Advances in Studies on Nonlinear Atmospheric Waves

Liu Shida (刘式达), *Liu Shikuo* (刘式适)
Department of Geophysics, Peking University, Beijing 100871
Received February 24, 1989

ABSTRACT

A review on the progress in the research of nonlinear atmospheric waves, especially the nonlinear Rossby waves is made in this paper. Many results reported here have been obtained in Peking University.

I. INTRODUCTION

In the last 20 years, the research on nonlinear cnoidal and solitary waves has been given much attention in the mathematics and physics, for which Scott et al. (1973) summarized systematically. They put forward that both the nonlinearity and dispersion might also produce the steady and periodic waves. In geophysical fluid dynamics, the research was first conducted by Long (1964) and Benney (1966), who studied the barotropic Rossby waves using horizontal shear velocity with the result that the amplitude of waves satisfies the KdV equation. Redekopp (1977) studied the baroclinic Rossby waves with the result that the amplitude of waves satisfies the modified KdV equation. Yamagata (1982) studied the weakly nonlinear quasi–geostrophic planetary waves, he also obtained the KdV equation. Clarke (1971), Smith (1972) and Grimshaw (1977) discussed the KdV dynamics of ageostrophic and semi–geostrophic long waves. Solitary equatorial waves were discussed by Boyd (1980). Chao et al. (1980) discussed the cnoidal in the barotropic atmosphere. Yano and Tsujimura (1987) classified the KdV–type solitary Rossby waves which are governed by the KdV equation.

All the above–mentioned researches use the multi–scale perturbation method to find the suitable approximate solutions when nonlinear and dispersive factors attain equilibrium by selecting small parameter. Though these studies are significant and can consider the blocking system as a soliton still, these studies lack physical analysis and select the small parameter arbitrarily and the method is complicated.

The authors (1982) tried to find the exact and asymptotic analytical solutions of the nonlinear atmospheric waves by a relatively simple method rather than the multi–scale perturbation method. We found that not only the Rossby waves satisfy the KdV equation but also satisfy dispersion relation of frequency–wavenumber–amplitude. This is known as the KdV–type solitary Rossby waves in the world today. It seems that the blocking systems which play an important role in the atmospheric circulation are a family of solitary Rossby waves.

Besides long waves and blocking system, there exist some symmetric pressure systems in the atmosphere. To illustrate this point Stern (1975) discussed firstly the planetary eddies and obtained the exact isolated solution of the steady state barotropic vorticity equation on infinite β–plane, he calls this solution a modon which is composed of a coupled cyclone–anticyclone systems. Larichev and Reznik(1976) found a moving barotropic solitary eddy solution. Flierl et al. (1980) derived a set of exact solution to the quasi–geostrophic equations by means of a two–layer model. McWilliams (1980) discussed a particular nonlinear analytical solution to the equivalent barotropic equations, he calls this solution an equivalent modon and suggests it as a model of the persistence for at least some blocking events. Swaters (1986) studied the barotropic modon propagation over slowly varying topography. These theories point out that the eddy systems are a family of modon eddies.

The soliton and modon are illustrated in Figs.1 and 2, respectively.

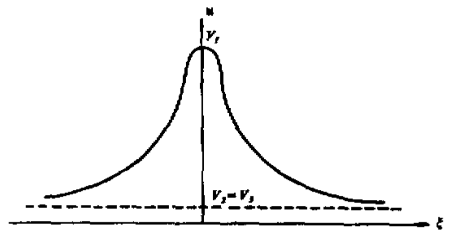

Fig.1. Soliton.

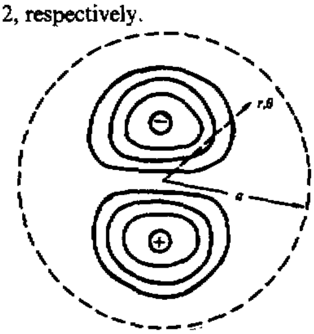

Fig.2. Modon (see Stern, 1975).

II. NONLINEAR ROSSBY WAVES; CNOIDAL WAVES AND SOLITARY WAVES (SOLITON)

1. *General Approach, KdV Equation*

The linear Rossby waves with horizontal divergent flow satisfy the dispersion relation.

$$\omega = -\beta_0 k / (k^2 + l^2 + \lambda_0^2) \tag{2.1}$$

for the case of no basic zonal flow within the β–plane approximation, where ω is the angular frequency, k and l the wavenumber in x and y directions respectively, β the Rossby parameter taken as a constant λ_0^{-1} the barotropic Rossby radius of deformation.

If longwave approximation $(k \to 0)$ is taken, that is to say, the condition

$$k^2 \ll l^2 + \lambda_0^2 \tag{2.2}$$

is satisfied, the dispersion relation (2.1) then is approximately written by

$$\omega - kc = -\gamma k^3 \tag{2.3}$$

where

$$c = -\frac{\beta_0}{l^2 + \lambda_0^2}, \qquad \gamma = -\frac{\beta_0}{(l^2 + \lambda_0^2)^2}. \tag{2.4}$$

The weak dispersion term $\Delta\omega = \gamma k^3$ in the dispersion relation (2.3) produces the term $\gamma \frac{\partial^3 A}{\partial \xi^3}$ in the governing equation of the nonlinear wave, where A is an appropriate normalized amplitude, ξ is an appropriate space coordinate. When the term $\gamma \frac{\partial^3 A}{\partial \xi^3}$ is balanced with a nonlinear term in the form of $A \frac{\partial^3 A}{\partial \xi^3}$, the nonlinear Rossby wave is governed by the KdV equation

$$\frac{\partial A}{\partial \tau} + A \frac{\partial A}{\partial \xi} + \gamma \frac{\partial^3 A}{\partial \xi^3} = 0, \tag{2.5}$$

where τ is an appropriate time scale.

Long (1964) first pointed out that the Rossby waves are governed by the KdV equation (2.5) under an appropriate condition. Redekopp (1977) revealed the nature of the KdV–type solitary Rossby waves.

Let us consider the quasi–geostrophic potential vorticity equation in the barotropic (shallow water) β–plane model, i.e.

$$\left(\frac{\partial}{\partial t}+u\frac{\partial}{\partial x}+v\frac{\partial}{\partial y}\right)(\nabla_h^2\psi-\lambda_0^2\psi)+\beta\frac{\partial\psi}{\partial x}=0, \tag{2.6}$$

where t is time, x and y the eastward and northward coordinate respectively, u and v the zonal and meridional velocities respectively, ψ the stream function of quasi–geostrophic flow, ∇_h^2 the horizontal Laplacian.

Setting that

$$u=\bar{u}(y)+u',\qquad v=v',\qquad u'=-\frac{\partial\psi'}{\partial y},\qquad v'=\frac{\partial\psi'}{\partial x} \tag{2.7}$$

where $\bar{u}(y)$ is the basic zonal current, ψ' the disturbed stream function, u' and v' the disturbed zonal and meridional velocities respectively. Equation (2.6) then can be reduced to

$$\left[\frac{\partial}{\partial t}+\left(\bar{u}(y)-\frac{\partial\psi'}{\partial x}\right)\frac{\partial}{\partial x}+\frac{\partial\psi'}{\partial x}\frac{\partial}{\partial y}\right]\left(\nabla_h^2\psi'-\lambda_0^2\psi'\right)+B\frac{\partial\psi'}{\partial x}=0, \tag{2.8}$$

where

$$B=\beta_0-\frac{\partial^2\bar{u}}{\partial y^2}\ . \tag{2.9}$$

(2.3) implies that when wavelength is large and dispersion is weak the appropriate slow space and time scales for the wave evolution in a frame moving with the non–dispersive motion are

$$\xi=\epsilon^{\frac{1}{2}}(x-ct),\qquad \tau=\epsilon^{\frac{3}{2}}t,\qquad y=y \tag{2.10}$$

where ϵ is the longwave parameter measuring the ratio of the waveguide scale to the wavelength. (2.10) is called the G–M (Garden–Morikawa) transform. using the G–M transform and perturbation method equation (2.8) can be reduced to KdV equation, the method is known as the reductive perturbation method.

