An Efficient Accurate Direct Solution of Poisson's Equation for Computation of Meteorological Parameters

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ABSTRACT

Poisson's equation is solved numerically by two direct methods, viz. Block Cyclic Reduction (BCR) method and Fourier Method. Qualitative and quantitative comparison between the numerical solutions obtained by two methods indicates that BCR method is superior to Fourier method in terms of speed and accuracy. Therefore, BCR method is applied to solve $\nabla^2\psi=\zeta$ and $\nabla^2\chi=D$ from observed vorticity and divergent values. Thereafter the rotational and divergent components of the horizontal monsoon wind in the lower troposphere are reconstructed and are compared with the results obtained by Successive Over-Relaxation (SOR) method as this indirect method is generally in more use for obtaining the streamfunction (ψ) and velocity potential (χ) fields in NWP models. It is found that the results of BCR method are more reliable than SOR method.

I. INTRODUCTION

It is well established that in low latitudes the wind field is more reliable than the geopotential field for various diagnostic and prognostic studies of weather systems. The streamfunction ψ and velocity potential χ which resolve the observed wind field into its non-divergent and irrotational parts are now widely used for diagnostic and prognostic studies. Further, in global spectral models the scalars ψ and χ fields are generally used as input than the vector wind fields. The main objective of the present study is to compute more reliable non-divergent streamfunction and irrotational velocity potential so that rotational and divergent component of wind fields can be constructed which can represent accurately the observed wind field. This objective can be achieved if an accurate, numerical method for the solution of Poisson equation is known.

II. ELLIPTIC EQUATIONS AND METHODS OF SOLUTION

Consider the equation

$$\nabla^2\varphi-\alpha(x,y)\varphi=F(x,y),$$

where ∇^2 is the two-dimensional Laplacian operator, $\alpha(x,y)$ is non-negative function and $F(x,y)$ is a forcing function. Above equation is called as Helmholtz type elliptic equation. In the above equation if $\alpha(x,y)$ is zero, then the equation is called as Poisson type elliptic equation. Poisson's equation is an important special case, particularly for meteorological problems. Barotropic vorticity equation is an ideal example for this purpose. In order to obtain numerical solution of the Poisson equation, it has to be approximated by finite difference analogue which will give us a system of linear equation,

$$AX=B.$$

For a finite difference grid with $M\times N$ interior points, 'A' is an $MN\times MN$ block tridiagonal matrix. There are different methods of solving a system of linear equations, which differ in speed, accuracy and efficiency. All these methods may be divided into two main categories, namely, the iterative or indirect method and the exact or direct method. The main problem in solving the Poisson equation in a limited area is to specify proper boundary conditions. At the boundary points if the dependent variables are specified, it is called as Dirichlet boundary condition and if the normal derivatives of the dependent variables are specified it is called as Neumann boundary condition.

1. *Indirect Methods*

Two iterative methods are very well known, namely, (1) Liebmann Successive Over–Relaxation (SOR) and (2) the Alternative Direction Implicit method (ADI). In these two methods ADI method is superior to SOR, but due to simplicity the SOR method is used widely. ADI method is rather difficult and requires large computer storage.

2. *Direct Methods*

(1) Matrix method: This method is based on the simple structure of the triangular matrices obtained by applying the well–known elimination procedure of Gauss to the matrix of the difference equation. It was proved by Karlqvist (1952) that the procedure is numerically stable i.e. there is no tendency for error growth. This method is generally less efficient than other direct methods.

(2) Dimension reduction method (DRM): This method gives a direct solution of Poisson's equation where boundary values are prescribed on the perimeter of the rectangular domain i.e. Dirichlet type boundary condition. The principle of the method is to transform the finite difference analogue of the Poisson equation to sets of independent one–dimensional finite difference equations by changing the coordinate system. Ogura (1969) had discussed this method in detail.

(3) Generalized Sweep–out method (GSM): This is an independent method given by Hirota et al. (1970). This method requires the determination of residuals swept out after an initial guess has been inserted on a row of the grid. This method can be applied to any arbitrarily shaped domain with boundary condition of Dirichlet or Neumann.

