proof of Theorem 6.2 is essentially the same as for Theorem 2.2 and 2.3. Especially, if for a given perturbation $(\delta\vec{v},\ \delta\varphi)$ $Q''(q^*)$ and φ^{**} are bounded from below by Q''_m and φ_m respectively and satisfy

$$Q''(q^*)\geqslant Q''_m\geqslant 0, \tag{6.32}$$

$$\begin{cases}\varphi^{**}\geqslant\varphi_m, \\ \text{and}\quad \varphi_m\geqslant(v_\lambda-a\dot{\lambda}_0\sin\theta)^2+v_\theta^2,\end{cases} \tag{6.33}$$

all the time $t>0$, we can take $\Delta^2 I$ or a simpler one,

$$\begin{aligned}\|\delta\vec{v},\delta\varphi\|_u^2=\varphi_m\Big[\ &\|\delta v_\lambda+(v_\lambda-a\dot{\lambda}_0\sin\theta)\frac{\delta\varphi}{\varphi^{**}}\|^2+\|\delta v_\theta+v_\theta\frac{\delta\varphi}{\varphi^{**}}\|^2\Big]\\ &+(1-Fr_m)\|\delta\varphi\|^2+\frac{1}{2}\varphi_m Q''_m\|\delta q\|^2,\end{aligned} \tag{6.34}$$

as the Liapounoff's norm and have

$$\| \delta \vec{v}, \delta \varphi \|_a^2 \leqslant \Delta^2 I^{(0)}, \tag{6.35}$$

where Fr_m is the upper bound of Froude number

$$Fr_m = \max\left(\frac{[v_\lambda - a\dot{\lambda}_0 \sin\theta]^2 + v_\theta^2}{\varphi^{**}} \right), \tag{6.36}$$

and $r_0 = r_1 = 1$. If $Fr_m \neq 1$, the uniform boundedness of $\|\delta\varphi\|^2, \|\delta v_\lambda + (v_\lambda - a\dot{\lambda}_0 \sin\theta)\delta\varphi / \varphi^{**}\|^2$ and $\|\delta v_\theta + v_\theta \delta\varphi / \varphi^{**}\|^2$ is guaranteed by (6.35), hence $\|\delta v_\theta\|^2$ and $\|\delta v_\lambda\|^2$ are also uniformly bounded because

$$\begin{aligned} \|\delta v_\lambda\| &= \|\delta v_\lambda + (v_\lambda - a\dot{\lambda}_0 \sin\theta)\frac{\delta\varphi}{\varphi^{**}} - (v_\lambda - a\dot{\lambda}_0 \sin\theta)\frac{\delta\varphi}{\varphi^{**}}\| \\ &\leqslant \|\delta v_\lambda + (v_\lambda - a\dot{\lambda}_0 \sin\theta)\frac{\delta\varphi}{\varphi^{**}}\| + \|(v_\lambda - a\dot{\lambda}_0 \sin\theta)\frac{\delta\varphi}{\varphi^{**}}\| \\ &\leqslant \|\delta v_\lambda + (v_\lambda - a\dot{\lambda}_0 \sin\theta)\frac{\delta\varphi}{\varphi^{**}}\| + \|v_\lambda - a\dot{\lambda}_0 \sin\theta\| \frac{1}{\varphi_m} \|\delta\varphi\|, \end{aligned}$$

$$\|\delta v_\theta\| \leqslant \|\delta v_\theta + v_\theta \frac{\delta\varphi}{\varphi^{**}}\| + \|v_\theta\| \frac{1}{\varphi_m} \|\delta\varphi\|.$$

The necessary condition for instability① is either (a), $Q''(q^*)$ is not a non-negative function, i.e. there exists some area where

$$Q''(q^*) < 0, \tag{6.37}$$

or (b), there exists some area where

$$1 - \frac{(v_\lambda - a\dot{\lambda}_0 \sin\theta)^2 + v_\theta^2}{\varphi^{**}} < 0, \tag{6.38}$$

Note 6.3 According to the classification of instabilities for the zonal and steady flow in a linearized model (Zeng, 1979; 1986), condition (6.37) gives rise of barotropic instability and inertial instability (symmetric instability is its special case), and condition (6.38) is the cause of super-critically high speed instability, which was first discovered by Lin (1955) in aerodynamics and Zeng (1962, unpublished paper) in rotating one-dimensional shallow water and then extended to the two-dimensional shallow water without Coriolis force by Blumen (1970) and Satomura (1981) and with Coriolis force by Zeng (1979).

Note 6.4 In the case of zonal and steady basic flow the necessary conditions (6.37) and (6.38) for the existence of instability are essentially the same as obtained by the linear theory but quantitatively different. Instead of condition (6.38) we obtain

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① Due to a carefulless omission of $\delta^2 q$ or $\Delta^2 q$ there is an additional term in the criteria of instability in the extended abstract published in the Proceedings of International Summer Colloquium on Nonlinear Dynamics of the Atmosphere, where one can also find the same error which leads to an incorrect sufficient condition of stability in the continuous baroclinic atmosphere.

$$1-\frac{(v_{\lambda}-a\dot{\lambda}_0\sin\theta)^2}{\varphi}<0 \tag{6.39}$$

by the linear theory (Zeng, 1979; 1986) or by the analysis of $\delta^2 I$. Comparison of (6.39) to (6.38) shows that when $Q''>0$ (while there may occur only the super–critically high speed instability) the domain of existence of instability is larger than that predicted by the linear theory. In fact, if the basic zonal flow is marginally stable, a small but negative $\delta\varphi$ might make condition (6.38) be satisfied, therefore stability of the basic flow might be broken down. Similar consideration can also be applied to the barotropic and inertial instabilities. This example tells us the importance of nonlinear consideration although the linearization of governing equations seems to be valid with respect to small perturbations.

VII. LAYER MODEL

Supposing there are J layers of shallow fluids, whose top surfaces, densities and velocities are denoted as Z_k, ρ_k and $\vec{v}_k$ respectively, $k=1,2,\ldots,J$, (see Fig. 3). We have the following governing equations (see Zeng, 1979):

$$\frac{\partial v_{\theta k}}{\partial t}+\vec{v}_k\cdot\nabla v_{\theta k}=-\frac{\partial\varphi_k}{a\partial\theta}+(2\omega\cos\theta+\frac{v_{\lambda}}{a}\mathrm{ctg}\theta)v_{\lambda}, \tag{7.1}$$

$$\frac{\partial v_{\lambda k}}{\partial t}+\vec{v}_k\cdot\nabla v_{\lambda k}=-\frac{\partial\varphi_k}{a\sin\theta\partial\lambda}-(2\omega\cos\theta+\frac{v_{\lambda}}{a}\mathrm{ctg}\theta)v_{\theta}, \tag{7.2}$$

$$\frac{\partial h_k}{\partial t}+\nabla\cdot\vec{v}_k h_k=0, \tag{7.3}$$

$$k=1,2,\ldots,J,$$

where $h_k\equiv Z_k-Z_{k+1}$ is the thickness of layer k, φ_k the reduced geopotential of the top surface Z_k, i.e.

$$\varphi_k\equiv\sum_{k'=1}^{k}\frac{\rho_{k'}-\rho_{k'-1}}{\rho_k}gZ_{k'}, \tag{7.4}$$

$\rho_0\equiv 0$, and $Z_{J+1}(\theta,\lambda)$ is a given bottom topography.

