On the Chaotic Behavior and Predictability of the Real Atmosphere①

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ABSTRACT

In this paper the concept of Chaos and its applications to the study of predictability theory is introduced. The author's attempt is to give a general overview of ideas and methods involved in this problem to scientists,who are interested in the problem of predictability but not familiar with the theory of chaos. The problem is discussed in 4 sections. In the first section, the concept of chaos and the study methods are outlined briefly; in the second section, the methods of quantitatively measuring the main characteristics of chaos which are the basis for the predictability theory are introduced; the third section discusses the time series analysis for directly studying chaotic phenomena in practical problems; and the last section presents some research results on the chaotic characteristics and the predictability of the real atmosphere.

I. CHAOS—INTRINSIC RANDOMNESS

27 years ago, Lorenz, a famous atmospheric scientist in the U.S., discovered nonperiodic motions in a determinate nonlinear dissipative system. Since then, a completely new concept —chaos has been introduced, and people's understanding of the universe has been changed. Although the chaos has not been accurately defined until now, it can be apparently regarded as non–regular behavior coming from determinate nonlinear systems, or the so–called intrinsic randomness. The discovery of chaos has broken up the universe of regular motions. People have no longer considered our universe as a constant one (stationary motions), or periodic one (periodic or quasi–periodic motions). Besides these two forms of motions, there is a more general form of motions—chaos. The discovery of chaos has also broken up the line dividing determinate motions and stochastic motions. It is the first time that people found totally irregular behavior in a deterministic nonlinear system which leads to the unpredictability of the long–term behavior of the system. In a word, in the viewpoint of chaos, many complex phenomena can be considered as purposeful structured behavior, rather than some external incidental behavior.

The most important characteristic of chaotic motions is that it is very sensitive to the change in initial conditions. That is to say that a slight change in the initial condition will completely change the final solution of the equation describing a deterministic system. In the field of weather forecasting, people often call it "butterfly effect"—a flying butterfly in the Wutai Mountains may affect the possibility of raining several months later in the Disneyland across the Pacific Ocean!

For reasons in the principle and technology, any measurements will unavoidably

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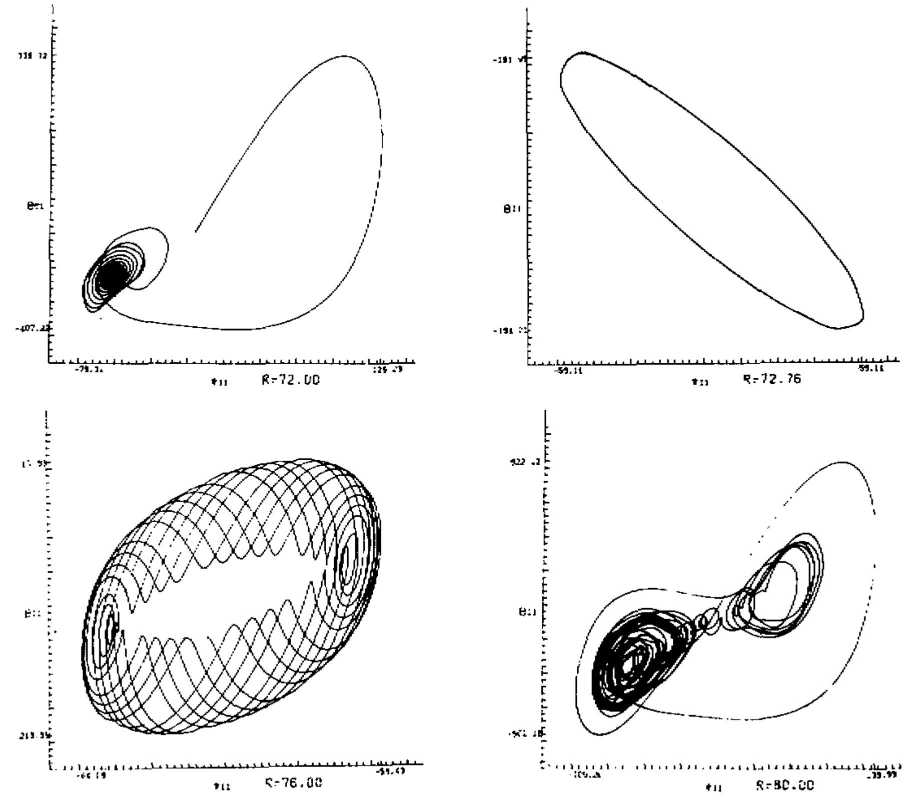