From (2.10) we have

$$\frac{\partial}{\partial t}=\epsilon^{\frac{3}{2}}\frac{\partial}{\partial\tau}-\epsilon^{\frac{1}{2}}c\frac{\partial}{\partial\xi},\qquad \frac{\partial}{\partial x}=\epsilon^{\frac{1}{2}}\frac{\partial}{\partial\xi}\qquad \frac{\partial}{\partial y}=\frac{\partial}{\partial y} \tag{2.11}$$

Substituting (2.11) into (2.8), we have

$$\left[\epsilon\frac{\partial}{\partial\tau}+(\bar{u}-c)\frac{\partial}{\partial\xi}-\frac{\partial\psi'}{\partial y}\frac{\partial}{\partial\xi}+\frac{\partial\psi'}{\partial\xi}\frac{\partial}{\partial y}\right]\left(\epsilon\frac{\partial^2\psi'}{\partial\xi^2}+\frac{\partial^2\psi'}{\partial y^2}-\lambda_0^2\psi'\right)+B\frac{\partial\psi'}{\partial\xi}=0 \tag{2.12}$$

Applying the regular perturbation expansion

$$\psi'=\epsilon\psi_1+\epsilon^2\psi_2+\cdots \tag{2.13}$$

and substituting it into (2.12) we then obtain the following first and second–order equation

$$\frac{\partial}{\partial\xi}\left[(\bar{u}-c)\frac{\partial^2\psi_1}{\partial y^2}-\lambda_0^2\psi_1+B\psi_1\right]=0 \tag{2.14}$$

$$\frac{\partial}{\partial\xi}\left[(\bar{u}-c)\frac{\partial^2\psi_2}{\partial y^2}-\lambda_0^2\psi_2+B\psi_2\right]+\frac{\partial}{\partial\tau}\left(\frac{\partial^2\psi_1}{\partial y^2}-\lambda_0^2\psi_1\right)+$$
$$(\bar{u}-c)\frac{\partial^3\psi_1}{\partial\xi^3}-\frac{\partial\psi_1}{\partial y}\frac{\partial}{\partial\xi}\left(\frac{\partial^2\psi_1}{\partial y^2}-\lambda_0^2\psi_1\right)+\frac{\partial\psi_1}{\partial\xi}\frac{\partial}{\partial y}\left(\frac{\partial^2\psi_1}{\partial y^2}-\lambda_0^2\psi_1\right)=0 \tag{2.15}$$

Integrating equation (2.14) with respect to ξ and taking the integral constant as zero, we obtain

$$(\bar{u}-c)\left(\frac{\partial^2\psi_1}{\partial y^2}-\lambda_0^2\psi_1\right)+B\psi_1=0. \tag{2.16}$$

If $\bar{u}-c\neq 0$, equation (2.16) yields

$$\frac{\partial^2 \psi_1}{\partial y^2} + \left[Q(y) - \lambda_0^2 \right] = 0, \tag{2.17}$$

where

$$Q(y) = \frac{B}{\bar{u} - c} = \frac{\beta_0 - \dfrac{\partial^2 \bar{u}}{\partial y^2}}{\bar{u} - c} \tag{2.18}$$

(2.17) is a second-order ordinary differential equation of $\psi_1(\tau,\xi,y)$ in y.
By setting

$$\psi_1 = A(\xi,\tau)G(y), \tag{2.19}$$

$G(y)$ then satisfies

$$\frac{d^2 G}{dy^2} + \left[Q(y) - \lambda_0^2 \right] G = 0. \tag{2.20}$$

Assuming ψ_1 vanishes along the parallels $y = y_1$ and $y = y_2$, hence

$$G|_{y=y_1} = 0, \qquad G|_{y=y_2} = 0. \tag{2.21}$$

The eigenvalue c can be determined when $\bar{u}(y)$ is given.

Substituting (2.19) into (2.15), yields

$$\frac{\partial \psi}{\partial \xi}\left[\frac{\partial^2}{\partial y^2}{}_2 + \left(Q(y) - \lambda_0^2 \right) \psi_2 \right] =$$

$$\frac{Q(y)}{\bar{u} - c} G \frac{\partial A}{\partial \tau} - G \frac{\partial^3 A}{\partial \xi^3} + \frac{1}{\bar{u} - c} \left(\frac{dG}{dy}\frac{d^2 G}{dy^2} - G\frac{d^3 G}{dy^3} \right) A \frac{\partial A}{\partial \xi}. \tag{2.22}$$

Multiplying (2.22) by $G(y)$ we have

$$G\frac{\partial}{\partial \xi}\left[\frac{\partial^2 \psi_2}{\partial y^2} + \left(Q(y) - \lambda_0^2 \right) \psi_2 \right] =$$

$$\frac{Q(y)}{\bar{u} - c} G^2 \frac{\partial A}{\partial \tau} - G^2 \frac{\partial^3 A}{\partial \xi^3} + \frac{G}{\bar{u} - c} \left(\frac{dG}{dy}\frac{d^2 G}{dy^2} - G\frac{d^3 G}{dy^3} \right) A \frac{\partial A}{\partial \xi}. \tag{2.23}$$

Integrating (2.23) with respect to y from y_1 to y_2 and applying the boundary condition (2.21) we then obtain the following KdV equation

$$\frac{\partial A}{\partial \tau} + RA\frac{\partial A}{\partial \xi} + S\frac{\partial^3 A}{\partial \xi^3} = 0, \tag{2.24}$$

where

$$R = \frac{\int_{y_1}^{y_2} \frac{1}{\bar{u} - c} \frac{dQ}{dy} G^3 \, dy}{\int_{y_1}^{y_2} \frac{1}{\bar{u} - c} Q G^2 \, dy} \tag{2.25}$$

$$S = -\frac{\int_{y_1}^{y_2} G^2 \, dy}{\int_{y_1}^{y_2} \frac{1}{\bar{u} - c} Q G^2 \, dy}. \tag{2.26}$$

2. *Nonlinear Rossby Waves with Nondivergent Flow*

Similar to the reductive perturbation method, by means of the nonlinear term expansion (Liu Shida and Liu Shikuo 1982, 1983, 1985, 1987, 1988) we can also obtain KdV equation and the new dispersion relation which contains the Rossby formula and amplitude is found.

The vorticity equation and continuity equation describing Rossby waves with the nondivergent flow can be written as

$$\frac{\partial}{\partial t}\left(\frac{\partial v}{\partial x}\right)+u\frac{\partial}{\partial x}\left(\frac{\partial v}{\partial x}\right)+\beta_0 v=0 \tag{2.27a}$$

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0 \tag{2.27b}$$

in (2.27a) $\frac{\partial v}{\partial y}$ and $u\frac{\partial}{\partial y}\left(\frac{\partial v}{\partial x}\right)$ are omitted.

We write the solution of the equation (2.27) as

$$u=\bar{u}+U(\theta),\quad v=V(\theta),\quad \theta=kx+ly-\omega t \tag{2.28}$$

Substituting (2.28) into (2.27) we get

$$k(-\omega+k\bar{u}+kU)V''+\beta_0=0 \tag{2.29a}$$

$$kU'+lV'=0 \tag{2.29b}$$

where the symbol primes denote the derivatives with respect to θ.