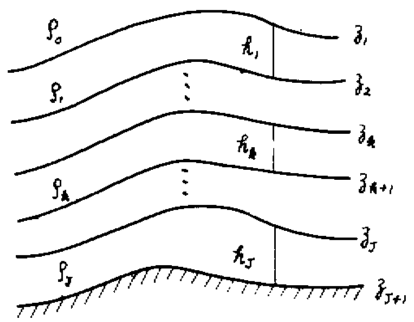

Fig. 3. A layer model.

This model is a good approximation of the oceanographic motions rather than atmospheric one. However, this model is very interesting because it partly represents the baroclinity, and hence by the help of this model we are able to explore the mechanism leading to the difference between the barotropic and the continuous baroclinic atmospheres.

Similarly to (6.1), (6.2) and (6.3), equations (7.1), (7.2) and (7.3) can easily be transformed into the followings:

$$\frac{\partial v_{\theta k}}{\partial t}-h_k q_k v_{\lambda k}=-\frac{\partial K_k}{a\partial\theta}, \tag{7.5}$$

$$\frac{\partial v_{\lambda k}}{\partial t}+h_k q_k v_{\theta k}=-\frac{\partial K_k}{a\sin\theta\partial\lambda}, \tag{7.6}$$

$$\frac{\partial q_k}{\partial t}+\vec{v}_k\cdot\nabla q_k=0, \tag{7.7}$$

where

$$K_k\equiv\varphi_k+\frac{1}{2}|\vec{v}_k|^2, \tag{7.8}$$

$$q_k\equiv\frac{1}{h_k}\left\{\frac{1}{a\sin\theta}\left(\frac{\partial v_{\lambda k}\sin\theta}{\partial\theta}-\frac{\partial v_{\theta k}}{\partial\lambda}\right)+2\omega\cos\theta\right\}. \tag{7.9}$$

We have an invariant functional

$$2I(\vec{v},\ \varphi)=2r_0E+2r_2M+\sum_{k=1}^{J}(r_{1k}F_k+r_{3k}Ma_k), \tag{7.10}$$

where $\vec{v}$ and φ consist of all components $\vec{v}_k$, φ_k, $k=1,2,...,J$, and

$$E\equiv\frac{1}{2}\iint_S\sum_{k=1}^{J}\rho_k\left(h_k|\vec{v}_k|^2+\frac{\rho_k-\rho_{k-1}}{\rho_k}gZ_k^2\right)dS, \tag{7.11}$$

$$M\equiv\iint_S\sum_{k=1}^{J}a\rho_k h_k\sin\theta(v_{\lambda k}+a\omega\sin\theta)dS, \tag{7.12}$$

$$F_k\equiv\iint_S\rho_k h_k Q_k(q_k)dS, \tag{7.13}$$

$$Ma_k\equiv\iint_S\rho_k h_k dS. \tag{7.14}$$

$Q_k(q_k)$ is an arbitrary function of its argument and depends on the index k too, and $r_2=0$ if $Z_s(\theta,\ \lambda)\equiv0$.

Now, we have

$$\begin{aligned}2\delta I=\sum_{k=1}^{J}\iint_S\Bigg\{&\left[2r_0h_kv_{\theta k}+r_{1k}Q''_k(q_k)\frac{\partial q_k}{a\sin\theta\partial\lambda}\right]\delta v_{\theta k}\\&+\left[2r_0h_kv_{\lambda k}+2r_2h_ka\sin\theta-r_{1k}Q''_k(q_k)\frac{\partial q_k}{a\partial\theta}\right]\delta v_{\lambda k}\\&+\Bigg[r_0\left(2\varphi_k+|\vec{v}_k|^2\right)+2r_2(v_{\lambda k}+a\omega\sin\theta)a\sin\theta\\&+r_1\left(Q_k-q_kQ'_k(q_k)\right)+r_{3k}\Bigg]\delta h_k\Bigg\}\rho_k dS,\end{aligned} \tag{7.15}$$

$$2\delta^2 I = \iint_S \sum_{k=1}^{J} \Big\{ r_0 h_k \left(\left[\delta v_{\lambda k} + U_k \frac{\delta h_k}{h_k} \right]^2 + \left[\delta v_{\theta k} + v_{\theta k} \frac{\delta h_k}{h_k} \right]^2 \right)$$

$$+ r_0 g \left[\left(\frac{\rho_k - \rho_{k-1}}{\rho_k} - \frac{|\vec{V}_k|^2}{g h_k} - \frac{\rho_{k-1} |\vec{V}|^2_{k-1}}{\rho_k g h_{k-1}} \right) (\delta Z_k)^2 + 2 \frac{|\vec{V}_k|^2}{g h_k} \delta Z_k \delta Z_{k+1} \right]$$

$$+ r_{1k} \frac{h_k}{2} Q''_k (q_k) \left(\delta^1 q_k \right)^2 \Big\} \rho_k dS, \tag{7.16}$$

$$2\Delta^2 I = \iint_S \sum_{k}^{J} \Big\{ r_0 h_k^{**} \left(\left[\delta v_{\lambda k} + U_k \frac{\delta h_k}{h_k^{**}} \right]^2 + \left[\delta v_{\theta k} + v_{\theta k} \frac{\delta h_k}{h_k^{**}} \right]^2 \right)$$

$$+ r_0 g \left[\left(\frac{\rho_k - \rho_{k-1}}{\rho_k} - \frac{|\vec{V}_k|^2}{g h_k^{**}} - \frac{\rho_{k-1} |\vec{V}|^2_{k-1}}{\rho_k g h_{k-1}^{**}} \right) (\delta Z_k)^2 + 2 \frac{|\vec{V}_k|^2}{g h_k^{**}} \delta Z_k \delta Z_{k+1} \right]$$

$$+ r_{1k} \frac{h_k}{2} Q''_k \left(q_k^* \right) (\delta q_k)^2 \Big\} \rho_k dS, \tag{7.17}$$

where

$$U_k \equiv v_{\lambda k} + a r_2 r_0^{-1} \sin\theta, \quad \vec{V}_k \equiv \vec{\theta^0} v_{\theta k} + \vec{\lambda^0} U_k, \tag{7.18}$$

$$h_k^{**} \equiv h_k + \delta h_k, \quad q_k^* \equiv q_k + r_k^* \delta q_k, \ 0 \leqslant r_k^* \leqslant 1. \tag{7.19}$$