Fig.1. Four attractors in the phase space. (a) stationary attractor; (b) periodic attractor; (c) quasi–periodic attractor; (d) strange attractor (Yang Peicai, 1985).

introduce some uncertainties. The "butterfly effect" of a chaotic system may amplify those uncertainties so quickly that after certain duration of time the real state of the system will be covered by those uncertainties, which results in the unpredictability of the long–term evolution of the system.This seems to be a pessimistic conclusion which means that science will never forecast the state of a chaotic system after a period of time. Of course, weather forecasters may also attribute some failures in the long–term forecast to chaos.

How do people describe and study chaotic phenomena?

Mathematicians have provided a convenient and effective tool—the concept of phase space. They call the system whose state changes with time a dynamical system and the space structured by its state variables the state space or phase space. Thus a one–to–one relationship is created between a state of the system and a point in the phase space. The state of the system at each moment corresponds to a point in the phase space, whereas the variation of the state with time corresponds to a curve in the phase space. The former is called a phase point, the latter a phase trajectory.

In the study of dynamical systems, what concerns people is the destination of the state of the system rather than the instant situation. In other words, what kind of structure will the system remain after a long time ($t \to \infty$), or where will the phase trajectory describing the evolution of the state of the system in the phase space lead to ? For a dissipative system, all the phase trajectories will eventually be attracted to a finite set of zero volume which is called an attractor. In fact, this attractor represents the asymptotic solution of the system when $t \to \infty$. As a dynamical system always corresponds to a problem in physics, we can convert a physics problem to a geometric problem by creating a phase space.

Up to now, 4 types of attractors have been found in dynamical systems (Liu Shida and Liu Shigua, 1989). They are : (1)the stationary attractor which is a stationary point in the phase space meaning that the state of the system is independent of time; (2) the periodic attractor which is a close trajectory in the phase space indicating that the state of the system varies periodically with time; (3) the quasi–periodic attractor which is a torus in the phase space indicating that the state of the system varies quasi–periodically at two (or three) incommensurable frequencies; (4) the strange attractor which is a fractal, a geometrical object having infinite area and zero volume and folded infinite times in the phase space, indicating that the state of the system varies with time non–periodically and irregularly. As the strange attractor possesses every property of chaos, it is sometimes called the chaotic attractor. These 4 types of attractors are shown in Fig.1.

The most interesting attractor here is the strange attractor. The following sections will discuss its main characteristics, measuring methods and applications to the study of the predictability theory.

II. THE MEASURE OF CHAOTIC ATTRACTORS—THEORETICAL BASIS OF PREDICTABILITY

As mentioned before, a chaotic attractor in a dissipative system has two main characteristics. One is its sensitivity to initial conditions. The other is its fractal structure. The former describes the informative property of the system and is closely related to the Lyapunov exponent and the entropy. The latter describes the geometric property of the system and is closely related to the fractal dimension. These two properties and three relative characteristic quantities are not only powerful tools in the diagnosis of chaotic behavior but also the main basis of the study of the predictability theory.

Starting from the following dynamical system, the theories and the measuring methods will be introduced.

$$\frac{dx_i}{dt} = f_i \ (x_1,\dots,x_m;\alpha) \qquad i = 1,2,\dots,n \tag{1}$$

where t is time; $\{x_i, i = 1,\dots,n\}$ is a n–dimensional state variable which forms a n–dimensional phase space; $f_i, i = 1,\dots,n$ is a n–dimensional vector which describes a velocity field in the n–dimensional space; α is a controlling parameter of the system which determines the type of attractors in the phase space. Assuming that Eq.(1) is a dissipative system, i.e., it is a shrinking flow in the phase space, therefore $\sum_i \frac{\partial f_i}{\partial x_i} < 0$.

1. *Lyapunov Exponent*

One sign of the sensitivity of a chaotic system to the initial value is the rapidly increase of initially–introduced errors with time. If the error of the system at time t is expressed as $\{\delta x_i(t), i = 1,\dots,n\}$, then the following equation may be derived from Eq.(1), as long as $\{\delta x_i\}$ is small enough,

$$\frac{d\delta x_i}{dt} = \sum_{j=1}^{n} A_{ij}\delta x_j \qquad i = 1 \ldots n \tag{2}$$

where $A_{ij} = \left.\frac{\partial f_i(x_1,\ldots,x_n,\alpha)}{\partial x_j}\right|_{(x_1,\ldots,x_n)}$, which is the Jacobi matrix of the function f_i on the right side of Eq.(1). The space formed by the error components $\delta x_i\,(i=1,\ldots,n)$ is called a tangent space.

