On a Class of Solitary Wave Solutions of Atmospheric Nonlinear Equations

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ABSTRACT

In this paper, an attempt is made to study some interesting results of the coupled nonlinear equations in the atmosphere. By introducing a phase angle function ξ, it is shown that the atmospheric equations in the presence of specific forcing exhibit the exact and explicit solitary wave solutions under certain conditions.

I. INTRODUCTION

A spectacular progress has been made by recent studies on nonlinear equations describing the atmospheric waves. It is well–known by now that in the stmosphere, there exist various types of waves which are classified (Panchev, 1985) depending on the nature of the boundary conditions. One class of waves of physical interest is the Rossby wave that ordinarily occurs due to the variation of the Coriolis parameter with latitude. Although it is not always possible to express the solutions of such equations in closed forms, it has been found that by employing suitable boundary conditions. The solutions may be expressible in terms of elliptic functions. As a result, useful insight may be gained into the nature and behavior of the solutions and in some cases by passing over to the solitary wave limit. It may be remarked that solitary wave disturbances are now found to be implicated as the primary causal factor in a significant number of wind–shear related aircraft accidents (Christie and Muirhead, 1983).

It has been observed that the usual techniques which reduce a typical nonlinear equation to a convenient integrable form are applicable (Huang & Zhang, 1988; Guha–Roy, 1990) to atmospheric weather equations. Indeed, in this way periodic and solitary waves have been obtained which are nondispersive in nature. In this communication, we wish to pursue the subject of nonlinear waves in the atmospheric setting and show the other variants of solutions which could not arrive earlier in a similar context.

II. MATHEMATICAL FORMALISM

We start our discussion by considering the nonlinear coupled equations in a barotropic nondivergent atmosphere in the absence of specific forcing in the following form (Huang & Zhang, 1987).

$$u_t + uu_x - fv = -\varphi_x + G_1 , \tag{1a}$$

$$v_t + uv_x + fu = -\varphi_y + G_2 , \tag{1b}$$

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$$u_x + v_y = 0 \ , \tag{1c}$$

where (u,v) is the fluid velocity at (x,y), f the Coriolis parameter and $G_1(= -vu_y)$, $G_2(= -vv_y)$ denote the advection effects of v.

Now if one assumes that the latitude variation of the Coriolis parameter is given by the linear approximation

$$f = f_0 + \beta y, \quad \beta = \text{const.} \tag{2}$$

then in the presence of specific forcing Q, Eq. (1) reduces (Huang & Zhang, 1987) to

$$(\partial_t + u\partial_x + u_x)(v_x - u_y) + \beta v = Q = \frac{\mu v}{v + \gamma} \tag{3a}$$

$$u_x + v_y = 0 \ , \tag{3b}$$

where μ and γ are parameters. Obviously there will be no specific forcing for $v = 0$.

The above set of equations (3a, b) may be further reduced by using the following transformations:

$$u = \bar{u} + u(\xi), \quad v = v(\xi), \quad \xi = kx + ly - \omega t, \tag{4}$$

the symbols have their usual meanings.

Thus, substitution of (4) into (3) and elimination of u results in

$$(\gamma + v)v'' + v'^2 = \zeta v - \frac{\Delta v}{\gamma + v} \ , \tag{5}$$

where $\gamma = \frac{k}{l}(\nu - \bar{u})$, $\zeta = \frac{k\beta}{1(k^2 + l^2)}$, $\nu = \frac{\omega}{k}$,

$$\Delta = \frac{k\mu}{l(k^2 + l^2)} \text{ and } \bar{u} > 0,\ k > 0,\ l < 0,\ \beta > 0,\ \mu > 0,\ (') \equiv \frac{d}{d\xi}$$

Integrating (5) $w.\,r.\,t.\ \xi$, we get,

$$[(\gamma + v)v']^2 = \frac{2}{3}\zeta v^3 + \zeta(\gamma - \delta)v^2 + c \ , \tag{6}$$

where c is an integration constant and $\delta[= (\mu / \beta)] > 0$.