Note that every quadratic form can easily be transferred into a diagonal one by an appropriate linear transformation. For example, by introducing

$$\delta\eta = \mathbf{X}^{-1} \delta \mathbf{Z}. \tag{7.20}$$

we have

$$\sum_{k=1}^{J} \rho_k \left\{ \left[\frac{\rho_k - \rho_{k-1}}{\rho_k} - \left(\frac{|\vec{V}_k|^2}{g h_k^{**}} - \frac{\rho_{k-1} |\vec{V}|^2_{k-1}}{\rho_k g h_{k-1}^{**}} \right) \right] (\delta Z_k)^2 + \right.$$

$$\left. 2 \frac{|\vec{V}_k|^2}{g h_k^{**}} \delta Z_k \delta Z_{k+1} \right\} \equiv (\mathbf{B}\delta\mathbf{Z}, \ \delta\mathbf{Z}) = \sum_{j=1}^{J} \mu_j (\delta\eta_j)^2, \tag{7.21}$$

where $\delta\eta$ and $\delta\mathbf{Z}$ are two vectors,

$$\delta\eta = (\delta\eta_1, \ldots, \delta\eta_j, \ldots, \delta\eta_J),$$
$$\delta\mathbf{Z} = (\delta Z_1, \ldots, \delta Z_j, \ldots, \delta Z_J),$$

$\mathbf{B}$ is a matrix. μ_j and $\mathbf{X}_j$ its eigenvalue and eigen–vector respectively,

$$\mathbf{B} \equiv \left\{ b_{kk'} \right\}, \quad (k,\ k' = 1,2,\ldots,J) \tag{7.22}$$

$$\begin{cases} b_{kk} = \rho_k \left[\dfrac{\rho_k - \rho_{k-1}}{\rho_k} - \left(\dfrac{|\vec{V}_k|^2}{gh_k^{**}} - \dfrac{\rho_{k-1}|\vec{V}|^2_{k-1}}{\rho_k g h^{**}_{k-1}} \right) \right], \\ b_{k,k+1} = b_{k+1,k} = \dfrac{|\vec{V}|}{gh_k^{**}} \rho_k, \\ b_{k,k'} = 0, \quad (|k-k'| \geq 2) \ , \end{cases}$$

$$\mathbf{BX}_j = \mu_j \mathbf{X}_j, \tag{7.23}$$

and matrix $\mathbf{X}$ consists of J eigen–vectors,

$$\mathbf{X} = \left[\mathbf{X}_1, \mathbf{X}_2, \ldots, \mathbf{X}_J \right]. \tag{7.24}$$

Similar to the results obtained in the previous section, we have the following theorems:

Theorem 7.1 Every propagating wave or steady flow solution to the layer model (7.1)–(7.3) corresponds to a stationary point of $I(\vec{v},\ \varphi)$, i.e., $\delta I = 0$, and vice versa.

Theorem 7.2 Sufficient conditions of stability of basic flow determined by $\delta I = 0$ with respect to perturbations are (1) r_0 and r_{lk} $Q''_k(q_k^*)$ all have the same sign, and (2) all $\mu_j > 0$, $j = 1,2,\ldots,J$.

In order to illuminate the classification of instabilities in layer model and their mechanism, let us take a two–layer model. We have

$$\mathbf{B} = \begin{bmatrix} \left\{ \dfrac{\rho_1 - \rho_0}{\rho_1} - \dfrac{|\vec{V}_1|^2}{gh_1^{**}} \right\} \rho_1 & \dfrac{|\vec{V}_1|^2}{gh_1^{**}} \rho_1 \\ \dfrac{|\vec{V}_1|^2}{gh_1^{**}} \rho_1 & \rho_2 \left\{ \dfrac{\rho_2 - \rho_1}{\rho_2} - \left(\dfrac{|\vec{V}_2|^2}{gh_2^{**}} + \dfrac{\rho_1 |\vec{V}_1|^2}{\rho_2 g h_1^{**}} \right) \right\} \end{bmatrix} ,$$

$$\mu_{1,2} = \frac{1}{2} \left\{ (a_1 + a_2 - b_1) \pm \left[(a_1 - a_2 + b_1)^2 + 4\rho_1^2 \frac{|\vec{V}_1|^4}{g^2 h_1^{**2}} \right]^{\frac{1}{2}} \right\}, \tag{7.25}$$

where

$$a_1 \equiv \left(1 - \frac{|\vec{V}_1|^2}{gh_1^{**}}\right) \rho_1, \quad a_2 \equiv \left(1 - \frac{|\vec{V}_2|^2}{gh_2^{**}}\right) \rho_2, \quad b_1 \equiv \left(1 + \frac{|\vec{V}_1|^2}{gh_1^{**}}\right) \rho_1. \tag{7.26}$$

From (7.25) it is clear that there is one negative eigenvalue if either

$$a_1 + a_2 - b_1 \leq 0, \tag{7.27}$$

or

$$(a_1 - a_2 + b_1)^2 + 4\left(\rho_1 \frac{|\vec{V}_1|^2}{gh_1^{**}}\right)^2 > (a_1 + a_2 - b_1)^2. \tag{7.28}$$

Condition (7.27) can be transformed into

$$1 - \left(\frac{|\vec{V}_2|^2}{gh_2^{**}} + 2\frac{\rho_1 |\vec{V}_1|^2}{\rho_2 g h_1^{**}}\right) \leqslant 0. \tag{7.27$'$}$$

