An Exact Solution for Two–Dimensional Frictionless Motion in the Atmosphere

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ABSTRACT

A solution of the nonlinear problem for determining the wind velocity in frictionless atmosphere (the gradient wind) under given geopotential (pressure) field is proposed. The approach is analytical and is based on quadratic polynomial approximation of the geopotential field and linear approximation of the wind velocity field with respect to x and y, the coefficients of the expansions being functions of the time t. The derived system of ordinary nonlinear differential equations is analyzed as a dynamical system. Exact analytical solutions are found for some particular cases. Some of their properties bear a resemblance to those or really existing atmospheric vortices (cyclones and anticyclones).

I. INTRODUCTION

Consider the equations of two–dimensional motion in isobaric (p) coordinates (Panchev, 1985)

$$\partial U / \partial t + U\partial U / \partial x + V\partial U / \partial y - fV = -\partial H / \partial x,$$
$$\partial V / \partial t + U\partial V / \partial x + V\partial V / \partial y + fU = -\partial H / \partial y, \tag{1.1}$$

where $H(x,y)$ is the geopotential of isobaric surface p = const. Equations (1.1) are nonlinear and for arbitrary $H(x,y)$ analytical solution is impossible. However, it has been long ago observed (see Young, 1986; Cushman–Roisin, 1987) that if

$$H(x,y) = H_{00} + H_{10}x + H_{20}y + \frac{1}{2}H_{11}x^2 + \frac{1}{2}H_{22}y^2 + H_{12}xy, \tag{1.2}$$

then

$$U(x,y,t) = U_0 + U_1(t)x + U_2(t)y,$$
$$V(x,y,t) = V_0 + V_1(t)x + V_2(t)y. \tag{1.3}$$

The gradient wind with constant horizontal shear (1.3), subject to investigation in the present paper, is a theoretical idealization (hypothetical motion) implied by the special mathematical structure of the momentum equations (1.1). Strictly speaking, such a motion does not exist in the real atmosphere. Nevertheless, more than a century this mathematical problem attracts the attention of the applied mathematicians and geophysical fluid dynamicists. Most of the publications based on the method of polynomial approximations (1.2), (1.3) are in oceanographic aspect. Very few concern the atmospheric dynamics: such as Bagrov (1947); Benton, Lipps and Tuann (1964), Panchev and Spassova (1988) etc. It seems to us that now, in the era of numerical methods, the interest to this method of solution is increasing mainly by two reasons—analytical attractiveness and proximity of the model results to some observed situations in the atmosphere.

It is well known in meteorology (Petterssen, 1956), that over limited area around the point (0,0), a large variety of geometrical configurations of the isobars or streamlines which are observed on the weather maps can be approximated rather well by the above expressions (1.2), (1.3). This empirical fact warrants the assumed mathematical structure of the solution (1.2), (1.3). To eliminate the simple translation motion we assume

$$H_{10} = H_{20} = 0 \quad \text{and} \quad U_0(t) = V_0(t) = 0 \tag{1.4}$$

Therefore, the isohypses are assumed essentially curvilineal and the geostrophic component of the motion is excluded from the solution.

In such a formulation this problem was studied by us earlier (Panchev and Spassova, 1988). We are continuing this study. In section II steady solutions are obtained. The nonlinear nonstationary problem is considered in section III. Each section contains also discussion of the results.

II. GRADIENT WIND–STATIONARY SOLUTION

Upon substituting (1.2), (1.3) into (1.1) one obtains

$$\dot{U}_k + U_k U_1 + V_k (U_2 - f) = -H_{1k},$$
$$\dot{V}_k + U_k (V_1 + f) + V_k V_2 = -H_{2k}, \tag{2.1}$$

where $k = 1,2$ and $(\,\dot{}\,)$ means d/dt. We shall use the variables proposed by Ball (1965)

$$D(t) = \partial U / \partial x + \partial V / \partial y = U_1(t) + V_2(t) - \text{divergence}$$
$$\zeta(t) = \partial V / \partial x - \partial U / \partial y = V_1(t) - U_2(t) - \text{vorticity}$$
$$F(t) = \partial U / \partial x - \partial V / \partial y = U_1(t) - V_2(t) - \text{deformation}$$
$$M(t) = \partial V / \partial x + \partial U / \partial y = V_1(t) + U_2(t) - \text{deformation} \tag{2.2}$$
$$S = H_{11} + H_{22}, \quad R = H_{11} - H_{22}, \quad Q = 2H_{12}. \tag{2.3}$$