Integrating (2.29b) and taking the integral constant as zero we have

$$U=-\frac{l}{k}V. \tag{2.30}$$

Substituting (2.30) into the (2.29a) and eliminating U we have

$$k(\omega-k\bar{u}+lV)V''=\beta_0 V, \tag{2.31}$$

which is the nonlinear equation of V.

If $\omega-k\bar{u}+lV\neq 0$, (2.31) can be written as

$$V''+\left[\frac{-\beta_0}{k(\omega-k\bar{u}+lV)}\right]V=0 \tag{2.32}$$

Setting $V'=W$,(2.32) can be rewritten as

$$W'=\frac{\beta_0 V}{k(\omega-k\bar{u}+lV)}\equiv F(V) \tag{2.33a}$$

$$V'=W \tag{2.33b}$$

where $F(V)$ is the nonlinear function of V. The equilibrium point of (2.33) is $(V, W)=(0,0)$. Expanding $F(V)$ in Taylor series near the equilibrium point we have

$$F(V)=\frac{\beta_0}{k(\omega-k\bar{u})}V-\frac{\beta_0 l}{k(\omega-k\bar{u})^2}V^2+\cdots \tag{2.34}$$

when we take (2.34) until the second order terms, (2.33) then can be reduced to

$$W'=\frac{\beta_0}{k(\omega-k\bar{u})}V-\frac{\beta_0 l}{k(\omega-k\bar{u})^2}V^2 \tag{2.35a}$$

$$V'=W \tag{2.35b}$$

The above equations can be rewritten as

$$V''=-\frac{\beta_0 l}{k(\omega-k\bar{u})^2}V^2+\frac{\beta_0}{k(\omega-k\bar{u})}V. \tag{2.36}$$

Derivating (2.36) with respect to θ we obtain the following equation

$$V'''+\frac{2\beta_0 l}{k(\omega-k\bar{u})^2}VV'-\frac{\beta_0}{k(\omega-k\bar{u})}V'=0, \tag{2.37}$$

which is the corresponding ordinary differential equation of KdV equation.

Therefore it may be concluded that the nonlinear Rossby waves are governed by the KdV equation.

The exact solution of (2.37) is

$$v(x,y,t)=V_2+(V_1-V_2)cn^2\sqrt{\frac{\beta_0 l}{6k(\omega-k\bar{u})^2}(V_1-V_3)}(kx+ly-\omega t), \tag{2.38}$$

where V_1, V_2, V_3 are three real zeros of the cubic polynomial

$$P(V) \equiv V^3 - \frac{3}{2}\frac{\omega - k\bar{u}}{l}V^2 + B, \tag{2.39}$$

while cn( ) represents the Jacobi elliptic cosine function. In this sense the Rossby waves represented by (2.38) are called Rossby cnoidal waves.

When $V_2 = V_3$, the solution (2.38) may be reduced to

$$v(x,y,t) = \frac{\omega - k\bar{u}}{l} - \frac{3(\omega - k\bar{u})}{2l} sech^2 \sqrt{-\frac{\beta_0}{4k(\omega - k\bar{u})}}\ (kx + ly - \omega t), \tag{2.40}$$

which is called Rossby solitary waves or Rossby soliton, as shown in Fig.1.

From (2.38) it can be seen that the wavelength of Rossby cnoidal waves is

$$L = \frac{2}{k}\sqrt{\frac{6k(\omega - k\bar{u})^2}{\beta_0 l(V_1 - V_3)}}\,K(m), \tag{2.41}$$

where

$$K(m) = \int_0^{\frac{\pi}{2}} \frac{1}{\sqrt{1 - m^2 sin^2 t}}\,dt \tag{2.42}$$

is the complete elliptic integral of the first kind. The modulus m satisfies

$$m^2 = \frac{V_1 - V_2}{V_1 - V_3}\ . \tag{2.43}$$

Formula (2.41) is a relation of the wave speed of the Rossby cnoidal waves. In addition, on the basis of the relation between the roots and coefficient in the cubic equation $P(V) = 0$

$$V_1 + V_2 + V_3 = \frac{3}{2}\frac{\omega - k\bar{u}}{l} < 0, \tag{2.44}$$

we then obtain another important relation of the wave speed

$$\omega - k\bar{u} = \frac{2l}{3}\left(V_1 + V_2 + V_3\right) \tag{2.45}$$

or

$$c \equiv \frac{\omega}{k} = \bar{u} + \frac{2l}{3k}\left(V_1 + V_2 + V_3\right). \tag{2.46}$$

According to (2.41) and (2.46) the wave speed can be determined. Taking $L = \frac{2\pi}{k}$ the wave speed is

$$c = \bar{u} + \frac{\beta_0}{k^2}\frac{\pi^2}{4K(m)}\left(\frac{V_1 - V_3}{V_1 + V_2 + V_3}\right) \tag{2.47}$$

When $m \to 0$, $K(m) \to \frac{\pi}{2}$ and $V_1 \to V_2$, the amplitude $V_1 - V_2 \to 0$. If taking $V_1 > 0$, $V_2 < 0$, certainly $V_1 \to V_2 \to 0$ and $V_3 \to \frac{3}{2}\frac{\omega - k\bar{u}}{l}$, so that (2.47) is reduced to

$$c = \bar{u} - \frac{\beta_0}{k^2}, \tag{2.48}$$

which is the Rossby formula.

It shows that the Rossby cnoidal waves reduce to the linear Rossby waves in case of infinitesimal amplitude.

3. *Nonlinear Rossby Waves with Semi-Geostrophic Flow*

Why $\frac{\partial u}{\partial y}$ and $v\frac{\partial}{\partial y}(\frac{\partial v}{\partial x})$ in (2.27a) are omitted? Why we can not solve the quasi–geostrophic potential vorticity equation (2.6) by means of (2.28)? This is due to that when $\frac{\partial u}{\partial y}$ and $v\frac{\partial}{\partial y}(\frac{\partial v}{\partial x})$ in (2.27a) are considered or (2.28) is substituted into (2.6) the nonlinear effect will disappear. That is to say, the horizontal nondivergent or quasi–geostrophic approximations can not reflect the nonlinear effect.

In order to overcome this shortcoming and solve the nolinear Rossby waves we apply the semi–geostrophic assumption that distinguishes between the advecting and advected quantities and consider that the advected quantity is geostrophic but the advecting one is non–geostrophic.

With the semi–geostrophic assumption the equations in the barotropic β–plane model (Liu Shikuo and Liu Shida, 1988) can be written as

$$\left(\frac{\partial}{\partial t}+u\frac{\partial}{\partial x}+v\frac{\partial}{\partial y}\right)\nabla_h^2+\beta_0\frac{\partial\varphi}{\partial x}+f_0^2\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)=0 \quad (2.49a)$$

$$\left(\frac{\partial}{\partial t}+u\frac{\partial}{\partial x}+v\frac{\partial}{\partial y}\right)\varphi+\left(c_0^2+\varphi\right)\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)=0 \quad (2.49b)$$

$$f_0\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right)=\nabla_n^2\varphi, \quad (2.49c)$$

where f_0 is the Coriolis parameter and taken as a constant, and

$$c_0^2=gH, \qquad \varphi=gh. \quad (2.50)$$

In (2.50) H and h are the undisturbed scale height and the departure from H respectively, g is the gravitational acceleration.