It is remarking (Huang & Zhang, 1987) in this context that $c$represents the pseudo–energy satisfying the inequality

$$0 < c < \frac{1}{3}[\zeta(\delta - \gamma)^3] \ . \tag{7}$$

Let us transform Eq. (6) by setting,

$$v = \frac{[(\lambda - \alpha\gamma) - \gamma W]}{(W + \alpha)} \ ,$$

where $\lambda(> 0)$ is a constant. Eq. (6) then becomes,

$$W'^2 = A(W + \alpha)^6 + B(W + \alpha)^5 + C(W + \alpha)^4 + D(W + \alpha)^3 \ . \tag{8}$$

In (8), the parameters A, B, C and D stand for

$$A = \frac{1}{\lambda^4}[(\frac{1}{3}\zeta\gamma^3 + c) - \delta\zeta\gamma^2], \quad B = \frac{2\delta\zeta\gamma}{\lambda^3} \ ,$$

$$C = -\frac{\zeta(\delta+\gamma)}{\lambda^2} \ , \quad D = \frac{2\zeta}{3\lambda} \ .$$

For convenience, one can express (8) as

$$W'^2 = \sum_{i=0}^{6} a_i W^i \ , \tag{9}$$

where $a'_i s$ are obviously a combination of the constants.

It is interesting to note (Guha-Roy, 1988) that for $\delta > [\frac{1}{3}\gamma + \frac{c}{\zeta\gamma^2}]$, by applying the Kano and Nakayama (1981) method, one can find the solutions of Eq.(9) in terms of Weierstrass function as,

$$W = \frac{ap(\xi+\theta;\ g_2, g_3)}{[1 + bp(\xi+\theta;\ g_2, g_3)]} \ , \tag{10}$$

where g_2 and g_3 are real parameters such that

$$g_2^3 - 27g_3^2 > 0, p'^2 = 4p^3 - g_2 p - g_3 \ ,$$

and a and b being constants, are determined in terms of $a'_i s$.

The exact bounded periodic solution of (9) is then given by

$$W(\xi) = \frac{a[e_3 + (e_2 - e_3)sn^2\left\{\sqrt{e_1 - e_3}\,\xi + \theta^*\right\}]}{1 + b[e_3 + (e_2 - e_3)sn^2\left\{\sqrt{e_1 - e_3}\,\xi + \theta^*\right\}]} \ , \quad \theta^* = \text{const} \ . \tag{11}$$

It may be noted that the modulus of the Jacobian function "sn" is

$$m = \frac{e_2 - e_3}{e_1 - e_3} \ ,$$

and e_1, e_2 and e_3 are real roots of

$$4z^3 - g_2 z - g_3 = 0 \ ,$$

such that $e_1 > e_2 > e_3$. It is needless to mention that the solitary wave is a wave having infinite periods and this happens when m is unity (Kano & Nakayama, 1981). As a result, in the solitary wave limit $e_1 = e_2$. Thus the solitary wave solution is obtained as

$$W(\xi) = \frac{a[e_1 - (e_1 - e_3)\text{sech}^2(\sqrt{e_1 - e_3}\,\xi + \theta^*)]}{1 + b[e_1 - (e_1 - e_3)\text{sech}^2(\sqrt{e_1 - e_3}\,\xi + \theta^*)]} \ , \tag{12}$$

since $e_1 + e_2 + e_3 = 0$.

Next we consider three possible cases that exhibit solitary wave solutions too. In order to see this, we reset (9) in the form

$$W'^2 = \sum_{i=2}^{6} a_i W^i \tag{13}$$

by adjusting the constants A, B, C, D and α.

Now for $\delta < [\frac{1}{3}\gamma + \frac{c}{\zeta\gamma^2}]$, one can write (13) as

$$W'^2 = a_6 W^2 (W + \lambda_1)^2 (W^2 + \lambda_2) \tag{14}$$

subject to the conditions:

$$a_6,\ \lambda_1,\ \lambda_2 < 0 \text{ and } \lambda_1 < \lambda_2 .$$

It may be noted that this is possible provided

$$A = \frac{2B}{3\alpha} = -\frac{c}{6\alpha^2} = \frac{2D}{7\alpha^3} . \tag{15}$$

Let us write Eq. (14) as

$$\frac{d\xi}{dW} = \pm\frac{1}{\lambda_1\sqrt{|a_6|}}\left[\frac{1}{W\sqrt{-(W^2+\lambda_2)}} - \frac{1}{(W+\lambda_1)\sqrt{-(W^2+\lambda_2)}}\right]$$

$$\text{or,}\ \int^W \frac{dW}{W\sqrt{-(W^2+\lambda_2)}} = \pm\lambda_1\sqrt{|a_6|}(\xi+\xi_0) + \int^W \frac{dW}{(W+\lambda_1)\sqrt{-(W^2+\lambda_2)}}$$

$$= \pm\lambda_1\sqrt{|a_6|}(\xi+\xi_0) + G(F(\xi)) ,$$

where

$$F(\xi) = [W(\xi) + \lambda_1] ,$$

$$G(F(\xi)) = \frac{1}{(\lambda_1^2+\lambda_2)^{1/2}} \text{aresin}[\frac{\lambda_1 F(\xi) - (\lambda_1^2+\lambda_2)}{F(\xi)\sqrt{-\lambda_2}}] .$$