Suppose that there is no vertical shear of the basic flow, i.e. $\vec{V}_1 = \vec{V}_2$, but a horizontal shear of $\dot{\lambda}_1 \equiv v_{\lambda 1}(a\sin\theta)^{-1}$ exists, i.e. $|\vec{V}_1| \equiv 0$ by every choice of r_2, but the velocity $|\vec{V}_1|$ in some area is large enough such that

$$|\vec{V}_1|^2 \geqslant \left[\frac{1}{gh_2^{**}} + \frac{2\rho_1}{\rho_2 g h_1^{**}}\right]^{-1}, \tag{7.29}$$

hence (7.27)$'$ is satisfied. This is the super–critically high speed instability. Next, suppose that there is no horizontal shear of $\dot{\lambda}_k$, but $\dot{\lambda}_1 \neq \dot{\lambda}_2$, we can choose r_2 (if there is no orography) such that makes $|\vec{V}_1| = 0$ but $|\vec{V}_2| \not\equiv 0$, or $|\vec{V}_2| = 0$ but $|\vec{V}_1| \not\equiv 0$. If

$$1 - |\vec{V}_2|^2 / gh_2^{**} \leqslant 0 \tag{7.30}$$

in the first case, or

$$1 - 2\frac{\rho_1 |\vec{V}_1|^2}{\rho_2 g h_1^{**}} \leqslant 0 \tag{7.31}$$

in the second case, (7.27)$'$ is also satisfied. This is the Helmholtz instability in a vertically shear flow of stratified fluid under the influence of gravity but modified by the shallow approximation. It is well known that according to Helmholtz theory, once there is a vertical shear, large or small, there are always unstable waves with small enough horizontal wavelength on one hand, but in order to make a wave with given horizontal wavelength unstable it is necessary that the vertical shear of the basic flow should exceed a critical value on the other hand. Now, the shallow water approximation can be applied only to long waves, that is why a critical velocity (vertical shear) is necessary for the occurrence of Helmholtz instability in our layer model.

Condition (7.28) can be transformed into

$$M_1^2 - M_2^2 > \frac{(1 - M_1^2)^2 \beta - M_1^4}{(1 - \beta)(1 - M_1^2)}, \tag{7.32}$$

where

$$M_k^2 \equiv |\vec{V}_k|^2 / gh_k^{**}, \quad k = 1,2; \quad \beta \equiv \frac{(\rho_2 - \rho_1)}{\rho_1}.$$

If $M_2 \equiv 0$, (7.32) leads to

$$M_1^2 > \frac{\beta}{(1+\beta)};\tag{7.33}$$

but if $M_1=0$, (7.32) leads to

$$(1-\beta)\ M_2^2 + \beta < 0.\tag{7.34}$$

(7.34) can be satisfied only if $\beta > 1$ or $\beta < 0$. In all the cases [(7.32), (7.33) and (7.34)], modified Helmholtz instability might occur only under a vertical shear exceeding a critical value, and another instability might occur under unstable stratification ($\beta < 0$). In general case a mixed Helmholtz—supercritically high speed instability might also occur.

In addition, mixed barotropic—baroclinic instability and inertial or symmetric instability might occur if at least one $r_{1k} Q''_k (q_k^*)$ changes its sign or its sign is opposite to r_0.

VIII. BAROCLINIC PRIMITIVE EQUATIONS

Taking hydrostatic approximation and writing the governing equations in coordinative system $(\theta,\ \lambda,\ \xi,\ t)$, where ξ is the entropy

$$\xi - \xi_0 = \ln\ (T/p^{R/C_p}),\tag{8.1}$$

and ξ_0 is a constant, we have

$$\frac{\partial v_\theta}{\partial t} - hqv_\lambda = -\frac{\partial K}{a\partial\theta},\tag{8.2}$$

$$\frac{\partial v_\lambda}{\partial t} + hqv_\theta = -\frac{\partial K}{a\sin\theta\partial\lambda},\tag{8.3}$$

$$\frac{\partial q}{\partial t} + \vec{v}\cdot\nabla q = 0,\tag{8.4}$$

where

$$h \equiv -\frac{\partial p}{\partial\xi},\tag{8.5}$$

$$K \equiv c_p T + \varphi + |\vec{v}|^2/2,\tag{8.6}$$

$$q \equiv \frac{1}{h}\left\{\frac{1}{a\sin\theta}\left(\frac{\partial v_\lambda \sin\theta}{\partial\theta} - \frac{\partial v_\theta}{\partial\lambda}\right) + 2\omega\cos\theta\right\},\tag{8.7}$$

p is the pressure, φ is the geopotential, q is the potential vorticity, $\partial/\partial t$, $\partial/\partial\theta$, and $\partial/\partial\lambda$, are the derivatives given at the isentropic surface.

Note that the continuity equation

$$\frac{\partial h}{\partial t} + \nabla\cdot h\vec{v} = 0\tag{8.8}$$

can be obtained from equations (8.1)–(8.4).

Assuming the atmosphere is bounded from below by a spherical surface with radius a (without orography) and has bounded total energy E and generalized enstrophy F, we have conservations of E, F, total angular momentum M and total mass Ma.

For simplicity we investigate only those basic and disturbed flows whose estropy ξ is monotonically increasing function with radius r, and whose bottom surface is an isentropic one with fixed constant ξ_s, $(\delta\xi_s = 0)$. In such case the invariant functional I and its variations take rather simple forms. Let

$$2I = 2r_0 E + r_1 F + 2r_2 M + r_3 Ma, \tag{8.9}$$

where

$$E \equiv \frac{1}{2}\iint_S \int_0^{\infty} (|\vec{v}|^2 + 2c_p T)\ hd\xi dS, \tag{8.10}$$

$$F \equiv \iint_S \int_0^{\infty} Q\ (q,\ \xi)\ hd\xi dS, \tag{8.11}$$

$$M \equiv \iint_S \int_0^{\infty} a\ (v_\lambda + a\omega\sin\theta)\ hd\xi dS, \tag{8.12}$$

$$Ma \equiv \iint_S \int_0^{\infty} hd\xi dS = \iint_S p_s dS, \tag{8.13}$$

where $\xi_s = 0$ has been taken for simplicity but without lost of generality, it is equivalent to $\xi_0 = -\ln(T_s / p_s^{R/C_p})$, and p_s and T_s are the variables at the bottom surface.

Now $\vec{v}$, h and p_s can be taken as the independent functions from which all other variables can be determined. Indeed, p, T, φ are represented by h and p_s as follows

$$p\ (\theta,\lambda,\xi,t) = p_s(\theta,\lambda,t) - \int_0^{\xi} h\ (\theta,\lambda,\xi',t)\ d\xi', \tag{8.14}$$

$$T\ (\theta,\lambda,\xi,t)\ = \Big[p(\theta,\lambda,\xi,t)\Big]^{\frac{R}{C_p}} e^{(\xi - \xi_0)}, \tag{8.15}$$

$$\varphi\ (\theta,\lambda,\xi,t)\ = \varphi_s(\theta,\lambda,t) - \int_0^{\xi} RT\ (\theta,\lambda,\xi',t)\ \frac{\partial}{\partial \xi'}\ (\ln\frac{p}{p_s})\ d\xi', \tag{8.16}$$