(4) Fourier method: In certain circumstances the fast Fourier transform can be applied to obtain direct solution of Poisson's equation. Hockney (1965) had given outline of this method. An example of the use of this method is given by Williams (1967) in his numerical experiments with a differentially heated rotating annulus, where he has applied the trigonometric interpolation method.

(5) Block cyclic reduction method: This method reduces the dimension of the matrix problem to be solved. Procedure of this method places a restriction on the number of interior points of the grid; either M or N should be equal to $2^{k+1}-1$ where k is an integer. A complete discussion of this method is given by Buzbee et al. (1970).

Out of the five direct methods mentioned above, only two methods, namely, Fourier method and Block Cyclic Reduction method are used in the present study. Therefore, some more details of these two methods (to obtain solution of Poisson's equation) are given below.

3. *Solution of Poisson's Equation*

(1) *Fourier method*

We consider here a square region in (x, y) plane covered by 48×48 grid points. Using the usual 5–point difference approximation, equation (1) with $\alpha = 0$ may be written in finite difference form as,

$$\varphi_{i-1,j}+\varphi_{i+1,j}+\varphi_{i,j-1}+\varphi_{i,j+1}-4\varphi_{i,j}=F_{i,j}$$
$$(i,\ j=0,\ 1,\cdots,\ 47).$$

The periodic boundary conditions are

$$\varphi_{i+48L,\ j+48L}=\varphi_{i,j},\quad F_{i+48L,\ j+48L}=F_{i,j},$$

where L is any integer. The set of 24 equations on the even lines of the mesh for any of the 48 harmonic amplitudes may be written as

$$\varphi_{j-2}+\lambda_k\varphi_j+\varphi_{j+2}=F_j,\quad (j=0,\ 2,\cdots,\ 46)$$

where

$$\lambda_k=-2(8-8\cos\frac{2\pi k}{48}+\cos\frac{4\pi k}{48}).$$

This involves the recursive application of the process of cyclic reduction and this process may be carried out recursively until a small number of equations are obtained which can be solved directly. The solution on the even lines of the mesh is obtained by the process of Fourier synthesis. The total solution is completed by solving on the odd lines of the mesh directly from the already obtained solutions on the even lines. The number of mesh points should be of the form 3×2^p, where p is positive integer >1.

(2) *Block–cyclic reduction method*

Block–cyclic reduction method can solve the general elliptic type equation like

$$\frac{\partial^2\varphi}{\partial y^2}+K_1(x)\frac{\partial^2\varphi}{\partial x^2}+K_2(x)\frac{\partial\varphi}{\partial x}-K_3(x)\varphi=F(x,\ y). \qquad (1)$$

In particular when $K_1(x)=1$ and $K_2(x)=K_3(x)=0$ it reduces to Poisson's equation $\nabla^2\varphi=F(x,\ y)$. We have used Dirichlet boundary condition to obtain solution of Poisson's equation i.e. φ values are prescribed at the boundaries. This method can also be applicable for Neumann boundary condition. The finite difference form of the Poisson's equation can be written as

$$A_i\varphi_{i-1,j}+B_i\varphi_{i,j}+C_i\varphi_{i+1,j}-\varphi_{i,j+1}-\varphi_{i,j-1}=G_{i,j}\ . \qquad (2)$$

where

$$A_i=-s^2,\quad =\Delta y/\Delta x,\quad G_{i,j}=-\Delta y^2F_{i,j},$$
$$B_i=2(1+s^2),\ \ \text{and}\ \ C_i=-s^2.$$

Further let $i=1,\ 2\cdots,\ M$ and $j=1,\ 2,\cdots,\ N$.
Under the condition $\Delta x=\Delta y$, the above system of matrix equations may be written as

$$E\varphi_j-I\varphi_{j-1}-I\varphi_{j+1}=g_j,\quad j=1,\ 2,\cdots,\ N. \qquad (3)$$

where the subscript i has been dropped for brevity, E is an $M\times N$ matrix given as follows.