Therefore, we have

$$W(\xi) = \pm\sqrt{|\lambda_2|}\,\text{sech}[\sqrt{|\lambda_2|}\left\{\pm\lambda_1\sqrt{|a_6|}(\xi+\xi_0) + G(F(\xi))\right\}] . \tag{16}$$

On the other hand, if we write (13) for

$$\delta > [\frac{1}{3}\gamma + \frac{c}{\zeta\gamma^2}], \quad \text{as}$$

$$W'^2 = a_6 W^2 (W+\lambda_1)^2 (W^2 + \lambda_2 W + \lambda_3) , \tag{17}$$

then by adopting the technique described here, a combined form solution can be obtained (Guha–Roy, 1987), namely,

$$W(\xi) = \pm\frac{4\lambda_3}{\sqrt{\lambda_2^2 - 4\lambda_3}\,[e^Y + e^{-Y} \pm (2\lambda_2/\sqrt{\lambda_2^2 - 4\lambda_3})]} , \tag{18}$$

where

$$F(\xi) = [W(\xi) + \lambda_1] ,$$

$$Y = \sqrt{\lambda_3}\,[\pm\lambda_1\sqrt{a_6}\,(\xi+\xi_0) + G(F(\xi))] , \text{ and}$$

$$G(F(\xi)) = -\frac{1}{\sqrt{\tau}}\ln\frac{[\left\{F^2(\xi) + (-2\lambda_1+\lambda_2)F(\xi) + \tau\right\}^{1/2} + \tau^{1/2}]}{F(\xi)} + \frac{(2\lambda_1+\lambda_2)}{\tau^{1/2}} ,$$

with $\tau = (\lambda_1^2 - \lambda_1\lambda_2 + \lambda_3) > 0$, a_6, $\lambda_3 > 0$ and λ_1, $\lambda_2 < 0$.

It is noteworthy that the possible form (17) will exist if

$$A = -\frac{B}{3\alpha} = -\frac{c}{3\alpha^2} = -\frac{D}{\alpha^3} \ .$$

Finally it is interesting to point out that when $\alpha = -(B/6A)$, we can express (9) as

$$W'^2 = a_6 W^6 + a_4 W^4 + a_2 W^2 \ , \tag{19}$$

provided $\frac{B}{A} = \frac{18C}{5B} = \frac{6\sqrt{6}D}{\sqrt{5}}$.

Now if we write (19) as

$$W'^2 = a_6 W^2 (W^2 + \lambda_1)(W^2 + \lambda_2) \ , \tag{20}$$

then it is not difficult to find that Eq. (19) has a solution of the form

$$\xi = \pm \frac{1}{\sqrt{8R}} \ln[(\frac{1}{W^2})(2R - NW^2 \pm 2\sqrt{RP})] + K \ ,$$

where

$$P = MW^4 - NW^2 + R \ , \tag{21}$$
$$M = a_6/2, \quad N = -a_6(\lambda_1 + \lambda_2)/2, \ R = (a_6 \lambda_1 \lambda_2)/2$$

and K is integration constant.

Setting $\theta = 2\sqrt{2R}(\xi - K)$, we have from (21),

$$W^2 = \frac{4R \ \exp(\pm\theta)}{\left[\left\{N + \exp(\pm\theta)\right\}^2 - 4RM\right]} \ . \tag{22}$$

It is thus observed that as ξ tends to infinity, W^2 approaches to zero for either sign in the exponent. As is shown in Ref. (Lakshmi, 1979) that W^2 represents a well–behaved finite energy, soliton solution whenever $N > \sqrt{4MR}$. The details may be found in lakshmi (1979) 's paper.

It may be remarked that special solutions of (13) are of different kinds than that obtained from (9). It is instructive to point out that the solution (16) may not be evaluated, owing to the causal factor δ, from the solution of (9) by vanishing a_0 and a_1. But the possibility of constructing other solutions of (13) from the solution of (9) seems to be clear. We shall deal this criteria together with its stability property in a subsequent paper.

III. CONCLUDING DISCUSSION

Summing up, we have reported a number of solitary wave solutions of the atmospheric equations. It is worthwhile to note that soliton like solution also exists for certain restriction of the parameters. To make a conclusion, we may say that these solutions may be useful in understanding the fundamental properties of the nonlinear waves in the atmosphere. It may be further remarked taht there will be no solitary wave solution in the absence of specific forcing. Finally, it is needless to mention that similar analysis can be made for $k > 0, \ l > 0$.

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