New Approach to Study the Evolution of Rossby Wave Packet

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ABSTRACT

The average variational principle was employed in this paper to study the evolution of large–scale and slowly varying Rossby wave packet with basic flow both in barotropic and baroclinic atmospheres. The evolution of the structure of Rossby wave packet with both time and space was studied. The results obtained in this paper are similar to the results of by WKBJ method. In addition, the dispersive process of the wave packet was analysed by taking Gaussian type wave packet as an initial disturbance. The valid time scale for application of wave packet theory in the atmosphere was obtained.

I. INTRODUCTION

As we all know that it is difficult to solve the governing equations of barotropic and baroclinic atmospheres, even in the linear case. By analysing the eigenvalues of the equations, we can get some ideas about whether the weather system is a develping one or a decaying one (Kuo 1949), but some puzzled difficulties would appear in this analysis. Some results show that the normal mode is not complete in the analysis of the eigenvalues of the barotropic equation; therefore, the method to study the Rossby wave packet is needing to give some pictures about the evolution of the large–scale Rossby wave structure in the atmosphere without solving the equations completely.

Since the individual weather system such as a Rossby wave trough and ridge can be more or less considered as a wave packet, the problem can be simplified, and more detailed conclusions can be obtained. Dickinson (1968), Grose and Hoskins(1979), Hoskins and Karoly (1982), and Held (1981) have investigated the propagation of forced stationary wave train. Then, Lu and Zeng (1981), Zeng (1983) studied the problem of evolution of Rossby wave packet by using WKBJ method. The investigating results of Zeng (1983) gave us a very clear relationship between the developments of disturbance and their structure as well as their spatial scale change.

Whithma (1974) developed a theoretic system named average variational principle (AVP) to study both linear and nonlinear waves and he also proved that the AVP is equivalent to WKBJ method in the lowest order approximation. In this paper we study the evolution process of Rossby wave packet by using AVP and find that it is more clear in physical picture and convenient in mathematical calculation by using AVP than by using WKBJ method. Furthermore, AVP can also be extended to study other kinds of wave types in the atmosphere. In the final part of this paper we point out that the wave packet is a proper concept in studying the propagation of the Rossby wave by using the Fourier transform on wave packet.

II. INTRODUCTION TO THE AVERAGE VARIATIONAL PRINCIPLE (AVP)

For a dynamical system, there exists a Lagrange L, from which the motion equation of the dynamic system can be derived. For a slowly varying wave–train we can make the variable have the form like $\Phi = |\Phi_0|\cos(\Theta+\eta)$, i.e. we suppose that the wavetrain evolves with quasi–period both in time and in space. On this basis, an average Lagrange $\mathcal{L}$ over a spatial period can be obtained.

$$\mathcal{L} = \mathcal{L}(-\Theta_t, \nabla\Theta, \Phi_0)$$

therefore we have

$$\delta \bar{J} = \delta \iint_D dxdydzdt\, \mathcal{L}(-\Theta_t, \nabla\Theta, \Phi_0) \tag{2.1}$$

where

$$D = R^3 \times R^1$$

The variation equation for variationals in Φ_0 is

$$\delta_{\Phi_0} \iint \mathcal{L}\, dD = 0 \tag{2.2}$$

i.e.

$$\mathcal{L}_{\Phi_0} = 0 \tag{2.3}$$

The variation equation for variationals in Θ is

$$\delta_{\Theta_0} \iint \mathcal{L}\, dD = 0 \tag{2.4}$$

i.e

$$\partial_t \, \mathcal{L}_{\Theta_t} + \partial_{x_j} \, \mathcal{L}_{\Theta_{x_j}} = 0 \tag{2.5}$$

Since

$$\Theta_t = \partial\Theta / \partial t = -\omega, \quad \nabla\Theta = \vec{k}$$