Assuming that the solutions to (2.49) are

$$u=(\theta), \quad v=V(\theta), \quad \varphi=\Phi(\theta), \quad \theta=kx+ly-\omega t \quad (2.51)$$

and substituting (2.51) into (2.49), we get

$$K_h^2(kU+lV-\omega)\Phi'''+\beta_0 k\Phi'+f_0^2(kU+lV)=0 \quad (2.52a)$$

$$-\omega\Phi'+\Big[\quad(kU+lV)\Phi\Big]'+c_0^2(kU+lV)'=0 \quad (2.52b)$$

$$f_0(kU-lV)'=-K_h^2\Phi'' , \quad (2.52c)$$

where the primes denote the derivatives with respect to θ and

$$K_h^2=k^2+l^2 \quad (2.53)$$

Integrating (2.52b) and (2.52c) with respect to θ and taking the integral constant to be zero we have

$$kU+lV=\frac{\omega\Phi}{c_0^2+\Phi} , \quad (2.54a)$$

$$kV-lU=-\frac{K_h^2\Phi'}{f_0} . \quad (2.54b)$$

Substituting (2.54a) into (2.52a) yields

$$-\omega K_h^2\left(1-\frac{\Phi}{c_0^2+\Phi}\right)\Phi'''+\left[\beta_0 k+\lambda_0^2\omega\left(1+\frac{\Phi}{c_0^2}\right)^{-2}\right]\Phi'=0, \quad (2.55)$$

which is an ordinary differential equation of Φ. From this we can find out Φ and then it is substituted into (2.54), U and V can be ditermined.

The equation (2.55) shows that the semi–geostrophic and barotropic vorticity equation can reflect the nonlinear effect and overcome the shortcoming of the quasi–geostrophic model.

By setting

$$\Phi' \equiv I, \tag{2.56}$$

the equation (2.55) then can be rewritten as

$$I'' + Q(\Phi, I) = 0, \tag{2.57}$$

where

$$Q(\Phi, I) = \frac{\beta_0 k \left(1 + \frac{\Phi}{c_0^2}\right)^2 + \lambda_0^2 \omega}{-K_n^2 \omega \left(1 + \frac{\Phi}{c_0^2}\right)} I = 0 . \tag{2.58}$$

This is essentially an eigenvalue problem of the nonlinear equation and can be solved approximately by Taylor expansion method.

Expanding $Q(\Phi, I)$ in Taylor series in the vicinity of the point $(\Phi, I) = (0,0)$, we have

$$Q(\Phi, I) = \frac{\beta_0 k + \lambda_0^2 \omega}{-\omega K_h^2} I + \frac{\beta_0 k - \lambda_0^2 \omega}{-\omega K_h^2 c_0^2} \Phi I + \cdots \tag{2.59}$$

If we take the right hand side of (2.59) until the second term, the equation (2.57) is then reduced to

$$I'' + \frac{\beta_0 k + \lambda_0^2 \omega}{-\omega K_h^2} I + \frac{\beta_0 k - \lambda_0^2 \omega}{-\omega K_h^2 c_0^2} \Phi I = 0 \tag{2.60}$$

or

$$\Phi''' + \frac{\beta_0 k - \lambda_0^2 \omega}{-\omega K_h^2 c_0^2} \Phi \Phi' + \frac{\beta_0 k + \lambda_0^2 \omega}{-\omega K_h^2} \Phi' = 0, \tag{2.61}$$

which is the ordinary differential equation corresponding to the KdV equation, it is formally the same as (2.37).

It is shown that the semi–geostrophic approximation not only filters out the inertio–gravity waves as the quasi–geostrophic approximation, but also finds out the solution to the nonlinear equations. The Rossby waves obtained by the semi–geostrophic approximation are also governed by the KdV equation.

Similar to (2.38) the solution to equation (2.61) is easily given by

$$\varphi(x,y,t) = \Phi_2 + (\Phi_1 - \Phi_2) cn^2 \sqrt{\frac{\beta_0 k - \lambda_0^2 \omega}{12 \omega K_h^2 c_0^2} (\Phi_3 - \Phi_1)} (kx + ly - \omega t) \tag{2.62}$$

and the angular frequency of the Rossby waves with semi–geostrophic flow is

$$\omega = - \frac{\beta_0 k \left(\frac{\pi}{2K(m)}\right)^2 \left(\frac{\Phi_3 - \Phi_1}{\Phi_1 + \Phi_2 + \Phi_3}\right)}{K_h^2 + \lambda_0^2 \left(\frac{\pi}{2K(m)}\right)^2 \left(\frac{\Phi_3 - \Phi_1}{\Phi_1 + \Phi_2 + \Phi_3}\right)}, \tag{2.63}$$

where Φ_1, Φ_2 and Φ_3 are three real zeros of the cubic polynomial

$$P(\Phi) \equiv \Phi^3 + \frac{3(\beta_0 k + \lambda_0^2 \omega)}{\beta_0 k - \lambda_0^2 \omega} c_0^2 \Phi^2 + B. \tag{2.64}$$

The dispersion relation (2.63) includes both the wavenumber and the wave parameter concerning the amplitude and when $m \to 0$ it is degenerated into

$$\omega = - \frac{\beta_0 k}{K_h^2 + \lambda_0^2}, \tag{2.65}$$

which is the angular frequency of the linear Rossby waves with quasi–geostrophic flow.

In analogy to the barotropic model we can also discuss the baroclinic Rossby wave with semi–geostrophic flow (see Liu Shikuo and Liu Shida, 1988).

III. NONLINEAR INERTIO–GRAVITY WAVES

1. *Genenal Approach, KdV Equation*

The dispersion relation of linear internal inertio–gravity waves is

$$\omega^2 = \frac{k^2N^2 + n^2f^2}{k^2 + n^2}, \tag{3.1}$$

where f is the Coriolis parameter, N is the Brunt–Vaisala frequency, k and n are the horizontal and vertical wavenumber respectively.

Considering only waves propagating to the right one obtains

$$\omega = \sqrt{\frac{k^2N^2 + n^2f^2}{k^2 + n^2}} = \sqrt{\frac{k^2N^2(1 + \frac{n^2f^2}{k^2N^2})}{n^2(1 + \frac{k^2}{n^2})}} . \tag{3.2}$$

If $f^2 \ll N^2$, $k^2 \ll n^2$ and $\frac{n^2f^2}{k^2N^2} \ll 1$ (3.2) then can be reduced to

$$\omega - kc = -\gamma k^3 + \frac{\alpha}{k}, \tag{3.3}$$

where

$$c = \frac{N}{n}, \qquad \gamma = \frac{N}{2n^3}, \qquad \alpha = \frac{nf^2}{2N}. \tag{3.4}$$

(3.3) is slightly different from (2.3). However, if the last term on the right hand side is negligible, then (3.3) is in the same from as (2.3). Therefore the internal gravity waves for $k^2 \ll n^2$ satisfy the KdV equation.

Long(1965), Benjamin(1966), Maslowe and Redekopp (1980), Ablowitz and Segur (1980), Ono (1975) et al. all pointed out that the internal gravity waves in stratified shear flow are governed by the KdV equation (2.5) under an appropriate condition. For instance, we consider two–dimensional motion of a Boussinesq fluid for which the equations are

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + w\frac{\partial u}{\partial z} = -\frac{1}{\rho_0}\frac{\partial p}{\partial x} \tag{3.5a}$$

$$\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + w\frac{\partial w}{\partial z} = -\frac{1}{\rho_0}\frac{\partial p}{\partial x} - g\frac{\rho}{\rho_0} \tag{3.5b}$$

$$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0 \tag{3.5c}$$

$$\frac{\partial \rho}{\partial t} + u\frac{\partial \rho}{\partial x} + w\frac{\partial \rho}{\partial z} - \frac{N^2}{g}\rho_0 w = 0, \tag{3.5d}$$

where ρ_0 is the undisturbed fluid density, ρ and p are the disturbed density and pressure, respectively.