$$E=\begin{bmatrix} 4 & -1 & 0 & . & . & . & . & . & . & . & . & . & 0 \\ -1 & 4 & -1 & 0 & . & . & . & . & . & . & . & . & 0 \\ 0 & -1 & 4 & -1 & 0 & . & . & . & . & . & . & . & . \\ 0 & 0 & -1 & 4 & -1 & 0 & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . & 0 & -1 & 4 & -1 \\ . & . & . & . & . & . & . & . & . & . & 0 & -1 & 4 \end{bmatrix},$$

φ_j and g_j are column vectors consisting of M elements (e.g. $\varphi_{1j}, \varphi_{2j}, \cdots, \varphi_{Mj}$) and I is the $M\times M$ unit matrix. The $g_{i,j}$ are equal to $G_{i,j}$ except for $g_{1,j}$, $g_{M,j}$, $g_{i,1}$ and $g_{i,N}$.

These relationships are given below.

$$\left.\begin{aligned} g_{1,j} &= G_{1,j} - A_1\varphi_{0,j} \\ g_{M,j} &= G_{M,j} - C_M\varphi_{M-1,j} \end{aligned}\right\} \quad j\neq 1,\quad j\neq N$$
$$\left.\begin{aligned} g_{i,1} &= G_{i,1} + \varphi_{i,0} \\ g_{i,N} &= G_{i,N} + \varphi_{i,N+1} \end{aligned}\right\} \quad i\neq 1,\quad i\neq M \tag{4}$$

Block cyclic reduction method allows the original matrix equations to be reduced in a recursive manner to a set of equations involving a much smaller matrix. This reduction procedure is repeated until only one block of the original matrix remains. After each reduction cycle the right hand side terms of matrix equation (3) are of the form

$$g_j^{(r+1)} = g_{j-h}^{(r)} + g_{j+h}^{(r)} + E^{(r)}g_j^{(r)},\quad h=2^r, \tag{5}$$

where

$$E^{(r+1)} = (E^{(r)})^2 - 2I. \tag{6}$$

The procedure places a restriction on the number of interior points of the $M\times N$ grid, either M or N should be equal to 2^k-1, where k is a positive integer. Generally a system with $N=2^k-1$ requires $k(N+1)/2$ Gaussian eliminations.

When the vectors g_j are calculated in direct fashion, increasingly round off errors and instability develop as N gets large. It was found that the errors were unacceptable for $N>31$ on EC−1040/CYBER−170. Buneman (1969) alleviated this problem by alternatively calculating the $g_j^{(r)}$. Buneman introduced the relationship

$$g_j^{(r)} = E^{(r)}P_j^{(r)} + q_j^{(r)}. \tag{7}$$

Substituting (7) into (5) and using (6) we get

$$\left[E^{(r)^2} - 2I\right]P_j^{(r+1)} + q_j^{(r+1)} = E^{(r)}\left(P_{j-h}^{(r)} + P_{j+h}^{(r)}\right) + q_{j-h}^{(r)} + q_{j-h}^{(r)} + E^{(r)^2}P_j^{(r)} + E^{(r)}q_j^{(r)}, \tag{8}$$

where the auxiliary vectors $P_j^{(r)}$ and $q_j^{(r)}$ are

$$P_j^{(r+1)} = P_j^{(r)} + (E^{(r)})^{-1}\left(P_{j-h}^{(r)} + P_{j+h}^{(r)} + q_j^{(r)} \right) ,$$
$$q_j^{(r+1)} = q_{j-h}^{(r)} + 2P_j^{(r+1)} + q_{j+h}^{(r)} . \tag{9}$$

This variant of the Buneman cuts the storage requirements in half but requires about twice as many additions in a computational algorithm. However, the extra computation time for these additions becomes negligible as N increases.

III. RESULTS

In order to evaluate efficiency of the above two methods, qualitative comparison of the two methods is carried out by obtaining numerical solution of the Poisson's equation,

$$\nabla^2 \varphi(x,y) = F(x,y). \tag{10}$$

Consider

$$\varphi(x,y) = \sum_{k=1}^{N/2-1} ky \ \sin\left(\frac{2\pi k \Delta x}{N} \right) . \tag{11}$$

where k and N represent zonal wavenumber and number of grid points respectively. The grid point values of $F(x,y)$ are analytically computed from $\varphi(x,y)$ and are used as input to solve $\varphi(x,y)$ by Fourier method and Block Cyclic reduction method.