(2.5) can be written as

$$\partial_t \, \mathcal{L}_{\omega} - \partial_{x_j} \, \mathcal{L}_{k_j} = 0 \tag{2.5$'$}$$

Furthermore, there exists the consistency equations for the existence of

$$\partial_t \vec{k} + \nabla\omega = 0, \quad \nabla \times \vec{k} = 0 \tag{2.6}$$

For linearized problem, the amplitude Φ_0 is independent of $\vec{k}$, ω, so we take $\mathcal{L}$ as the following form

$$\mathcal{L} = \mathcal{L}(\omega, \vec{k})|\Phi_0|^2 \tag{2.7}$$

Substituting (2.7) into (2.3) gives the dispersion relation

$$G(\omega, \vec{k}) = 0 \tag{2.8}$$

From this result we can draw a conclusion that the average Lagrange function can be obtained without doing much more mathematical calculation if we know the dispersion relation.

Substituting the dispersion relation (2.8) into (2.5)′ gives

$$\frac{\partial}{\partial t}(G_\omega |\Phi_0|^2) - \frac{\partial}{\partial x_j}(G_{k_j} |\Phi_0|^2 = 0$$

From (2.8), ω can be expressed as the function of $\vec{k}$

$$\omega = \tilde{W}(\vec{k})$$

or

$$G\{\tilde{W}(k),k\} = 0 \tag{2.9}$$

It is easy to get from (2.9) that

$$G_\omega \frac{\partial \tilde{w}}{\partial k_j} + \frac{\partial G}{\partial k_j} = 0 \tag{2.10}$$

which is just the definition of group velocity, so the group velocity can be written as

$$\vec{C}_g = -\nabla_{\vec{k}} G / G_\omega$$

where $\nabla_{\vec{k}} = \sum_{q=1}^{3} \partial / \partial k_q \hat{q}$ is the gradient operator in wave number space.

Finally we get the following equation by substituting $g(\vec{k}) = G_\omega$ into (2.10)

$$\frac{\partial}{\partial t}(g(\vec{k})|\Phi_0|^2) + \nabla \cdot (\vec{C}_g g(\vec{k})|\Phi_0|^2) = 0 \tag{2.11}$$

This is an important equation and it is our basis for following investigation.

III. ROSSBY WAVE PACKET IN THE BAROTROPIC ATMOSPHERE

The linear equation of barotropic atmosphere is

$$\partial_t q + (\vec{v} \cdot \nabla)q + \bar{\beta}_y \tilde{\nabla}^{-2} q_x = 0 \tag{3.1}$$

where $q = (\nabla^2 - F)\psi$, $\tilde{\nabla}^{-2} = (\nabla^2 - F)^{-1}$, F is some constant.
Let $q = \Phi_x$, then the Lagrange L can be written as

$$L = \Phi_x \Phi_t + (\vec{v} \cdot \nabla\Phi)\Phi_x + \bar{\beta}_y (\nabla\psi \cdot \nabla\psi + F\psi^2)$$

If the basic flow is a zonal flow, then

$$L = \Phi_x \Phi_t + \bar{u}\Phi_x^2 + \bar{\beta}_y (\nabla\psi \cdot \nabla\psi + F\psi^2) \tag{3.2}$$

In the following analysis we only consider the zonal basic flow and neglect the variation of $\bar{u}$ and $\bar{\beta}_y$ in one period, so the last two terms in (3.2) can be neglected in average Lagrange $\mathcal{L}$.