Setting that

$$u = \bar{u}(z) + u', \qquad w = w', \qquad u' = \frac{\partial \psi}{\partial z}, \qquad w' = -\frac{\partial \psi}{\partial x}, \tag{3.6}$$

and

$$\sigma = \frac{g\rho}{\rho_0}, \tag{3.7}$$

the equations (3.5) can be written in the form

$$(\frac{\partial}{\partial t} + \bar{u}\frac{\partial}{\partial x})\nabla^2\psi + J(\nabla^2\psi, \psi) = \frac{\partial \sigma}{\partial x} \tag{3.8a}$$

$$(\frac{\partial}{\partial t}+\bar{u}\frac{\partial}{\partial x})\sigma+J(\sigma,\psi)+N^2\frac{\partial\psi}{\partial x}=0, \tag{3.8b}$$

where

$$\nabla^2=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial z^2} \tag{3.9}$$

and

$$J(A,B)=\frac{\partial A}{\partial x}\frac{\partial B}{\partial z}-\frac{\partial A}{\partial z}\frac{\partial B}{\partial x} \tag{3.10}$$

Applying the transformation

$$\xi=\epsilon^{\frac{1}{2}}(x-ct),\quad \tau=\epsilon^{\frac{3}{2}}t,\quad z=z \tag{3.11}$$

and the boundary condition

$$\psi|_{z=z_1}=0\ ,\qquad \psi|_{z=z_2}=0 \tag{3.12}$$

and seeking a solution in terms of the regular expansion

$$\psi=\epsilon\psi_1+\epsilon^2\psi_2+\cdots \tag{3.13a}$$

$$\sigma=\epsilon\sigma_1+\epsilon^2\sigma_2+\cdots \tag{3.13b}$$

and admitting the separable solution

$$\psi_1=A(\xi,\tau)G(z), \tag{3.14}$$

we then obtain the following KdV equation

$$\frac{\partial A}{\partial\tau}+RA\frac{\partial A}{\partial\xi}+S\frac{\partial^3 A}{\partial^3\xi}=0\ , \tag{3.15}$$

where

$$R=\frac{\int_{z_1}^{z_2}\frac{G^3}{(\bar{u}-c)^4}\left[(\bar{u}-c)^2\frac{\partial^3\bar{u}}{\partial z^3}-(\bar{u}-c)\frac{\partial\bar{u}}{\partial z}\frac{\partial^2\bar{u}}{\partial z^2}-2(\bar{u}-c)\frac{\partial N^2}{\partial z^2}+3N^2\frac{\partial\bar{u}}{\partial z}\right]dz}{\int_{z_1}^{z_2}\frac{G^2}{(\bar{u}-c)^3}\left[2N^2-(\bar{u}-c)\frac{\partial^2\bar{u}}{\partial z^2}\right]dz} \tag{3.16}$$

$$S=-\frac{\int_{z_1}^{z_2}G^2\,dz}{\int_{z_1}^{z_2}\frac{G^2}{(\bar{u}-c)^3}\left[2N^2-(\bar{u}-c)\frac{\partial^2\bar{u}}{\partial z^2}\right]dz}\ . \tag{3.17}$$

As the method for obtaining such solutions is essentially the same as for the Rossby waves we omit its description here.

Notice that the coefficient R vanishes if there is no shear and if N is a constant. It implies that for such environments the KdV equation emerges only if the non−Boussinesq terms are contained.

2. *Nonlinear Internal Gravity Waves*

Similar to the Rossby waves, by means of the nonlinear term expansion we can also discuss the internal gravity waves and other waves (cf Liu Shida and Liu Shikuo, 1982, 1983).

Disregarding the convective terms in (3.5) and assuming that

$$u=U(\theta),\quad w=W(\theta),\quad p=P(\theta),\quad \frac{\rho}{\rho_0}=\Pi(\theta),\quad \theta=kx+nz-\omega t \tag{3.18}$$

we get

$$(-\omega+kU)U'=-\frac{1}{\rho_0}kP' \tag{3.19a}$$

$$(-\omega+kU)W' = -\frac{1}{\rho_0}nP' - g\Pi \tag{3.19b}$$

$$kU' + nW' = 0 \tag{3.19c}$$

$$(-\omega+kU)\Pi' - \frac{N^2}{g}W = 0, \tag{3.19d}$$

where the symbol prime represents the derivative with respect to θ. Integrating (3.19c) and taking the integral constant as zero we get

$$W = -\frac{k}{n}U \ . \tag{3.20}$$

Eliminating P from (3.19a) and (3.19b) we have

$$(-\omega+kU)(nU' - kW') = kg\Pi. \tag{3.21}$$

Substituting (3.20) into (3.21) and (3.19d) and assuming that $\omega - kU \neq 0$, we then obtain

$$U' = \frac{kng}{(k^2+n^2)(-\omega+kU)}\Pi \equiv F(U,\Pi) \tag{3.22a}$$

$$\Pi' = -\frac{kN^2}{gn(-\omega+kU)}U \equiv G(U), \tag{3.22b}$$

where F and G are the nonlinear functions.
The equilibrium point, which makes U' and Π' vanish, is $(U,\Pi)=(0,0)$. Expanding F and G in Taylor series near the equilibrium point we obtain

$$F(U,\Pi) = -\frac{kng}{(k^2+n^2)\omega}\Pi - \frac{k^2ng}{(k^2+n^2)\omega^2}\Pi U + \cdots \tag{3.23a}$$

$$G(U) = \frac{kN^2}{gn\omega}U + \frac{k^2N^2}{gn\omega^2}U^2 + \cdots \tag{3.23b}$$

If we take F and G until the second terms (3.23) then becomes

$$U' = -\frac{kng}{(k^2+n^2)\omega}\Pi - \frac{k^2ng}{(k^2+n^2)\omega^2}\Pi U + \cdots \tag{3.24a}$$

$$\Pi' = \frac{kN^2}{gn\omega}U + \frac{k^2N^2}{gn\omega^2}U^2 \tag{3.24b}$$

Eliminating Π from (3.24a) and (3.24b) and neglecting the third terms of Π and U we get

$$U'' = -\frac{3k^2N^2}{(k^2+n^2)\omega^3}U - \frac{k^2N^2}{(k^2+n^2)\omega^2}U + \frac{g^2k^3n^2}{(k^2+n^2)\omega^3}A, \tag{3.25}$$

where A is an integral constant.

Derivating (3.25) with respect to θ we obtain the following equation which is the corresponding ordinary differential equation of KdV equation

$$U''' + \frac{6k^3N^2}{(k^2+n^2)\omega^3}UU' + \frac{k^2N^2}{(k^2+n^2)\omega^2}U' = 0. \tag{3.26}$$

Therefore the solutions of cnoidal waves and solitary waves for the internal gravity waves are

$$u(x,z,t) = U_2 + (U_1 - U_2)\mathrm{cn}^2\sqrt{\frac{k^3N^2}{2(k^2+n^2)\omega^3}(U_1-U_3)}\,(kx+nz-\omega t) \tag{3.27}$$

$$u(x,z,t) = -\frac{\omega}{2k} + \frac{\omega}{k}sech^2\frac{kN}{\sqrt{2\omega^2(k^2+n^2)}}(kx+nz-\omega t), \tag{3.28}$$

where U_1, U_2, U_3 are three real zeros of the cubic polynomial

$$P(U) = U^3 + \frac{\omega}{2k}U^2 - \frac{g^2n^2}{(k^2+n^2)N^2}AU + B. \tag{3.29}$$

(3.28) denotes that the amplitude is in direct proportion to the wave speed.