Obviously

$$2U_1 = D + F, \qquad 2V_1 = M + \zeta,$$
$$2U_2 = M - \zeta, \qquad 2V_2 = D - F. \tag{2.4}$$

In these variables the transformed equations are not simpler than (2.1), but in some aspects are more convenient for analysis and interpretation:

$$\dot{\zeta} = -D(f + \zeta),$$
$$\dot{F} = -DF + fM - R, \tag{2.5}$$
$$\dot{M} = -DM - fF - Q,$$
$$\dot{D} = f - S - \frac{1}{2}(D^2 + M^2 + F^2 - \zeta^2).$$

In case of closed isolines of the H–field, additional rotation of the coordinate axes can make $Q = 0$. We assume this and hereafter the coordinate system is fixed.

The steady solution of (2.5) will give the gradient wind characteristics $\bar{\zeta}$, $\bar{F}$, $\bar{M}$, $\bar{D}$, and through (2.4) – the coefficients $\bar{U}_k$ and $\bar{V}_k$ in (1.3). With $\dot{\zeta} = \dot{F} = \dot{M} = \dot{D} = 0$, the vorticity equation in (2.5) (the first one) implies two mathematically possible cases:

CASE A. Nondivergent gradient wind – D = 0. Then

$$\bar{F} = 0, \quad \bar{M} = \bar{R}/f, \quad \bar{\zeta}^2 + 2f\bar{\zeta} - (2S + R^2/f^2) = 0. \tag{2.6}$$

Only the solution corresponding to "+" sign is physically reasonable:

$$\bar{\zeta} = -f + (f^2 + 2S + R^2/f^2)^{1/2}. \tag{2.7}$$

For $S > 0$ (cyclonic pressure distribution) $\bar{\zeta}$ exists always and is positive ($\bar{\zeta} > 0$). However in case of anticyclonic distribution ($S = -|S| < 0$) some restrictions are to be imposed on R and S for $\bar{\zeta}$ from (2.7) to be real and negative:

$$R^2/f^2 \leqslant 2|S| \leqslant f^2 + R^2/f^2. \tag{2.8}$$

CASE B. Divergent gradient wind ($D \neq 0$) with vanishing absolute vorticity $\bar{\zeta}_a = f + \bar{\zeta} = 0$. In this case

$$\bar{M} = fR / (f^2 + \bar{D}^2), \quad F = -\bar{D}R / (f^2 + \bar{D}^2) \tag{2.9}$$

and $\bar{D}^2 = Y$ satisfies the quadratic equation

$$Y^2 + 2(f^2 + S)Y + (f^4 + 2Sf^2 + R^2) = 0.$$

A positive solution exists only if $S = -|S| < 0$,, anticyclonic pressure distribution. Therefore the solution for $\bar{D}$ must be positive too. We find

$$\bar{D} = [(S^2 - R^2)^{1/2} + S - f^2]^{1/2}$$

and the restriction for existence is

$$|S| + (S^2 - R^2)^{1/2} \geqslant f^2.$$

However this mathematical solution is not recognized to have meteorological value and is not considered further.

Linear stability analysis of the steady solution (2.6) shows that it is stable, which is not surprising.

III. THE NONLINEAR PROBLEM

a) Nondivergent flow In this case the continuity equation holds

$$\partial U / \partial x + \partial V / \partial y = 0. \tag{3.1}$$

In view of (1.3) this equation implies that

$$D(t) = U_1(t) + V_2(t) = 0 \quad \text{and} \quad \bar{D} = 0. \tag{3.2}$$

Then the system (2.5) becomes overdetermined—three equations for two unknown functions $F(t)$ and $M(t)$:

$$\dot{F} = fM - R, \quad \dot{M} = -fF,$$
$$F^2 + M^2 = \zeta_0^2 + 2f\zeta_0 - 2S = a_0^2 \geqslant 0 \tag{3.3}$$

where $\dot{\zeta} = 0$, i.e. $\zeta = \zeta_0 = \text{const}$. Moreover, the initial vorticity ζ_0 is constrained to ensure that $a_0^2 \geqslant 0$. The only solution which can meet these conditions is

$$F(t) = F_0 \cos ft + M_0 \sin ft,$$
$$M(t) = M_0 \cos ft - F_0 \sin ft, \tag{3.4}$$

provided that $R = 0$ (circular isohypses). Comparing (3.3) with (2.6) we conclude that if $\zeta_0 = \bar{\zeta}$, then $a_0^2 = M_0^2 + F_0^2 = 0$, i.e. $F_0 = M_0 = 0$. Thus we assume $\zeta_0 \neq \bar{\zeta}$.