For slowly varying wavetrain, let ψ be the form like

$$\psi = |\psi_0| \cos(\Theta + \eta) \tag{3.3}$$

where

$$\Theta = mx + ny - \omega t$$

The average Lagrange function can be expressed as

$$\mathcal{L} = -\frac{1}{m}(v^4 \omega - \bar{u} v^4 m + \bar{\beta}_y m v^2)|\psi_0|^2 \tag{3.4}$$

where

$$v^4 = (m^2 + n^2 + F)^2$$

The dispersion relation is

$$\omega = \tilde{W}(\vec{k}) = m\bar{u} - m\bar{\beta}_y / v^2 \tag{3.5}$$

or

$$G(\omega,\vec{k})=\omega v^4-m\bar{u}v^4+m\bar{\beta}_y v^2.$$

Substituting (3.5) into (2.11) gives

$$\frac{\partial}{\partial t}(v^4|\psi_0|^2)+\nabla\cdot(\vec{c}_g v^4|\psi_0|^2)=0 \tag{3.6}$$

The variation of m has been neglected which was induced by Φ_x.

Integrating (3.6) over the whole area, we have

$$\frac{\partial}{\partial t}\iint_\Omega dxdy(v^4|\psi_0|^2)=\oint_{\partial\Omega}\int dl\hat{n}\cdot\vec{c}_c v^4|\psi_0|^2 \qquad (\Omega\in R^2) \tag{3.6$'$}$$

For steady basic flow, there also exists a conserved wave action

$$\frac{\partial}{\partial t}\left(\frac{v^4}{4\bar{\beta}_y}|\psi_0|^2\right)+\nabla\cdot\left(\vec{c}_g\frac{v^4}{4\bar{\beta}_y}|\psi_0|^2\right)=0 \tag{3.7}$$

Let $v^4|\psi_0|^2/4\bar{\beta}_v$ denote A, (3.7) can be written as

$$\frac{\partial}{\partial t}A+\nabla\cdot(\vec{c}_g A)=0 \tag{3.7$'$}$$

Based on (3.6) the energy equation can be derived.

From (3.6)

$$v^2\frac{\partial}{\partial t}(v^2|\psi_0|^2)+v^2\nabla\cdot(\vec{c}_g v^2|\psi_0|^2)+v^2|\psi_0|^2(\frac{\partial v^2}{\partial t}+\vec{c}_g\cdot\nabla v^2)=0$$

i.e.

$$\frac{\partial}{\partial t}(v^2|\psi_0|^2)+\nabla\cdot(\vec{c}_g v^2|\psi_0|^2)=-|\psi_0|^2 D_g v^2/Dt \tag{3.8}$$

for

$$\frac{\partial v^2}{\partial t}+\vec{c}_g\cdot\nabla v^2=\nabla_{\vec{k}}v^2\cdot[\frac{\partial\vec{k}}{\partial t}+(\vec{c}_g\cdot\nabla)\vec{k}]=-\nabla_{\vec{k}}v^2\cdot\nabla\tilde{w}. \tag{3.9}$$

Here we have used the relation

$$\frac{\partial\vec{k}}{\partial t}+(\vec{c}_g\cdot\nabla)\vec{k}=-\nabla\tilde{w}, \tag{3.10}$$

therefore (3.8) can be written as

$$\frac{\partial}{\partial t}(v^2|\psi_0|^2)+\nabla\cdot(\vec{c}_g v^2|\psi|_2^0)=\nabla_{\vec{k}}v^2\cdot\nabla\tilde{W}|\psi_0|^2. \tag{3.11}$$

Because

$$\nabla\tilde{w}=\hat{i}\left(m\frac{\partial\bar{u}}{\partial x}-m\frac{\partial\bar{\beta}_y}{\partial x}/v^2\right)+\hat{j}\left(\frac{\partial\bar{u}}{\partial y}-m\frac{\bar{\beta}_{yy}}{v^2}\right)$$

$$\nabla_{\vec{k}}v^2=2m\hat{i}+2n\hat{j}$$

(3.11) can be also written as

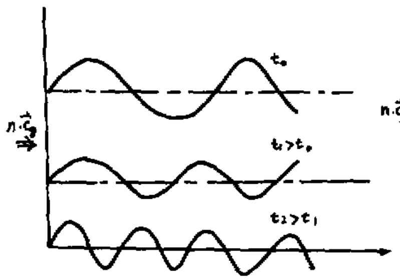

Fig. 1. The variation of wave length.