Similarly, the equations of nonlinear inertia waves may be written as

$$U' = \frac{f}{KU - \omega} V \tag{3.30a}$$

$$V' = -\frac{f}{KU - \omega} U, \tag{3.30b}$$

where the symbol prime denotes the derivatives with respect to the phase function θ.

IV. COMMON CHARACTER OF NONLINEAR WAVES

It is clear that (2.33), (3.22) and (3.30) can be transformed into the plane autonomous systems as follows:

$$\dot{X} = F(X, Y) \tag{4.1a}$$

$$\dot{Y} = G(X, Y), \tag{4.1b}$$

where X and Y are the two physical quantities, F and G nonlinear functions of X and Y, and $\dot{X}$ and $\dot{Y}$ the derivatives with respect to θ.

On the basis of our analysis (1987) we conclude that the common characters of the nonlinear atmospheric waves are:

(1) The family of the orbit on the phase plane (X, Y) is closed and the orbit is a circle or ellipse. Precisely because of this, it implies that X and Y are periodic functions and represent the nonlinear waves. The origin $(X, Y) = (0,0)$ is called the centre point.

(2) The nonlinear functions F and G satisfy

$$F(X, -Y) = -F(X, Y) \tag{4.2a}$$

$$G(X, -Y) = G(X, Y), \tag{4.2b}$$

which are known as the symmetric theorem and tubular centre theorem. The two theorems state that the origin (0,0) is a centre point and the motion around the origin is periodical.

(3) The functions X and Y vary periodically with θ, and the periodical solutions have a general form.

Because F and G are in the same form for different waves and the orbits are also similar, therefore the ways in which the waves vary with θ are also similar.

(4) Near by equilibrium point (0,0), the periodical solution is the cnoidal waves. The limited case of the cnoidal waves is the solitary waves.

V. MODON THEORY

A modon differs from a soliton, it is usually an exact analytic solution of the quasi–geostrophic potential vorticity equation and it is an isolated two–dimensional eddies. But the potential vorticity has a discontinuous derivative at some order.

we consider the barotropic quasi–geostrophic potential vorticity equation for the steady motion

$$J(\psi, q) = 0, \tag{5.1}$$

where $J(A, B)$ is the Jacobi operator, while

$$q = \nabla_h^2 - \lambda_0^2 \psi + f \tag{5.2}$$

is the quasi–geostrophic potential vorticity. In β–plane approximation f can be written as

$$f = f_0 + \beta_0 y. \tag{5.3}$$

(5.1) implies that q must be constant on a streamline. Thus q must be a function of ψ i.e.

$$q = F(\psi), \tag{5.4}$$

where F( ) is an arbitrarily chosen function.

Substituting (5.2) into (5.4) we have

$$\nabla_h^2 - \lambda_0^2\psi + \beta_0 y = G(\psi), \tag{5.5}$$

where

$$G(\psi) \equiv F(\psi) - f_0 \tag{5.6}$$

is also an arbitrary function.

If we choose $G(\psi)$ as a linear function of ψ, i.e.

$$G(\psi) = -\lambda_1^2\psi, \tag{5.7}$$

the equation (5.5) then reduces to

$$\nabla_n^2\psi + \lambda^2\psi = -\beta_0 y, \tag{5.8}$$

where

$$\lambda^2 = \lambda_1^2 - \lambda_0^2. \tag{5.9}$$

Thus the problem may be reduced to the solution of the linear inhomogeneous Helmholtz equation (5.8).

In polar coordinates (r, θ), we apply the following boundary condition on $r = a$

$$\psi|_{r=a} = 0, \qquad \frac{\partial^2\psi}{\partial r\partial\theta}|_{r=a} = 0, \tag{5.10}$$

where the former denotes the condition of free streamline, the latter implies that $\vec{v}^2 = (\nabla\psi)^2$ is uniform on $r = a$.

The general solution of (5.8) is

$$\psi = -\frac{\beta_0}{\lambda^2} r\sin\theta + \sum_{m=0}^{\infty} J_m(\lambda r)(A_m r\cos\theta + B_m \sin m\theta), \qquad (r \leqslant a) \tag{5.11}$$

where J_m is the Bessel function of the $m'th$ order, A_m and B_m are the arbitrary (at present) constants.

The first term on the right hand side of (5.11) is a particular solution corresponding to the inhomogeneous term, $-\frac{\beta_0}{\lambda^2} r\sin\theta$, the second term is the general solution of the homogeneous equation of (5.8).

By means of the boundary condition in (5.10) we then obtain readily

$$B_1 = \frac{\beta_0 a}{\lambda^2 J_1(\lambda a)}, \quad B_0 = B_2 = \cdots = 0, \qquad A_0 = A_1 = B_2 = \cdots = 0 \tag{5.12}$$

and

$$J_2(\lambda a) = 0 \tag{5.13}$$

Assuming that μ_n (n=1,2,...) is the roots of (5.13), we have

$$\lambda a = \mu_n \tag{5.14}$$

Consequently, the solution of the equation (5.8) satisfying the boundary condition (5.11) is

$$\psi = -\frac{\beta_0 a\sin\theta}{\mu_n^2}\left[r - a\frac{J_1(\frac{\mu_n r}{a})}{J_1(\mu_n)}\right] \qquad (r \leqslant a), \tag{5.15}$$

while the relative vorticity is obtained from (5.8) and (5.12), and for any one of the root of (5.13) we find

$$\zeta \equiv \nabla_h^2\psi = -\beta_0 a\sin\theta \frac{J_1(\frac{\mu_n r}{a})}{J_1(\mu_n)} \ . \tag{5.16}$$

The streamline diagram of the modon obtained from (5.15) is illustrated in Fig.2.

Similarly, the modon solution can also be obtained in plane Cartesian and spherical coordinates (see Zeng, 1979).

When the nonsteady term is considered the quasi–geostrophic potential vorticity equation is given by

$$\frac{\partial q}{\partial t}+u\frac{\partial q}{\partial x}+v\frac{\partial q}{\partial y}=0. \tag{5.17}$$

Introducing the transformation

$$\xi=x-ct, \qquad y=y, \tag{5.18}$$

the equation (5.17) reduces to

$$J(\psi+cy,q)=0, \tag{5.19}$$

which is the same as (5.1) in form, only in a reference frame moving zonally with speed c.

Notice that the modon solution can also be found in exterior region $(r>a)$ and it has patterns of all kinds.

VI. STABILITY ANALYSIS OF NONLINEAR WAVES

The authors (1983, 1984, 1985) discussed the stability of the nonlinear internal gravity waves, internal inertio–gravity waves and Rossby waves and analyzed their topological structure.