1. *Numerical Solution for Analytical Input*

(1) *Fourier method*

As already mentioned in Section (II. 3.(2)) the number of grid points (N) should be 3×2^p, $p > 1$. In this case $p = 3$ is chosen. Equation $\nabla^2 \varphi(x,y) = F(x,y)$ is solved on (24 $\times$ 24) mesh (i.e. 576 grid points). Forcing $F(x,y)$ is supplied at every grid point. The RMS error between the actual values of $\varphi(x,y)$ obtained from expression (11) and the solution $\varphi(x,y)$ at the grid points using Fourier method varies from 1.2 to 1.8.

(2) *Block–cyclic reduction method*

Using computed values of $F(x, y)$ from Eq. (11), the solution of Eq. (10) for $\varphi(x,y)$ is obtained by Block–cyclic reduction method on (38 $\times$ 15) mesh (i.e. 570 grid points). It was found that the solution of $\varphi(x,y)$ at the grid points is correct up to six decimal places. Number of grid points involved for computation in both the methods (i.e. present method and Fourier method) are comparable but Fourier method requires 50% more time as compared to Block–cyclic reduction method. Further, Block–cyclic method is also utilized to solve Poisson equation for non–periodic function. The solution is found to be quite accurate. Block–cyclic reduction method is superior to Fourier technique both in accuracy and speed. Hence, Block–cyclic reduction method is ultimately chosen to obtain streamfunction ψ and velocity potential χ from the given vorticity and divergence fields, respectively.

2. *Numerical Solution for Observed Input*

The basic data for computation of ψ and χ fields consist of observed zonal and meridional wind components obtained from FGGE level IIIb data over the area 80.625°E to 99.375°E and Equator to 15.0°N at 1.875 regular latitude longitude grid points for 4th July 1979.

The block cyclic reduction method discussed in Section (II. 3(2)) requires the boundary values of ψ and χ to be prescribed. After supplying the boundary values, the method was used to solve $\nabla^2\psi=\zeta$ and $\nabla^2\chi=D$ where ζ and D are vorticity and divergence, in the interior of the domain respectively. Since total wind consists of rotational and divergent part of wind, the wind fields are reconstructed using the relations

$$u=u_\psi+u_\chi \quad \text{and} \quad v=v_\psi+v_\chi$$

i.e. $$u=-\partial\psi/\partial y+\partial\chi/\partial x \quad \text{and} \quad v=\partial\psi/\partial x+\partial\chi/\partial y. \qquad (12)$$

We have used two cases of boundary conditions.

Case 1: Assume $\psi=0$, $\chi=0$ at the boundary.

Case 2: Assume $\chi=0$, at the boundary. ψ at the boundary points were obtained using $u=-\partial\psi/\partial y$, and $v=\partial\psi/\partial x$ where u and v are observed zonal and meridional wind values.

The boundary condition for Case 2 is assumed on the hypothesis that during monsoon season horizontal wind is mainly dominated by its non–divergent part. This is also supported by the study of Krishnamurti, (1971) which indicates that about 80 percent of the variance of the motion field is described by the rotational part. Using observed wind data for 850 hPa, 700 hPa and 200 hPa levels during 4 to 7 July 1979 (Depression case during second phase of MONEX period), ψ and χ fields are obtained from the vorticity and divergence fields and thereafter with the help of expression (12) zonal and meridional wind fields are reconstructed.