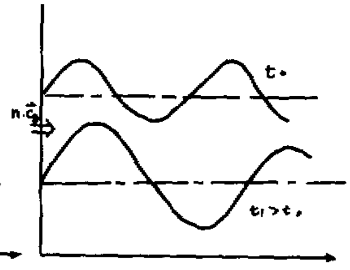

Fig.2. The variation of amplitude.

$$\frac{\partial E}{\partial t}+\nabla\cdot(\vec{C}_g E)=m^2\frac{\partial \bar{u}}{\partial x}+mn\frac{\partial \bar{u}}{\partial y}-m^2\frac{\bar{\beta}_{xy}}{v^2}-mn\frac{\bar{\beta}_{yv}}{v^2} \tag{3.12}$$

where $E=\frac{1}{2}v^2|\psi_0|^2$.

By neglecting the variation of $\bar{\beta}_y$ and integrating (3.12), we have

$$\frac{\partial}{\partial t}\iint_\Omega dxdyv^2|\psi_0|^2+\oint_{\partial\Omega}dl\hat{n}\cdot\vec{c}_g v^2|\psi_0|^2=\iint_\Omega dxdy(2m\frac{\partial\bar{u}}{\partial x}+2mn\frac{\partial\bar{u}}{\partial y}) \tag{3.12$'$}$$

Under steady condition, (3.8) becomes

$$\oint_{\partial\Omega}dl\hat{n}\cdot\vec{c}_g v^2|\psi_0|^2=-\iint_\Omega dxdyD_g v^2/Dt|\psi_0|^2. \tag{3.13}$$

If there exists energy flux flowing into the area of steady wave,

$$\iint_\Omega dxdyD_g v^2/Dt|\psi_0|^2=-\oint_{\partial\Omega}dl\hat{n}\cdot\vec{c}_g v^2|\psi_0|^2>0 \tag{3.13$'$}$$

(3.13)′ means that the wave length would become short in some place within the steady flow area, otherwise the wave length would become long (see Fig.1). Generally speaking, if there is energy flux flowing into the area of the wavetrain, both the amplitude and the wave number would grow. From the meteorological point of view, it means that the trough and ridge would be strengthened and at the same time, the wave length would become short. (see Fig.2). Eq.(3.10) also tells us some information about the evolution of the structure of trough and ridge.

In the end of this section, we summarized all the integral equation obtained above

$$\frac{\partial}{\partial t}\iint_\Omega dxdyv^4|\psi_0|^2=0$$

$$\frac{\partial}{\partial t}\iint_\Omega dxdyv^4/4\bar{\beta}_y|\psi_0|^2=0 \qquad \text{(conservation of dimension)}$$

$$\frac{\partial}{\partial t}\iint_\Omega dxdy\frac{1}{2}v^2|\psi_0|^2=-\frac{1}{2}\iint_\Omega dxdyD_g v^2/Dt|\psi|_0{}^2$$

(conservation of wave action)

$$\frac{\partial}{\partial t}\iint_\Omega dxdy\frac{1}{2}v^2|\psi_0|^2=\iint_\Omega dxdy(m^2\partial\bar{u}/\partial x+mn\partial\bar{u}/\partial y)|\psi_0|^2$$

(transport of energy)

IV. THE ROSSBY WAVE PACKET IN THE BAROCLINIC ATMOSPHERE

The linear equation of baroclinic atmosphere is (Zeng, 1979)

$$\partial_t q + (\vec{v} \cdot \nabla) q + \nabla B \cdot \nabla \tilde{\nabla}^{-2} q = 0$$

where

$$q = (\nabla^2 - H(z))\psi, \quad H(z) > 0$$

$$\tilde{\nabla}^{-2} = (\nabla^2 - H(z))^{-1}, \quad \nabla^2 = \left(\frac{\partial}{\partial x}\right)^2 + \left(\frac{\partial}{\partial y}\right)^2 + \left(\frac{\partial}{\partial z}\right)^2.$$

Let $q = \Phi_x$ the Lagrange function L is

$$L = \Phi_x \Phi_t + \Phi_x (\vec{v} \cdot \nabla \Phi) + \nabla B [\nabla \psi \cdot \nabla \psi + H(z) \psi^2].$$

In the following analysis we only consider the zonal basic flow and neglect the variation of basic flow $\bar{u}$ and B in one period in the average Lagrange L.