1. *Stability of Nonlinear Internal Gravity Waves*

Let us consider two–dimensional (x,z) motion of Boussinesq fluid with stratification $\bar{\rho}(z)$ and shear $\bar{u}(z)$, then the nonlinear equation of stratified shear flow can be written as

$$\frac{\partial u}{\partial t}+(\bar{u}+u')\frac{\partial u}{\partial x}+w\frac{\partial \bar{u}}{\partial z}=-\frac{1}{\bar{\rho}}\frac{\partial p}{\partial x} \tag{6.1a}$$

$$\frac{\partial w}{\partial t}+(\bar{u}+u')\frac{\partial w}{\partial x}=-\frac{1}{\bar{\rho}}\frac{\partial p}{\partial z}-g\frac{\rho}{\bar{\rho}} \tag{6.1b}$$

$$\frac{\partial u}{\partial x}+\frac{\partial w}{\partial z} \tag{6.1c}$$

$$\frac{\partial}{\partial t}\left(\frac{\rho}{\bar{\rho}}\right)+(\bar{u}+u')\frac{\partial}{\partial x}\left(\frac{\rho}{\bar{\rho}}\right)-\frac{N^2}{g}w=0, \tag{6.1d}$$

where u and w are the components of disturbed velocity in x and z directions respectively, p and ρ are the disturbed pressure and density respectively, N is the Brunt–Vaisala frequency.

Setting that

$$u=U(\theta),\ w=W(\theta),\ p=P(\theta),\ \frac{\rho}{\bar{\rho}}=\Pi(\theta),\quad \theta=kx+nz-\omega t \tag{6.2}$$

and substituting (6.2) into (6.1) then yields

$$(-\omega+kU+k\bar{u})U'+\frac{\partial \bar{u}}{\partial z}W=-\frac{1}{\bar{\rho}}kP' \tag{6.3a}$$

$$(-\omega+kU+k\bar{u})W'=-\frac{1}{\bar{\rho}}nP'-g\Pi \tag{6.3b}$$

$$kU'+nW'=0 \tag{6.3c}$$

$$(-\omega+kU+k\bar{u})\Pi'-\frac{N^2}{g}W=0 \tag{6.3d}$$

where the prime denotes the derivative with respect to θ.

Assuming $\bar{u}(z)$, $\frac{\partial \bar{u}}{\partial z}$ and $N^2(z)$ to be the slowly varying variables and eliminating W and P from (6.1), we then obtain the following autonomous dynamic systems

$$U' = \frac{kng\Pi + kn\frac{\partial \bar{u}}{\partial z}U}{(k^2+n^2)(-\omega + k\bar{u} + kU)} \equiv F(U,\Pi) \tag{6.4a}$$

$$\Pi' = -\frac{kN^2 U}{gn(-\omega + k\bar{u} + kU)} \equiv G(U,\Pi), \tag{6.4b}$$

where both F and G are nonlinear functions.

The equilibrium point which makes U' and Π vanish is $(U,\Pi) = (0,0)$. Clearly, it represents the undisturbed state. From (6.4) we can see that the phase path equation near the equilibrium point on phase plane (U,Π) is

$$\frac{dU}{d\Pi} = -\frac{n\frac{\partial \bar{u}}{\partial z} + ng\Pi}{\frac{N^2}{gn}(k^2+n^2)U} . \tag{6.5}$$

Expanding F and G in Taylor series near the equilibrium, (6.4) then can be written as

$$U' = \frac{kn\frac{\partial \bar{u}}{\partial z}}{(k^2+n^2)(-\omega + k\bar{u})}U + \frac{kng}{(k^2+n^2)(-\omega + k\bar{u})}\Pi - \frac{k^2 n\frac{\partial \bar{u}}{\partial z}}{(k^2+n^2)(-\omega + k\bar{u})^2}U^2 - \frac{kng}{(k^2+n^2)(-\omega + k\bar{u})}\Pi U + \cdots \tag{6.6a}$$

$$\Pi' = -\frac{kN^2}{gn(-\omega + k\bar{u})}U + \frac{k^2N^2}{gn(-\omega + k\bar{u})^2}U^2 + \cdots \tag{6.6b}$$

only the linear part on the right hand side of (6.6) is taken, it reduces to

$$U' = \frac{kn\frac{\partial \bar{u}}{\partial z}}{(k^2+n^2)(-\omega + k\bar{u})}U + \frac{kng}{(k^2+n^2)(-\omega + k\bar{u})}\Pi \tag{6.7a}$$

$$\Pi' = -\frac{kN^2}{gn(-\omega + k\bar{u})}U , \tag{6.7b}$$

which is referred to as the linear internal gravity waves. The phase path equation obtained from (6.7) is also (6.5). It implies that the phase path of nonlinear internal gravity waves is the same as that of linear internal gravity waves.

Generally $|\omega| \gg k\bar{u} > 0$, the characteristic equation of (6.7) is given by

$$\begin{vmatrix} \frac{kn\frac{\partial \bar{u}}{\partial z}}{(k^2+n^2)(-\omega)} - \lambda & \frac{kng}{(k^2+n^2)(-\omega)} \\ \frac{kN^2}{gn\omega} & 0 - \lambda \end{vmatrix} = 0, \tag{6.8}$$

namely

$$\lambda^2 + A\lambda + B = 0, \tag{6.9}$$

where

$$A \equiv \frac{kn\frac{\partial \bar{u}}{\partial z}}{\omega(k^2+n^2)}, \quad B \equiv \frac{k^2N^2}{\omega^2(k^2+n^2)} . \tag{6.10}$$

The characteristic roots of (6.9) are

$$\lambda=\frac{A}{2}\{-1\pm\sqrt{1-4Ri}\},\tag{6.11}$$

where

$$Ri\equiv N^2/(\frac{\partial\bar{u}}{\partial z})^2\tag{6.12}$$

is the Richardson number.

On the parameter plane $(\frac{\partial\bar{u}}{\partial z},\ N^2)$ the integral curves in the neighbourhood of the equilibrium point divide the plane into some domains in which the stability is very different from each other. It is illustrated in Fig.3, in which "." represents the stable equilibrium point, "o" the unstable equilibrium point. That associated with each path is the direction, indicated by an arrow in the figure. It shows how the state of the system changes with the increasing time. The dashed line represents $Ri=1/4$.

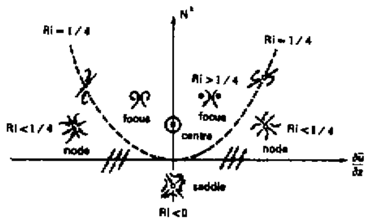

Fig.3. Stability of nonlinear internal gravity waves.

According to Poincare–Bendixson's theory, the qualitative property of the second–order systems near the origin is identical with the first–order systems of equations (6.6). We have the following conclusions:

the flow is stable for $\frac{\partial\bar{u}}{\partial z}<0,\quad N^2>0$

the flow is unstable for $\frac{\partial\bar{u}}{\partial z}<0,\quad N^2<0$

the flow is unstable for $\frac{\partial\bar{u}}{\partial z}>0$

2. *Stability of Nonlinear Internal Inertio–Gravity Waves*

Similarly, we can discuss the stability of nonlinear inertia–internal gravity waves, the conclusion is as follows:

the flow is stable for $\frac{\partial\bar{u}}{\partial z}<0,\quad L_0^2<0\ (N^2>0)$ or $L\geqslant L_0$

the flow is unstable for $\frac{\partial\bar{u}}{\partial z}$ arbitrary and $L<L_0$

the flow is unstable for $\frac{\partial\bar{u}}{\partial z}>0,\quad L_0^2<0\ (N^2>0)$ or $L\geqslant L_0$,

where L is the horizontal scale, and L_0 is the baroclinic Rossby deformation radius, it satisfies

$$L_0^2 \equiv -\frac{N^2H^2}{f_0^2} \quad . \tag{6.13}$$

3. *Stability of nonlinear Rossby Waves*

For the nonlinear baroclinic Rossby waves in two–layer model, the conclusion of stability is that

the flow is stable for $p>0$, and $q>0$
the flow is unstable for $p<0$, and $q>0$
the flow is unstable for $q<0$,

where

$$p \equiv -\frac{2\mu^2(c-c_1)(c-c_2)}{k^2(\bar{u}_1-c)(\bar{u}_3-c)}, \qquad q \equiv \frac{2\beta_0\mu^2(c-c_0)}{k^4(\bar{u}_1-c)(\bar{u}_3-c)} \tag{6.14}$$

while μ^{-1} is the baroclinic Rossby radius of deformation and

$$c_0 = \bar{u}^2 - \frac{\beta_0}{2\mu^2}, \qquad c_1 = c_0 + \sqrt{(\frac{\beta_0}{2\mu^2})^2 - \bar{u}_s^2}, \qquad c_2 = c_0 - \sqrt{(\frac{\beta_0}{2\mu^2})^2 - \bar{u}_s^2}, \tag{6.15}$$

$$\bar{u}_2 = \frac{1}{2}(\bar{u}_1 + \bar{u}_3) \qquad \bar{u}_s = \frac{1}{2}(\bar{u}_1 + \bar{u}_3) \ .$$