Since $D(t) = 0$ one can introduce the stream function $\psi(x,y,t)$ such that $U = -\partial\psi / \partial y$, $V = \partial\psi / \partial x$ and compare $H(x,y)$ with $\psi(x,y,t)$ corresponding to (3.4). For simplicity, if $F_0 = 0$, then

$$H(x,y) = \text{const} + \frac{1}{2} H_* (x^2 + y^2), \quad (H_* = H_{11} = H_{22}), \tag{3.5}$$

$$\psi(x,y,t) = \frac{1}{4}(M_0 \cos ft + \zeta_0)x^2 - \frac{1}{4}(M_0 \cos ft - \zeta_0)y^2 - \frac{1}{2} M_0 xy \sin ft. \tag{3.6}$$

Therefore, the isohypses are circles, but the streamlines are not unless $M_0 = 0$.

b) Divergent flow ($D(t) \neq 0$). In this case the full continuity equation

$$\frac{\partial H}{\partial t} + U\frac{\partial H}{\partial x} + V\frac{\partial H}{\partial y} = -H\left(\frac{\partial U}{\partial x} + \frac{\partial V}{\partial y}\right) \tag{3.7}$$

should be associated with (1.1), assuming that $H = H(x,y,t)$, i.e. the shallow water system of equations should be treated with the present technique. Then three additional equations about $R(t)$, $S(t)$, $Q(t)$ can be derived (Ball, 1965). The resulting system of seven nonlinear ordinary differential equations can be solved only numerically. However, ignoring equation (3.7), one can consider the system (2.5) as a mathematical problem only which, despite of the nonlinearity of each equation, admits exact analytical solution, provided that $R = Q = 0$, $S = S(t) \neq 0$.

Actually, denoting

$$f + \zeta = \zeta_a, \quad S + \frac{1}{2}f^2 = S_*,$$

$$E(t) = M^2(t) + F^2(t) - \zeta_a^2(t), \tag{3.8}$$

we reduce the system (2.5) to two equations

$$\dot{E} = -2DE,$$

$$\dot{D} = -S_* - \frac{1}{2}(E + D^2). \tag{3.9}$$

Eliminating $E(t)$ we reach the single equation

$$\ddot{D} + 3D\dot{D} + D^3 + 2S_* D + \dot{S}_* = 0, \tag{3.10}$$

first derived and studied by Spassova (1988) for a simpler case ($f = 0$).

According to Kamke (1959), the substitution

$$D = \dot{W} / W \tag{3.11}$$

converts (3.10) into linear equation

$$\dddot{W} + 2S_*(t)\dot{W} + \dot{S}_*(t)W = 0. \tag{3.12}$$

The general solution of the latter is (Kamke, 1959)

$$W(t) = C_1 g_1^2(t) + C_2 g_1(t) g_2(t) + C_3 g_2^2(t), \tag{3.13}$$

where $g_1(t)$ and $g_2(t)$ are two linearly independent solutions of the equation

$$\ddot{g} + \frac{1}{2}S_*(t)g = 0. \tag{3.14}$$

However, the latter one is the well known classical equation of linear oscillator, which can be solved analytically for some particular form of the function $S_*(t)$, including the simplest one, $S_*(t) = S_* = \text{const}$ (positive or negative).

Having determined the divergence $D(t)$, the remaining equations in (2.5) (for ζ, F, M) become linear and in principle can be integrated (at $R = Q = 0$).

IV. CONCLUSION

In the present study we stressed on the mathematical aspect of the problem rather than on the physical one. In doing so we introduced maximum simplification allowing use of analytical tools only. The main result obtained in the paper concerns the exact (deterministic) solution of the nonlinear system (2.5) at $R = Q = 0$, via the equations (3.9) and (3.10). It can be used to test numerical schemes for solution.

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