(see Zeng, 1979). Therefore, for a given set of variations ($\delta\vec{v}$, δh, δp_s) we have increments of p, T and q as follows,

$$\delta p = \delta p_s - \int_0^{\xi} \delta h d\xi'. \quad (or\ \delta h = -\frac{\partial \delta p}{\partial \xi}), \tag{8.17}$$

$$\delta T = \delta^1 T + \delta^2 T + \cdots = \delta^1 T + \Delta^2 T, \tag{8.18}$$

$$\delta q = \delta^1 q + \delta^2 q + \cdots = \delta^1 q + \Delta^2 q, \tag{8.19}$$

where

$$c_p \delta^1 T = RT\frac{\delta p}{p}, \quad c_p \delta^2 T = -\frac{c_v}{2c_p} RT(\frac{\delta p}{p})^2. \tag{8.20}$$

$$\delta^1 q = \frac{1}{ha\sin\theta}\ (\frac{\partial v_\lambda \sin\theta}{\partial \theta} - \frac{\partial \delta v_\theta}{\partial \lambda})\ - q\frac{\delta h}{h}, \quad \delta^2 q = -\delta^1 q(\frac{\delta h}{h}), \tag{8.21}$$

$$c_p \Delta^2 T = -\frac{c_v}{2c_p} RT_*\ (\frac{\delta p}{p_*})^2, \tag{8.22}$$

$$(T_* = p_*^{R/c_p} e^{(\xi - \xi_0)},\ p_* = p + r_* \delta p,\ \ 0 \leqslant r_* \leqslant 1)$$

$$\Delta^2 q = \delta q - \delta^1 q = -\delta q\ (\frac{\delta h}{h})\ . \tag{8.23}$$

From these formulas it is easy to obtain δI, $\delta^2 I$ and $\Delta^2 I$. and we have

$$2\delta I = \iint_S \int_0^\infty \Big\{\Big[2r_0 h v_\theta + r_1 \frac{\partial}{a\sin\theta\partial\lambda}\ (\frac{\partial Q}{\partial q})\ \Big]\delta v_\theta + \Big[2h\ (r_0 v_\lambda + r_2 a\sin\theta)\ - r_1 \frac{\partial}{a\partial\theta}\ (\frac{\partial Q}{\partial q})\ \Big]\delta v_\lambda + \Big[2r_0 K + r_1\ (Q - q\frac{\partial Q}{\partial q})\ + 2r_2 a\sin\theta\ (v_\lambda + a\omega\sin\theta)\ + r_3\Big]\delta h\Big\} d\xi dS \tag{8.24}$$

$$2\delta^2 I = \iint_S \int_0^\infty h\Big\{r_0\Big[\delta v_\lambda + (v_\lambda + \frac{r_2}{r_0} a\sin\theta)\ \frac{\delta h}{h}\Big]^2 + r_0\Big[\delta v_\theta + v_\theta \frac{\delta h}{h}\Big]^2 - r_0\Big[\ (v_\lambda + \frac{r_2}{r_0} a\sin\theta)^2 + v_\theta^2\Big](\frac{\delta h}{h})^2 + r_0 C^2 (\frac{\delta p}{p})^2 + r_1 \frac{\partial^2 Q}{\partial q^2}\Big[\delta^1 q\Big]^2\Big\} d\xi dS + \iint_S r_0 \frac{RT_s}{p_s}\Big[\delta p_s\Big]^2 dS, \tag{8.25}$$

$$\begin{aligned}2\Delta^2 I = &\iint_S \int_0^\infty \Big\{r_0\Big[h^{**}|\delta\vec{v}|^2 + 2\ (\vec{v}\cdot\delta\vec{v})\ \delta h + c_p \delta^1 T\delta h + c_p h^{**}\Delta^2 T\Big] + r_1\Big[\frac{\partial Q\ (q,\xi)}{\partial q}\ (h\Delta^2 q + \delta q\delta h) + (\frac{h^{**}}{2})\ \frac{\partial^2 Q\ (q^*,\xi)}{\partial q^2}(\delta q)^2\Big] + r_2 a\delta h\delta v_\lambda \sin\theta\Big\} d\xi dS \\ = &\iint_S \int_0^\infty \Big\{r_0 h^{**}\Big[\delta v_\lambda + (v_\lambda + \frac{r_2}{r_0} a\sin\theta)\ \frac{\delta h}{h^{**}}\Big]^2 + r_0 h^{**}\Big[\delta v_\theta + v_\theta \frac{\delta h}{h^{**}}\Big]^2 \\ &- r_0 h^{**}\Big[\ (v_\lambda + \frac{r_2}{r_0} a\sin\theta)^2 + v_\theta^2\Big]\Big[\frac{\delta h}{h^{**}}\Big]^2 + r_0 h C^2_{**}\Big[\frac{\delta p}{p}\Big]^2 \\ &+ r_1 h\frac{\partial^2 Q\ (q^*,\ \xi)}{2\partial q^2}\Big[\delta q\Big]^2\Big\} d\xi dS + \iint_S r_0 \frac{RT_s}{p_s}\Big[\delta p_s\Big]^2 dS,\end{aligned} \tag{8.26}$$

where

$$\begin{cases} q^* = q + r^*\delta q,\quad 0 \leqslant r^* \leqslant 1, \\ h^{**} = h + \delta h, \end{cases}$$
$$C^2 \equiv R\ (\gamma_a + \frac{\partial T}{\partial r})\ \frac{RT}{g}, \tag{8.27}$$

$$C^2_{**} \equiv C^2 + \frac{c_v}{c_p}\left[\left\{1 - \left(\frac{p}{p_*}\right)^{1-c_v/c_p}\right\}RT + RT_*\left(\frac{p}{p_*}\right)^2\frac{\delta h}{h}\right]. \tag{8.28}$$

C and C_{**} have the dimension of velocity, namely the characteristic velocity of propagation of gravity wave in the continuous baroclinic atmosphere. In our case C and C_{**} must be positive due to the assumption of stable stratification (the enstropy monotonically increases with height), and $C^2_{**} - C^2 = O(\delta h)$.