Now, let us follow the temporal variation of an infinitesimal ball of diameter ε_0. It is easy to see from Eq.(2) that eigenvalues of the matrix $\{A_{ij}\}$ give the exponential change rate of the phase volume in each eigen–direction at certain time. As values and signs of the eigenvalue vary, the volume will stretch or contract at different rates in different directions. Therefore, after a period of time, the ball will change to an ellipsoid of decreasing volume. The exponential increasing rate in different directions in the tangent space after long enough time T may be expressed by the average change of the length $\varepsilon_i(T)$ of the basic axis relative to the initial diameter ε_0 of the ball as follows,

$$\lambda_i \approx \frac{1}{T}\log_2\left(\frac{\varepsilon_i(T)}{\varepsilon_0}\right) \qquad (i=1,\ldots,n)\ .$$

These equations define the Lyapunov exponents (the unit is bit / time). The Lyapunov exponents arranged in an order from the largest to the smallest are called the spectrum of Lyapunov exponents, i.e.,$\lambda_1 \geqslant \lambda_2 \geqslant \ldots \geqslant \lambda_n$.

It is easy to prove that the summation of all the Lyapunov exponents is equal to the divergence of the flow defined in Eq.(1), i.e.,$\sum \lambda_i = \sum_i \frac{\partial f_i}{\partial x_i}$.

In practice, for calculating λ_1, only one–dimensional expansion, i.e., the separation of two trajectories, needs to be considered. In this case, λ_1 may be expressed as follows,

$$\lambda_1 = \lim_{t\to\infty}\frac{1}{t}\log_2\frac{l(t)}{l_0} \ , \tag{3}$$

where $l(t)$ is the distance between the phase points on two differential trajectories at time t and l_0 is the distance between the initial points of these two trajectories. Similarly, the summation $\lambda_1+\lambda_2$ of the largest two Lyapunov exponents can be obtained by a two–dimensional expansion, i.e.,

$$\lambda_1+\lambda_2 = \lim_{t\to\infty}\frac{1}{t}\log_2\frac{s(t)}{s_0} \ , \tag{4}$$

where $s(t)$ and $s_0(t)$ are areas of triangles formed by three separate trajectories at time t and at the initial time, respectively.

Analogizing to that, one can find the summation of the largest j Lyapunov exponents.

In principle, it is possible to find any Lyapunov exponent by the same method. But it is very hard to distinguish the shrinking direction of a negative exponent, which often results in unstability in the calculation. Therefore, generally, only non–negative Lyapunov exponents are calculated.

The practically most important parameters are λ_1 and $\sum\lambda_i(\lambda_i>0)$, the summation of all the positive Lyapunov exponents. The former is not only a good criterion for chaotic behavior, but also the component easiest to calculate in the Lyapunov spectrum. The latter describes the average exponential increase rate of an infinitesimal volume in its expanding direction in the phase space, and is sometimes called the degree of chaos. As both λ_1 and $\sum\lambda_i$ can describe quantitatively the amplifying rate of a chaotic system versus the initial uncertainty, they can be used as a "ruler" for measuring the predictability of the state of the system. Usually, their reciprocals are used to define the temporal scale of the predictability of a system, i.e., the time needed for doubling state errors.

2. *Fractal*

The previous section has discussed how to convert a problem in physics to a problem in geometry using the concept of phase space and indicated that the asymptotic solution of a system is an attractor in the phase space.

To study attractors in the point of view of geometry, one must measure the dimension of those geometrical objects to find the difference in the dimension for various attractors.