In order to decide the choice of boundary condition depending on its accuracy in the results, both the cases of boundary conditions are used and wind field is reconstructed for 700 hPa level. Results of the two cases ($\psi=0$ or $\psi\neq0$ at the boundary) differ marginally. The root mean square difference or error between observed wind and computed rotational as well as divergent component of wind is separately carried out. These computations for zonal and meridional wind are presented in Table 1. The results show that in the case of zonal wind, the error between observed and computed divergent component is higher than the rotational component, but in the case of meridional wind, both the values i.e. errors between observed and computed divergent or computed rotational components are comparable. Therefore, we may conclude that the low level monsoonal zonal wind is dominated by its rotational part, where as for construction of meridional wind both divergent and rotational components are equally important. In order to verify the validity of this conclusion for different areas and different seasons similar computations will be carried out by varying observed wind data. Further, it is noted that the RMS error between observed and computed divergent part of wind (for both u and v) does not alter by varying the boundary condition. This may be due to the fact that in both the cases we have used $\chi=0$ at the boundary.

3. *Quantitative Comparison between Direct and Indirect Methods*

Various authors (Hawkins and Rosenthal 1965; Shukla and Saha, 1974) have used root mean square vector error (r.m.s.v.e.) between the observed wind and reconstructed wind field as a criterion of determining the goodness of a particular method. We have also followed the same approach and made quantitative intercomparison between the results of direct block cyclic reduction (BCR) method and indirect SOR method. While comparing two methods, it is necessary to have the same boundary conditions in both the methods. Therefore the boundary condition of Case 1 (i.e. $\psi=0$, $\chi=0$) is used in both BCR and SOR methods. In this case the computations are carried out for the same area at 850 hPa level (Table 2) as the results of SOR method were easily available to the authors for 850 hPa. The over–relaxation parameter was chosen to be 0.75. Following notations are used in the Table.

$$E_u^{\psi\chi} = \sqrt{\frac{1}{M \times N}\sum[u_o - u_c]^2},$$

$$E_v^{\psi\chi} = \sqrt{\frac{1}{M \times N}\sum[v_o - v_c]^2},$$

$$E_{\omega}^{\psi\chi} = \sqrt{\frac{1}{M \times N}\sum[(u_o - u_c)^2 + (v_o - v_c)^2]},$$

where suffix 'o' is for observed wind and 'c' is for total constructed wind components obtained from expression (12).

M × N = Total number of grid points.

It is seen from Table 2 that the error is small in the case of BCR method than in the case of SOR method. Further, both the methods show that error in the meridional flow (i.e. E_v) is smaller than that in the zonal flow (i.e. E_u). At this stage it is important to note that though the actual magnitude of error E_v is small due to weak meridional flow, the percentage of error in the meridional flow is much higher than that of zonal flow.

IV. CONCLUSION

On the basis of above results following conclusions may be drawn.

(1) Qualitative and quantitative comparison between the solutions obtained from two direct methods indicated that BCR method is superior to Fourier method in terms of speed and accuracy.

(2) In the lower troposphere monsoonal zonal flow is found to be dominated by its rotational part but for the meridional flow both rotational and divergent parts are equally important.

(3) Computations of reconstructed wind field showed that results of BCR method are more reliable than SOR method.

(4) Since BCR method is found to be more accurate and also easy to adopt, the method is useful to solve other meteorological prognostic and diagnostic equations like

(i) non–divergent barotropic vorticity equation, $\nabla^2(\partial\psi / \partial t) = J(\nabla^2\psi + f,\ \psi)$.

(ii) The linear balance equation $f\nabla^2\psi + \nabla f.\nabla\psi = \nabla^2\varphi$.

(iii) Omega equation.

In the present study BCR method is discussed for the case of solution of Poisson's equation only. It is proposed to extend this method for general elliptic equation.

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Table 1. Root Mean Square Difference of Wind at 700 hPa (m s^{-1})

Boundary condition \ Observed and constructed wind	u_o & u_ψ	u_o & u_χ	v_o & v_ψ	v_o & v_χ
$\psi = 0$ and $\chi = 0$	1.49	9.71	2.71	3.57
$\psi \neq 0$ and $\chi = 0$	1.45	9.71	3.26	3.57

Table 2. Root Mean Square Vector Error ($m\ s^{-1}$) at 850 hPa

Method of Solution	$E_u^{\psi\chi}$	$E_v^{\psi\chi}$	$E_{uv}^{\psi\chi}$
BCR	2.2	1.56	2.7
SOR	3.9	1.63	4.2

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