After finishing the same process as that in Section 3, we have the following integral equations:

$$\frac{\partial}{\partial t} \iint_\Omega dxdydz v^4 |\psi_0|^2 = 0 \qquad \text{(conservation of dimension)}$$

$$\frac{\partial}{\partial t} \iiint_\Omega dxdydz v^4 / 4B_y |\psi_0|^2 = 0 \quad \text{(conservation of wave aetion)}$$

$$\frac{\partial}{\partial t} \iiint_\Omega dxdydz \frac{1}{2} v^2 |\psi_0|^2 = -\frac{1}{2} \iiint_\Omega dxdydz D_g v^2 / Dt |\psi_0|^2$$

$$\frac{\partial}{\partial t} \iiint_\Omega dxdydz \frac{1}{2} v^2 |\psi_0|^2 = \iiint_\Omega dxdydz (m \partial \bar{u} / \partial x + mn \partial \bar{u} / \partial y + n^2 \partial \bar{u} / \partial z) |\psi_0|^2 \qquad \text{(transport oftotal energy)}$$

V. ESTIMATION OF VALID TIME SCALE FOR ROSSBY WAVE PACKET

The Rossby wave is a dispersive wave, each single wave moves with different group velocity, and after certain time scale, the wave packet would become smooth then the concept of wave packet is invalid after this time range. If we estimate this valid time scale from mathematical point of view and compare it with the real situation, then we can see whether this concept is a proper one.

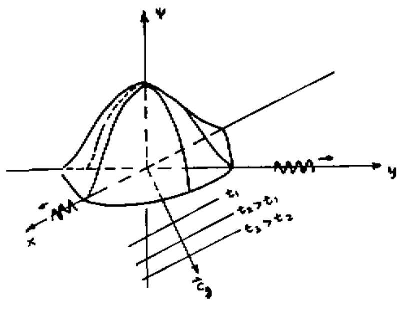

Fig. 3. The initial disturbance.

Consider the barotropic atmosphere and take the initial disturbance as a Gauss–type wave packet (see Fig. 3).

$$\psi(x,y,0) = A\exp[-\frac{1}{2}(ax^2 + by^2)].$$

Taking Fourier transform on

$$\psi(x,y,0) = \frac{1}{\sqrt{2\pi}}\iint_{-\infty}^{+\infty} dmdn\varphi(m,n)\exp[-i(mx+ny)], \tag{5.1}$$

$$\varphi(m,n) = \frac{1}{\sqrt{2\pi}}\iint_{-\infty}^{+\infty} dxdy\psi(x,y,0)\exp[i(mx+ny)] \tag{5.2}$$

$$= \frac{1}{|ab|}\exp[-\frac{1}{2}(m^2/a^2 + n^2/b^2)],$$

it is easy to see from above that when

$$(x,y) = (a^{-1}, b^{-1}),$$

the wave packet structure has remarkable change. Note that this conclusion is not restricted to Gauss–type initial disturbance only.