The dimension of a geometrical object can be defined in many different ways. Therefore, there are many different scales. The most famous two are the Hausdorff dimension D_H and the capacity dimension D_c. The capacity dimension is more apparent and easier to understand, and it is very similar to the Hausdorff dimension. The following discussion will focuses on it. It can be briefly demonstrated as follows: Let E express a subset in a d–dimensional space. Using some d–dimensional balls of diameter ε to cover E, if the minimal number of balls needed for completely covering E is $N(\varepsilon)$, then the capacity dimension of E may be defined as

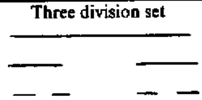
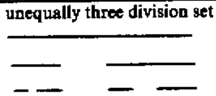
$$D_c = -\lim_{\varepsilon\to 0}\frac{\log N(\varepsilon)}{\log\varepsilon} \tag{5}$$

or

$$N(\varepsilon) = \lim_{\varepsilon\to 0}\varepsilon^{-D_c} .$$

It is easy to find from the definition that the dimension of a piont is zero, of a line is one, of a surface is two and of a solid is three. The capacity dimensions of those familiar geometrical entities are the same as those obtained empirically, i.e., they are all positive integers. But D_c is not necessarily an integer from the definition of Eq.(5). For some so–called fractals, D_c can be a positive fraction or an irrational number (Mandelbrot, 1982). People often use the shape of the coast line to illustrate this. Although the coast line is very apparent, it is not easy to calculate. To give a quantitative illustration of dimension, we use the Cantor sets which are familiar to most readers as examples. The dimensions and measures (denoted as L_c) of the Cantor sets are shown in Table 1.

Table 1. The Capacity Dimension of Cantor Sets

	Three division set		5 division set		unequally three division set	
						
	ε	N(ε)	ε	N(ε)	ε	N(ε)
	1	1	1	1	1	1
	1/3	2	1/5	3	1/4	3
	1/9	4	1/25	9	1/16	9
	3^{-n}	2^n	5^{-n}	3^n	4^{-n}	3^n
D_c	ln2 / ln3 = 0.63		ln3 / ln5 = 0.68		ln3 / ln4 = 0.79	
L_c	$1-\frac{1}{2}\sum_{k=1}^{\infty}(\frac{2}{3})^k=0$		$1-\frac{2}{3}\sum_{k=1}^{\infty}(\frac{3}{5})^k=0$		$1-\frac{1}{3}\sum_{k=1}^{\infty}(\frac{3}{4})^k=0$	

It can be found from Table 1 that the dimensions of several Cantor sets are between 0 and 1. This shows that the Cantor set has been made into a line segment with thousands and thousands holes. It is neither a point of zero–dimension, nor a line of one–dimension. It is a geometric object which has zero length and is situated between points and lines. It can also be found from Table 1 that if the division is more broken or irregular in constructing the Cantor set, then the corresponding dimension will be larger. This shows that the dimension can describe quantitatively the degree of complexity of a geometric structure. Since many problems can be converted to geometric problems, the concept of dimension can be used to compare the degree of complexity of different things.

In fact, a chaotic attractor is also a geometric object with fractal dimension. As mentioned above, the phase volume will expand infinitely in certain directions because of the divergence of the trajectories of a chaotic system. On the other hand, the phase volume will shrink to zero because the system is dissipative. In this contradictory case, the phase volume can meet both requirements only by stretching (caused by the divergence) and folding (caused by the convergence) infinite times. Therefore, the chaotic attractor is a complex geometric object with zero volume and infinite area. It is neither a solid nor a surface. Its dimension can only be a fraction between the dimensions of a solid and a surface.

Back to the calculation of dimension, although the definition Eq.(5) of the capacity dimension takes a calculable form, it is not realistic to calculate the dimension by repeatedly counting of the covering. Especially when the dimension is high, the calculation can be too heavy to carry out. Therefore, other practical methods have to be sought.

In 1983, two mathematicians Grassberger and Procaccia gave another definition of dimension, which is called the correlation dimension. It is defined as follows:

Let $E=\{x_i, i=1,2,\ldots,n\}$ express a subset in a d–dimensional space. Make a small d–dimensional ball taking point $x_i\,(i=1,\ldots,N)$ in E as the center and any positive number ε as the diameter. If we use the following formula

$$C_d(\varepsilon)=\frac{1}{N^2}\sum_{\substack{i\neq j\\ i,j=1}}^{N}\theta\left(\|x_i-x_j\|-\varepsilon\right)$$

to measure the average spatial correlation of points in E, then, the correlation dimention is defined as

$$D_2=\lim_{\varepsilon\to 0}\frac{\ln C_d(\varepsilon)}{\ln\varepsilon} \tag{6}$$

or

$$C_d(\varepsilon)\lim_{\varepsilon\to 0}\varepsilon^{D_2} \quad ,$$

where $C_d(\varepsilon)$ is called the correlation function, $\|x_i-x_j\|$ is the distance between two points on the attractor. $\theta\ (\alpha)$ is the Heaviside function which reaches 0 or 1 when $\alpha>0$ or $\alpha\leqslant 0$, respectively. The subscript of the sign of the correlation dimension D_2 is 2 because it is equivalent to the case for $q=2$ in another definition of dimension, the generalized dimension D_q (Grassberger and Procaccia,1984).