The procedure of deriving (8.24), (8.25) and (8.26) essentially is the same as the deriving of (6.18), (6.19) and (6.22) but with the following considerations:

$$hd\xi = -\frac{\partial p}{\partial \xi}d\xi = -dp = \rho d\varphi.$$

$$\int_0^\infty hc_p\delta^2 Td\xi = \int_0^\infty \delta p d\varphi = -\int_0^\infty \varphi\frac{\partial \delta p}{\partial \varphi}d\varphi = \int_0^\infty \varphi\delta h d\xi,$$

$$\int_0^\infty c_p\left[\delta^1 T\delta h + h\delta^2 T\right]d\xi = \frac{RT_s}{2p_s}(\delta p_s)^2 + A,$$

$$\int_0^\infty c_p\left[\delta^2 T\delta h + h^{**}\Delta^2 T\right]d\xi = \frac{RT_s}{2p_s}(\delta p_s)^2 + A_{**},$$

where

$$A \equiv \int_0^\infty \frac{1}{2}\left[\frac{\partial}{\partial \xi}\left(\frac{RT}{p}\right) - \left(\frac{c_v}{c_p}\right)\frac{RT}{p^2}h\right](\delta p)^2 d\xi = \int_0^\infty \frac{C^2 h}{2p^2}(\delta p)^2 d\xi,$$

$$A_{**} \equiv \int_0^\infty \frac{1}{2}\left[\frac{\partial}{\partial \xi}\left(\frac{RT}{p}\right) - \left(\frac{c_v}{c_p}\right)\frac{RT_*}{p_*^2}h^{**}\right](\delta p)^2 d\xi = \int_0^\infty \frac{h}{2p^2}C^2_{**}(\delta p)^2 d\xi,$$

because

$$\begin{aligned}\frac{\partial}{\partial \xi}\left(\frac{RT}{p}\right) - \left(\frac{c_v}{c_p}\right)\frac{RT}{p^2}h &= \frac{R}{p}\left[\frac{\partial T}{\partial \xi} - \frac{T}{p}\left(1 - \frac{c_v}{c_p}\right)\frac{\partial p}{\partial \xi}\right]\\ &= \frac{R}{p}\left[\frac{\partial T}{\partial p} - \frac{RT}{c_p p}\right]\frac{\partial p}{\partial \xi}\\ &= \frac{R^2 T}{gp^2}\left[-\frac{\partial T}{\partial r} - \frac{g}{c_p}\right]\frac{\partial p}{\partial \xi}\\ &= \frac{C^2}{p^2}h,\end{aligned}$$

$$\begin{aligned}\frac{\partial}{\partial \xi}\left(\frac{RT}{p}\right) - \left(\frac{c_v}{c_p}\right)\frac{RT_*}{p_*^2}h^{**} &= \frac{C^2}{p^2}h - \left(\frac{c_v}{c_p}\right)\left[\frac{RT_*}{p_*^2}h^{**} - \frac{RT}{p^2}h\right]\\ &= \frac{c^2}{p^2}h - \left(\frac{c_v}{c_p}\right)\frac{RTh}{p^2}\left[\frac{T_*}{T}\left(\frac{p}{p_*}\right)^2\left(1 + \frac{\delta h}{h}\right) - 1\right],\end{aligned}$$

and for every isentropic surface we have

$$\frac{T_*}{T}=\left(\frac{p_*}{p}\right)^{R/c_p}.$$

Theorem 8.1 Every propagating wave solution to the baroclinic primitive equations (8.2)–(8.4) with (i) a constant entropy at the bottom surface, and (ii) every isentropic surface enveloping the Earth corresponds to a stationary point of I in the subset consisting of functions which possess the characteristics mentioned above, and is determined by the following equations:

$$\begin{cases} 2(r_0 v_\lambda + r_2 a\sin\theta)h - r_1 \dfrac{\partial}{a\partial\theta}\left(\dfrac{\partial Q}{\partial q}\right) = 0, \\ 2r_0 v_\theta h + r_1 \dfrac{\partial}{a\sin\theta\partial\lambda}\left(\dfrac{\partial Q}{\partial q}\right) = 0, \\ 2r_0 K + r_1\left(Q - q\dfrac{\partial Q}{\partial q}\right) + 2r_2 a\sin\theta(v_\lambda + a\omega\sin\theta) + r_3 = 0, \end{cases} \tag{8.29}$$

where Q is an arbitrary function of q and ξ; r_0, r_1, r_2, and r_3, are arbitrary parameters. The phase angular velocity $\dot{\lambda}_0 = -r_2/r_0$.

The proof of Theorem 8.1 is essentially the same as that of Theorem 6.1.

However, from (8.25) or (8.26) it is difficult to find sufficient condition of stability. In fact, the term with $(\delta h/h)^2$ (or $(\delta h/h^{**})^2$) is negative, but all other terms with r_0 on the right hand side of (8.25) or (8.26) are positive provided $r_0>0$. Therefore, there is no guarantee that $\delta^2 I$ or $\Delta^2 I$ is positively definitive even if $r_1\partial^2 Q(q,\xi)/\partial q^2$ or $r_1\partial^2 Q(q^*,\xi)/\partial q^2$ is also positive, provided $[v_\lambda+(r_2/r_0)a\sin\theta]^2+v_\theta^2\neq 0$. In fact, according to a general integral inequality one can always find function δp with small enough vertical scale such that

$$\int_0^{\xi_T}(\delta h)^2 d\xi \gg \int_0^{\xi_T}(\delta p)^2 d\xi. \tag{8.30}$$

One example is $\delta p=(\delta p_s)\exp(-n\xi)$, from which we obtain

$$\int_0^{\xi_T}(\delta h)^2 d\xi = n^2\int_0^{\xi_T}(\delta p)^2 d\xi,$$

and

$$\lim_{n\to\infty}\int_0^{\xi_T}(\delta h)^2 d\xi\to\infty,$$

but

$$\lim_{n\to\infty}\int_0^{\xi_T}(\delta p)^2 d\xi = 0.$$

Another example is $\delta p=(\delta p_s)\exp(-\xi)\sin^2 m\xi$, we have

$$\lim_{m\to\infty}\int_0^{\xi_T}(\delta p)^2 d\xi=\frac{1}{4}(\delta p_s)^2,$$

but

$$\lim_{m\to\infty}\int_0^{\xi_T}(\delta h)^2 d\xi\to\infty.$$

This means that there is no upper bound of $\|\delta\vec{v}\|_3^2$, $\|\delta h / h\|_3^2$, $\|\delta p / p\|_3^2$, $\|\delta p_s\|^2$, and $\|(\partial^2 Q / \partial q^2)^{1/2}\delta q\|_3^2$, hence the stability is not guaranteed.