Now we begin to study the dispersion process of the wave packet.

$$\psi(x,y,t) = \frac{1}{\sqrt{2\pi}}\iint_{-\infty}^{+\infty} \varphi(m,n)\exp[i(mx+ny-\omega(\vec{k})t]dmdn, \tag{5.3}$$

$$\varphi(m,n) = \frac{1}{\sqrt{2\pi}}\iint_{-\infty}^{+\infty} \psi(x,y,t)\exp[-i(mx+ny-\omega(\vec{k})t]dxdy, \tag{5.4}$$

since ω is the function of wave vector $\vec{k}$, ω can be expressed as the Taylor series near a suitable wave number. $\vec{k}_0 = (m_0, n_0)$

$$\omega(\vec{k}) = \omega(\vec{k}_0) + \vec{C}_g(\vec{k} - \vec{k}_0) + \frac{1}{2}\Omega_1(m-m_0)^2 + \frac{1}{2}\Omega_1(n-n_0)^2\cdots\cdots.$$

where $\vec{c}_g$ is the group velocity. $\Omega_1 = \partial c_{gx}/\partial m$, $\Omega_1 = \partial c_{gy}/\partial n$.

For Gauss wave packet we have

$$\psi(x,y,t) = [(1+ia^2\Omega_1 t)(1+ib^2\Omega_2 t)]^{-1/2}|ab|Ae^{-i\omega(\vec{k}_0)t}$$

$$\exp\left[\frac{i}{2}\left(\frac{(x-c_{gx}t)^2}{1+ia^2\Omega_1 t}a^2 + \frac{(y-c_{gy})^2}{1+ib^2\Omega_2 t}b^2\right],\right.$$

$$\Delta x \sim \frac{1}{a}\sqrt{1+\Omega_1^2 a^4 t^2} = (\Delta x_0)\sqrt{1+\Omega_1^2 t^2/(\Delta x_0)^4},$$

$$\Delta y \sim \frac{1}{b}\sqrt{1+\Omega_2^2 b^4 t} = (\Delta y_0)\sqrt{1+\Omega_2^2 t^2/(\Delta y_0)^4},$$

when $t > t_c = \min\left(\frac{(\Delta x_0)^2}{|\Omega_1|}, \frac{(\Delta y_0)^2}{|\Omega_2|}\right)$, the structure of the wave packet has remarkable change.

Taking $F=0$, from Section 2, $\vec{C}_g$ and $\vec{\Omega}$, are

$$\vec{C}_g = (\bar{u} + \frac{m^2 - n^2}{v^4}\bar{\beta}_y, \quad \frac{2mn}{v^4}\bar{\beta}_y),$$

$$\vec{\Omega} = (\Omega_1, \Omega_2) = \left(\frac{4mn}{v^6}\bar{\beta}_y, \quad \frac{2m(m^2 - n^2)}{v^6}\bar{\beta}_y\right).$$

In the real atmosphere $[\bar{u}] \sim 10^{-2}$ (km / s), $[L] \sim 10^3$ (km), $[D] \sim 10^2$ (km), so $[n / m] \sim [L / D] \sim 10$, $[m] \sim 2\pi \times 10^{-3}$, $[\bar{\beta}_y] \sim 10^{-6}$, taking $(\Delta x_0) \sim 10^2$(km), $(\Delta y_0) \sim 10$ (km) where L, D are the length of area in x and y direction respectively. Therefore

$$t_{c_1} = (\Delta x_0)^2 v^6 / 4mn\bar{\beta}_y \sim 10^4 \times 10^6 m^6 / 10^{-6} \times 4 \times 10^{-2} m^3$$
$$= 16\pi^3 \times 10^5 \sim 100(\text{days}), \tag{5.5}$$
$$t_{c_2} = (\Delta y_0)^2 v^6 / 2mn^2\bar{\beta}_y \sim 10^2 \times 10^6 \times m^6 / 2 \times 10^{-6} \times 10^2 m^3$$
$$= 32\pi^3 \times 10^3 \sim 10(\text{days}), \tag{5.6}$$

The time scale above is large enough for disturbance travelling from middle latitude to low latitude, therefore we can conclude that the wave packet theory is a proper theory for investigating the disturbance in middle latitude.

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