This new definition of dimension replaces the tedious counting of covering by measuring the distance between phase points that it makes the calculations easier. It can be proved that the following relationship exists between D_2, D_H and D_c:

$$D_c\geqslant D_H\geqslant D_2 \ .$$

Moreover, D_2 is a good estimation of other two in most cases.

3. *Entropy*

Besides Lyapunov exponents and dimensions, there is another important characteristic quantity describing the information characteristics of a dissipative system, i.e., Kolmogorov entropy.

According to the theory of information, entropy is a quantity expressing the change rate of the information flow. If the gain of information in interval τ is $I(\varepsilon,\tau)$, where ε is the measurement accuracy, then the Kolmogorov entropy is defined as

$$K_1 = \lim_{\varepsilon\to 0}\lim_{\tau\to 0}\frac{I(\varepsilon,\tau)}{\tau} \quad . \tag{7}$$

In fact, Eq.(7) represents the average change rate of information in a long enough time interval when the accuracy is high enough. Its unit is also bit / time.

Information is a quantity describing the degree of uncertainty of the system states. The more the information, the more the difficult it is to make a correct prediction. Therefore, the information on a predictable system does not increase or decrease with time, i.e., $K_1 = 0$, whereas for a completely unpredictable random system $k_1 \to \infty$. In a chaotic system, since any initial uncertainty will be amplified at certain exponential rates, information increases at a constant rate, i.e., K_1 is equal to a positive constant.

How to determine the entropy of a chaotic attractor? First of all, according to the definition of Eq.(7), the gain of information in a time interval should be determined. Divide an attractor into several elements of size ε, and time into some interval of length τ, and mark those spatial–temporal elements by i_k $(k = 1,2,...)$. Considering a piece of trajectory of length $T = j\tau$ which contains j phase points denoted by $X(k\tau)$, $(k = 1,...,j)$, let $P(i_1,i_2,...,i_j)$ be the joint probability which is equal to the chance that the points $x(\tau)$, $x(2\tau)...x(j\tau)$ fall into elements $i_1,i_2,...i_j$, respectively then, the gain of information and the Kolmogorov entropy may be expressed as

$$K_1 = \lim_{j\to\infty}\lim_{\varepsilon\to 0}\lim_{\tau\to 0}\frac{\sum P(i_1,i_2...i_j)\log_2 P(i_1,i_2...i_j)}{j\tau} \quad . \tag{8}$$

However, in practice, Eq.(8) is not a feasible formula for calculating the Kolmogorov entropy. As in the problem of calculating the dimension, it is very difficult to determine the distribution function of probability by accounting elements. Therefore, people started to seek other definitions of entropy. Grassberger and Procaccia (1984) proposed an easier method to estimate the Kolmogorov entropy which is called order–2 Renyi entropy. Its definition is:

$$K_2 = \lim_{d\to\infty}\lim_{\varepsilon\to 0}\frac{1}{m\tau}\ln\frac{C_d(\varepsilon)}{C_{d+m}(\varepsilon)} \quad , \tag{9}$$

where

$$C_d(\varepsilon) = \lim_{N\to\infty}\frac{1}{N^2}\sum_{\substack{m\neq n\\ m,n=1}}^{N}\theta\left(\sum_{i=1}^{d}|x_{n+i} - x_{m+i}|^2 - \varepsilon^2\right)$$

is also called the correlation function which is roughly equal to the probability of two pieces of the near–by trajectories falling into the same spatial–temporal element shown in Fig.2. The subscript of K_2 in Eq.(9) is 2 because it corresponds to the case $q = 2$ in the generalized entropy.