We still can make a classification of instabilities in the baroclinic atmosphere. Except convective instability which is characterized by $\partial\xi / \partial z < 0$ (or $C^2 < 0$) and excluded by our assumption in this section, we have (a) mixed barotropic–baroclinic instability characterized by $\partial^2 Q(q^*, \xi) / \partial q^2$ in some region, (b) *inertial or symmetric instability which might* occur under condition $\partial^2 Q(q^*, \xi) / \partial q^2$ and other additional conditions, (c) Helmholtz instability and (d) super–critically high speed instability, both they might occur if

$$\int_0^{l} \left\{ hC_{*}^{2}\left(\frac{\delta p}{p}\right)^2 - h^{*}\left[\left(v_{\lambda} - \frac{r_2}{r_0}a\sin\theta\right)^2 + v_\theta^2\right]\left(\frac{\delta h}{h}\right)^2 \right\} d\xi < 0, \qquad (8.31)$$

in some region and some time. Roughly speaking, (8.31) is always the case that when internal inertio–gravity wave with small enough vertical wavelength takes place, because its characteristic phase velocity is much smaller than C and can easily be exceeded by the speed of basic flow $[(v_\lambda + r_2 a\sin\theta / r_0)^2 + v_\theta^2]^{1/2}$. Therefore the stability of a flow in a continuous baroclinic atmosphere can be guaranteed only if the perturbation super–imposed on it has always a simple structure in the vertical, otherwise a Helmholtz instability might eventually arise, and the stability might break down.

The problem on the stability of flow in baroclinic fluid needs more investigations.

APPENDIX [1]

The key point of variational method is to find an invariant functional consisting of integrals. Therefore, in the case that the fluid occupies infinitive space and the integrals are no longer bounded, the generalized variational method has to be modified. In this note we will deal with such modifications, taking the two–dimensional incompressible fluid in a β–plane as an example.

1. *Periodic channel*

The governing equation for the two–dimensional incompressible fluid in a β–plane is the conservation of the absolute vorticity

$$\frac{\partial q}{\partial t} + \vec{v} \cdot \nabla q = 0, \qquad (A.1.1)$$

where $\vec{v} = \vec{k} \times \nabla\psi \equiv \vec{i}u + \vec{j}v$, ψ is the stream function and

$$q = \Delta\psi + f, \qquad (A.1.2)$$

$$f = f_0 + \beta y. \qquad (A.1.3)$$

β is taken as a constant; and f_0 is another constant. The classical model without Coriolis force corresponds to the case with $f_0 = \beta = 0$.

Let the channel be parallel to x–axis with two rigid walls at $y = y_1$ and y_2, $y_2 > y_1$, and the motions including the basic ψ and perturbation $\delta\psi$ be all periodic along x–axis with period 2L. We have the conservations of total absolute momentum, M, total en-

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ergy, E, and total "generalized enstrophy", F, and the conservations of total zonal momentums B_1 and B_2 at the two boundaries y_1 and y_2 respectively, where

$$M \equiv \int_{-L}^{L}\int_{y_1}^{y_2}\left[u+\int_{y}^{y_3} f(y')dy'\right]dydx, \tag{A.1.4}$$

$$E \equiv \int_{-L}^{L}\int_{y_1}^{y_2}\frac{1}{2}(u^2+v^2)dydx, \tag{A.1.5}$$

$$F \equiv \int_{-L}^{L}\int_{y_1}^{y_2} Q(q)dydx. \tag{A.1.6}$$

$$B_1 \equiv \int_{-L}^{L} u(x,\ y_1,\ t)dx. \tag{A.1.7}$$

$$B_2 \equiv \int_{-L}^{L} u(x,\ y_2,\ t)dx, \tag{A.1.8}$$

$y_3 > y_2$, and Q is an arbitrary function of argument q. Hence we have an invariant functional. $I(\psi)$. $dI/dt=0$, where

$$I(\psi) \equiv 2r_0 E + r_1 F + r_2 M + r_3 B_1 + r_4 B_2 \tag{A.1.9}$$

depends on an arbitrary function Q and certain parameters r_0, r_1, and r_2. The parameters r_3 and r_4 will be determined later.

Giving a perturbation $\delta\psi$, the first and second variations, δI and $\delta^2 I$, and the difference between $I(\psi+\delta\psi)$ and $I(\psi)$ are obtained after elementary calculations as follows:

$$\begin{aligned}\delta I = &\int_{-L}^{L}\int_{y_1}^{y_2}\left\{-2r_0\psi + r_1 Q'(q) + r_2 y\right\}\delta q dydx \\ &+\int_{-L}^{L}\left\{(2r_0\psi - r_2 y_2 + r_4)\frac{\partial\delta\psi}{\partial y}\right\}_{y=y_2}dx \\ &+\int_{-L}^{L}\left\{(-2r_0\psi + r_2 y_1 + r_3)\frac{\partial\delta\psi}{\partial y}\right\}_{y=y_1}dx,\end{aligned} \tag{A.1.10}$$

$$\delta^2 I = \int_{-L}^{L}\int_{y_1}^{y_2}\left\{r_0|\delta\vec{v}|^2 + \frac{r_1}{2}Q''(q)(\delta q)^2\right\}dydx, \tag{A.1.11}$$

$$\Delta^2 I = \int_{-L}^{L}\int_{y_1}^{y_2}\left\{r_0|\delta\vec{v}|^2 + \frac{r_1}{2}Q''(q^*)(\delta q)^2\right\}dydx, \tag{A.1.12}$$

$$(q^* = q + r^*\delta q,\quad 0 \leqslant r^* \leqslant 1).$$

and

$$I(\psi+\delta\psi) - I(\psi) = \delta I + \Delta^2 I. \tag{A.1.13}$$

We can always keep ψ at $y=y_1$ as a constant ψ_1, and prove that ψ at $y=y_2$ is another constant ψ_2, independent of t.

Taking

$$\begin{cases} r_3 = 2r_0\psi_1 - r_2 y_1, \\ r_4 = -2r_0\psi_2 + r_2 y_2, \end{cases} \tag{A.1.14}$$

the last two integrals in (A.1.10) vanish, hence δI is simply expressed by a double integral, and we have

Theorem A.1. Every zonal flow $\psi(y)$ or propagating wave $\psi(x-ct,y)$ in a two-dimensional incompressible fluid channel $y_1 \leqslant y \leqslant y_2$ of β-plane is a stationary point of I in the functional space ψ, i.e. $\delta I=0$, where r_3 and r_4 are determined by (A.1.14), and $c=-r_2/2r_0$. The inverse is also true.

Theorem A.2. A flow $\psi(x-ct,y)$ which is determined by $\delta I=0$ with function Q and parameters r_0, r_1, and r_2 is always stable with respect to every small perturbation $\delta\psi$ which has a common x-period with $\psi(x-ct,y)$, if $\Delta^2 I$ is sign-definitive functional i.e. r_0 and $r_1 Q''/2$ both are either non-negative or non-positive everywhere in the channel.

Theorem A.3. A flow $\psi(x-ct,y)$ might be unstable if either $r_1 Q''(q^*)/2$ is a sign-nondefinitive function in the channel or its sign is opposite to r_0.