VII. CONCLUSION REMARKS

From the developmental stages of the studies on the soliton and modon we may see that on the one hand the researches of atmospheric waves have stepped into the nonlinear stages, on the other hand the researches of nonlinear waves are gradually deepening, from the single solitary wave to the modon eddies. And a close packed array of non–overlapping modons, which is called a modon–sea, can also be obtained. The soliton and modon may explain not only the blocking systems in the atmosphere, but also the oceanic eddies (e.g., blocking events or Gulf stream rings).

A number of further studies are needed to advance the assessment of blocks and modons. The studies on the stability of soliton and modon and robustness of these solution are underway. However, as Flierl et al. (1980) have remarked, it appears that the basic solitary eddy solutions, unlike plane Rossby waves, are stable and moderately robust. This leads further credence to the idea that these solitary wave solutions may be important in the evolution of isolated disturbances in the atmosphere or the ocean.

REFERENCES

Ablotz, M.J. and Segur, H.(1980), Long internal waves in fluids of great depth, stud. *Apppl. Math.* **62**: 249–262.

Benjmin, T.B. (1966), Internal waves of finite amplitude and permanent form, *J.Fluid Mech,* **25**: 241–270.

Benny, D.J. (1966), Long nonlinear waves in fluid flows, *J.Math. and Phys.,* **45**: 52–63.

Boyd, J.P. (1977), Equatorial solitary waves, Part I: Rossby solitons, *J. Phys. Oceanogr.* **10**: 1699–1717.

Chao, J.P. and Huang, R.X. (1980), Cnoidal waves in rotating barotropic atmosphere, *Scientia Sinica,* **7**: 696–705 (in Chinese).

Clake, R.A. (1971), Solitary and cnoidal planetary waves, *Geophys. Fluid Dyn.* **2**: 343–354.

Flierl, G.R., Larichev, V.D., Mcwilliams, J.C., Reznik,G.M. (1980), The dynamics of baroclinic and barotropic solitary eddies, *Dyn. Atmos. and Oceans,* **5**: 1–41.

Grimshaw,R. (1977), Nonlinear aspects of long shelf waves, *Geophys. and Astrophys., Fluid Dyn.,* **8**: 3–16.

Larichev, V.D. and Reznik, G. (1976a), Strongly nonlinear two–dimensional isolated Rossby waves, *Oceanologia.*

16: 547–550.

Larichev, V.D. and Reznik, G.(1976b), Two–dimensional Rossby soliton: an exact solution, Rep. U.S.S.R. Acad. Sci., 231(5), Also Polymode News, 19.

Liu S.D. and Liu S.K. (1982), Analytic solutions of nonlinear wave equations in the atmosphere, *Acta Meteorologica Sinica*, **40**: 279–288(in Chinese).

Liu S.D. and Liu S.K. (1983), The nonlinear internal gravity waves in stratified Fluid, *Acta Oceanologica Sinica*, **2**: 45–51.

Liu S.D. and Liu S.K.(1982), Nonlinear cnoidal waves and solitary waves in atmosphere, *Scientia Sinica*(B), **25**: 1103–1117.

Liu S.D. and Liu S.K. (1983), The classification of a family of stratified shear flow and some relevant nonlinear solution, *Proceedings of the second Asian congress of fluid Mech.*, 945–952.

Liu S.K. Liu S. D., Bu Y.K. (1984), The stability of nonlinear inertio–internal gravity waves for stratified shear flow. *Acta Meteorologica Sinica*, **42**: 24–34. (in Chinese).

Liu S.D. and Liu S.K. (1987), The common character of nonlinear waves in the atmosphere, *Kexue Tongbao*, **32**: 325–329.

Liu S.K. and Liu S.D. (1985), Linear and nonlinear stability problems for the baroclinic Rossby waves, *Scientia Sinica*(B), **28**: 1311–1320.

Liu S.K. and Liu S.D. (1983), Nonlinear waves in geophysical fluid, *Scientia Sinica*(B), **26**: 1176–1189.

Liu S.K. and Liu S.D. (1985), Nonlinear waves in barotropic model, *Advances in Atoms. Sci.* **2**: 147–157.

Liu S.K. and Liu S.D. (1988), Nonlinear waves with semi–geostrophic flow, *Acta Meteorologica*, **2**: 517–526 (in Chinese).

Long. R.R. (1964), Solitary waves in the westerlies, *J. Atmos. Sci.* ,**21**: 197–200.

Long, R.R. (1965), On the Boussinesq approximation and its role in the theory of internal waves, *Tellus*, **17**: 46–52.

Maslowe, S.A. and Redekopp, L.G. (1980), Long Nonlinear waves in stratified shear flows, *J. Fluid Mech.*, **101**: 321–348.

Mcwilliams, J.C. (1980), An application of equivalent modons to atmospheric blocking, *Dyn. Atmos. and Oceans*, **5**: 43–66.

Ono, H. (1975), Algebraic solitary waves in stratified fluid, *J. Phys. Soc. Japan*, **39**: 1082–1097.

Qin Yuanxun (1959), *Definite integral curves of the differential equations*,Science Press, Beijing(in Chinese).187pp.

Redekopp,L.G. (1977), On the theory of solitary Rossby waves, *J. Fluid Mech.*, **82**: 725–745.

Scott, A.C., Chu, F.Y.F. and McLanghlin, D.W. (1973), The Soliton a new concept in applied science, Proceedings, IEEE, **61**: 1443–1458.

Smith, R. (1972), Nonlinear Kelvin and continental–shelf waves, *J. Fluid Mech.*, **52**: 376–397.

Stern,M. (1975), Minimal properties of planetary eddies, *J. Mar. Res.*, **33**: 1–13.

Swaters, G.E. (1986), Barotropic modon propagation over slowly varying topography, *Geophys. Astrophy. Fluid Dyn.* **36**: 85–113.

Yamagata, T. (1982), On nonlinear planetary waves: a class of solutions missed by the traditional quasi–geostrophic approximation, *J. Oceanogr. Soc. Japan*, **38**: 236–244.

Yano, J.I. and Tsujimura, Y.N. (1987), The domain of validity of the KdV–type solitary Rossby waves in the shallow water–plane model, *Dyn. Atmos. and Oceans*, **11**: 101–129.

Yih. C.S. (1980), *Stratified flow*, Academic Press, New York, America. 418pp.

Zeng Q.C. (1979), *The physical–mathematical basis of numerical weather prediction*, Vol 1, Beijing, Science Press (in Chinese), 543pp.