It can be proved that there is the following relation between the Kolmogorov entropy, order–2 entropy and the summation of all positive Lyapunov components, i.e.,

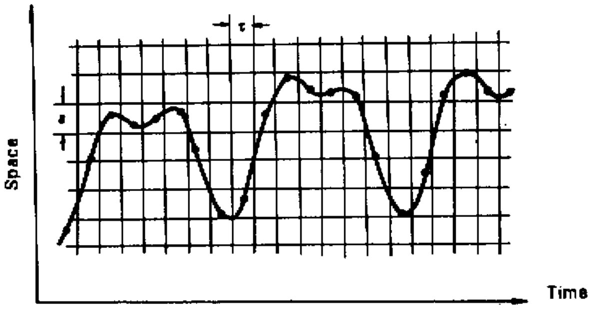


Fig.2. The temporal and spatial division of attractors and the trajectories (Grassberger et al., 1984).

$$\sum_{\lambda_i > 0} \lambda_i \geqslant K_1 \geqslant K_2 \ .$$

This shows that K_2 is a lower limit of $\sum \lambda_i$ and K_1. Therefore, $1 / K_2$ may be used as an upper limit of the average predictable time scales.

Up to now, three main characteristic parameters of a non–linear dynamical system have been introduced. As a summary of this section, the relationship between these three parameters and the dynamical characteristics of the system is shown in Table 2.

Table 2. The Characteristic Parameters and Dynamical Characteristics of Various Systems

	Deterministic system				stochastic system
	stationary motion	periodic motion	quasi–periodic motion	chaotic motion	
Lyapunov exponent	$\lambda_k < 0$ $(k = 1,...,n)$	$\lambda_1 = 0$ $\lambda_k < 0$ $(k = 2,...,n)$	$\lambda_1 = \lambda_2 = 0$ $\lambda_k < 0$ $(k = 3,4,...,n)$	at least $\lambda_1 > 0$	at least $\lambda_1 \to \infty$
dimension	0	1	2 or 3	positive non–integer	∞
entropy	0	0	0	positive	∞

III. CHAOTIC TIME SERIES—A COMMON CHAOTIC SYSTEM

For most problems in practice, dynamical systems are not such dynamical systems as described by Eq.(1), but a time series of a single variable, i.e.,

$$x_i = x(t_0 + i\Delta t) \quad (i = 1,2,...,N) \tag{10}$$

where x is the observed state variable, t_0 is the initial time of the observation, Δt is the time interval of the observation, and N is the capacity of samples or the length of the time series.

In practice, the series Eq.(10) contains traces of all the variables of the original system. It is possible to study the dynamical characteristics of the original system based on the knowledge or situation provided by Eq.(10). At least, it is possible to know whether an attractor exists based on Eq. (10), or to determine whether motions described by Eq. (10) is deterministic or random. If the attractor exists, it is possible to find its dimension and the minimal dimension of the phase space containing it, and thus determine the number of degree of freedom. Besides, it is also possible to find all the positive Lyapunov exponents and order–2 entropy of the system and from them the predictable time scale of the system.

After Packared's (1980) technique of reconstructing the phase space, Takens' (1981) embedding theory and the above–mentioned methods of calculating Lyapunov exponents, dimension and entropy were proposed, it has been possible to reach the above goals.

The following sections will discuss some basic steps.

By the method proposed by Packard et al. and with the data provided by Eq.(10), we can reconstruct an m–dimensional phase space R^m and restore the image of the attractor with the aid of an embedding dimensionality m and a time delay parameter t, i.e., to establish the following phase pattern:

$$X_m(t_i) = \{x(t_i), x(t_i + \tau), ..., x(t_i + (m-1)\tau)\} \ , \qquad (11)$$

where $X_m(t)$ is a phase point in R^m, $x(t_i + (j-1)\tau)$, $j = 1,...,m$, is the j–th coordinate component in the phase space.

Usually in order to ensure that the coordinate components are linearly independent, the time–delay parameter τ should be greater than decorrelation time of Eq.(10), and $\tau / \Delta t$ should be an integer.

It can be proved that as long as m is large enough (usually $m > 2d + 1$, d is the dimension of the attractor), the reconstructed system (11) is equivalent geometrically and *informatively to the original system.*

Next, based on the above reconstructed system, we can calculate the estimates of the maximal Lyapunov exponent, the correlation dimension and the order–2 entropy using Eq.(3), Eq.(6) and Eq.(9). In fact, those estimates vary with m. The steps of reconstruction and calculation must be repeated with increasing m until the calculated estimates do not change significantly with the increase of m (within pemissive errors range). The thus obtained estimates are the desired non–negative Lyapunov exponents, correlation dimension and the order–2 entropy. The corresponding embedding dimension is called the saturated embedding dimension (m_∞), i.e.,

$$\lambda_1(m_\infty) \approx \lambda_1(m_\infty + 1) \approx ... \approx \lambda_1 \ ,$$
$$D_2(m_\infty) \approx D_2(m_\infty + 1) \approx ... \approx D_2 \ ,$$
$$K_2(m_\infty) \approx K_2(m_\infty + 1) \approx ... \approx K_2 \ .$$