The method and the theorems described above are essentially the same as for spherical surface. Therefore, we have omitted the proof of these theorems. Note that zonal flow $\psi(y)$ is a special class of $\psi(x-ct,y)$, hence it is included in the Theorems A.2 and A.3.

2. *Infinitive channel with $\delta\vec{v}$ and $\delta q \in L_2$*

If the flow is not periodic along x, let $L\to\infty$, the functional I defined by (A.1.9) generally is unbounded, hence the method and results described in the previous section should be modified.

Theorem A.4. Every zonal flow or propagating wave $\psi(x-ct,y)$ in a two-dimensional incompressible fluid channel $y_1 \leqslant y \leqslant y_2$ of β-plane is given by the following equation

$$-2r_0\psi + r_1 Q'(q) + r_2 y = 0, \tag{A.2.1}$$

and the boundary conditions

$$\begin{cases} \psi = \dfrac{1}{2r_0}(r_2 y_1 + r_3) & \text{at} \quad y = y_1, \\ \psi = \dfrac{1}{2r_0}(r_2 y_2 - r_4) & \text{at} \quad y = y_2, \end{cases} \tag{A.2.2}$$

where Q is an arbitrary function, and r_s, $s=0,1,2,3,4$ are some parameters. The inverse theorem is also true.

Theorem A.4 can be verified by direct calculations. Note that Theorem A.4 and Theorem A.1 are the same if the words about I and $\delta I=0$ in theorem A.1 are replaced by the relevant equation, i.e. (A.2.1).

Taking a zonal flow or propagating wave as the basic flow, constructing a functional I as (A.1.9) with a fixed finite L, we have (A.1.12) and (A.1.13), but δI expressed as follows

$$\delta I=\int_{r_1}^{r_2}\left(\left[2r_0\psi-r_2 y\right]\delta v\right)\Bigg|_{x=-L}^{x=L}dy. \qquad (A.2.3)$$

Therefore, we have $I(\psi+\delta\psi)-I(\psi)$ expressed by the following formula

$$\Delta\equiv I(\psi+\delta\psi)-I(\psi)=\int^{r_2}\left(\left[2r_0\psi-r_2 y\right]\delta v\right)\Bigg|_{x=-L}^{x=L}dy$$
$$+\int_{-L}^{L}\int_{r_1}^{r_2}\left\{r_0|\delta\vec{v}|^2+\frac{r_1}{2}Q''(q^*)(\delta q)^2\right\}dydx$$
$$\equiv\Delta_1-\Delta_2. \qquad (A.2.4)$$

In addition, from equation (A.1.1) and its equivalent equations

$$\begin{cases}\dfrac{d\vec{v}}{dt}=-\nabla\varphi+\vec{k}\times f\vec{v},\\ \nabla\cdot\vec{v}=0,\end{cases} \qquad (A.2.5)$$

we can calculate dF/dt, dE/dt and so on. Finally, we have

$$\frac{dI}{dt}=-\int^{r_2}\left\{2r_0 uK+r_1 uQ-r_2 u\left(u+\int_{r}^{y}f(y')dy'\right)+r_2\varphi\right\}\Bigg|_{x=-L}^{x=L}dy$$
$$-r_3\left[\left(\varphi-\frac{u^2}{2}\right)_{y=r_1}\right]\Bigg|_{x=-L}^{x=L}-r_4\left[\left(\varphi-\frac{u^2}{2}\right)_{y=r_2}\right]\Bigg|_{x=-L}^{x=L}, \qquad (A.2.6)$$

where $K=\varphi+|\vec{v}|^2/2$ and φ is the pressure determined by ψ from solving the so-called balance equation.

Now, assume that both $|\delta\vec{v}|$ and $\delta q\in L_2$ are given over the whole infinitive channel. In this case, denoting the limit of Δ by $\tilde{\Delta}$ as $L\to\infty$, we have

$$\tilde{\Delta}=\int_{-\infty}^{\infty}\int_{r_1}^{r_2}\left\{-r_0|\delta\vec{v}|^2+\frac{r_1}{2}Q''(q^*)(\delta q)^2\right\}dydx$$
$$=\lim_{L\to\infty}\Delta_2, \qquad (A.2.7)$$

and can prove that

$$\frac{d\tilde{\Delta}}{dt}=\lim_{L\to\infty}\frac{d\Delta}{dt}=0. \qquad (A.2.8)$$

Therefore, Theorem A.2 and Theorem A.3 are both valid with $\Delta^2 I$ replaced by $\tilde{\Delta}$, becasue $\tilde{\Delta}$ is an quadratic functional of $\delta\psi$ and invariant, although I does not exist as $L\to\infty$.

3. *Infinitive channel. General case*

In the case of non–existence of $\tilde{\Delta}$, we define

$$e \equiv \frac{\Delta}{S}, \qquad (S \equiv \int_{-L}^{L} \int_{y_1}^{y_2} dy dx) \ . \tag{A.3.1}$$

It is not difficult to prove the existence of the following limits

$$\tilde{e} \equiv \lim_{L \to \infty} e$$

$$= \lim_{L \to \infty} \frac{1}{S} \int_{-L}^{L} \int_{y_1}^{y_2} \left\{ r_0 |\delta \vec{v}|^2 + \frac{r_1}{2} Q''(q^*)(\delta q)^2 \right\} dy dx, \tag{A.3.2}$$

$$\frac{d\tilde{e}}{dt} = \lim_{L \to \infty} \frac{de}{dt} = \lim_{L \to \infty} \frac{d}{dt} \left(\frac{\Delta_2}{S} \right) = 0. \tag{A.3.3}$$

Now, we take a generalized definition of stability:

Definition A flow is stable with respect to the small perturbations provided there exists a norm $\|\delta\psi\|_N$ for the perturbation, and the norm is always bounded.

The norm we can take is $|\Delta_2|^{1/2}$ (with L equal to the half period) in the periodic channel or $|\tilde{\Delta}|^{1/2}$ in the infinitive channel with finite $\tilde{\Delta}$. In the case of non–existence of $\tilde{\Delta}$ we take

$$\|\delta\psi\|_N^2 = \lim_{L \to \infty} \left| \frac{\Delta_2}{S} \right| . \tag{A.3.4}$$

Therefore, we have the following theorems valid for both periodic and infinitive channels and all cases.

Theorem A.5. A basic flow described in Theorem A.4 is stable with respect to the small perturbations if r_0 and $r_1 Q''(q^*)$ have the same definitive sign everywhere in the fluid.

Theorem A.6. The necessary conditions for the instability of basic flow which is described in Theorem A.4 are either r_0 and $r_1 Q''(q^*)$ having opposite sign or $Q''(q^*)$ changing its sign in the fluid.

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