The saturated embedding dimension is an important quantity which determines the characteristics of a time series. If m_∞ does not exist, then the estimate of D_2 will increase infinitely with the increase of m. This means that no attractor exists and the diagnosed time series is a random system. Besides, the magnitude of m_∞ gives the number of the degree of freedom of the system, which together with the dimension defines the upper and lower limits of the number of basic variables necessary to simulate the dynamical behavior of the system.

In practice, the quality and quantity of data affect the results significantly.

The quality of the data depends mainly on the level of noise. It is hard generally to estimate the effects of errors in calculating non–negative Lyapunov exponents. However, results of some numerical experiments showed that data with high noise level or with low resolution would usually cause an under–estimated Lyapunov exponent. Wolf et al. (1985) proposed an empirical requirement for the accuracy of data. They pointed out that a fairly good estimate is possible as long as the resolution of data is better than 8 bit. The dependence of the Lorenz strange attractor on the resolution of data is shown in Fig.3.

For the calculation of the correlation dimension and the order–2 entropy, the effect of noise can be shown by the plotting of the correlation function $C_d(\varepsilon)$. Based on the definition of the correlation dimension (Eq.(6)), it is clear that $C_d(\varepsilon)$ is a straight line in the logarithemic coordinates(ln $C_d(\varepsilon)$ vs. ln ε) with D_2 being its slope. If there is noise ε_n

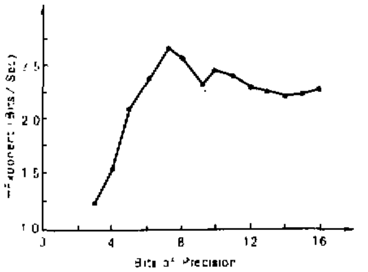


Fig.3. The dependence of the maximum Lyapunov exponent on the accuracy of data.

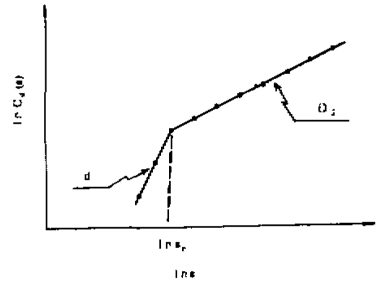


Fig.4. The correlation function of the chaotic attractor with noise.

superimposed on data, then this straight line will bend downward at $\varepsilon = \varepsilon_n$ (Fig.4). The slope of the line will still be D_2 for $\varepsilon < \varepsilon_n$, but the slope will be equal to the embedding dimension d for $\varepsilon < \varepsilon_n$ (Ben–Mizrachi et al., 1984), i.e.,

$$C_d(\varepsilon) = \varepsilon^{D_2} \quad \text{when} \quad \varepsilon > \varepsilon_n \ ,$$
$$C_d(\varepsilon) = \varepsilon^{d} \quad \text{when} \quad \varepsilon < \varepsilon_n \ .$$

The problem of quantity of data often causes doubts and controversy, especially when the attractor has higher dimension (usually $D > 4$). The quantity of data is usually called a trap in the calculation of characteristics of attractors. The requirement for enough quantity of data is based on the consideration ensuring reasonable density of phase points on the attractor so that an accurate spatial distribution can be obtained. Wolf (1986) pointed out that the number of data should be about $10^D - 30^D$ (where D is the dimension of the attractor) for a good image of the reconstructed attractor. Recently, Eckmann and Ruelle (1990) has proposed a fundamental relation between the number of sample N and the dimension D:

$$D = \frac{2\log N}{\log\left(\frac{1}{\rho}\right)} \ ,$$

where $1/\rho$ presents the size of grid mesh dividing the attractor. If $\rho = 0.1$, then $N = 10^{D/2}$. The required quantity of data is much smaller than that proposed by Wolf. The author (1988) used the boundary layer wind data of Beijing area to study the dependence of the estimate of the correlation dimension on the quantity of data. Results showed that the estimate converges stably to 6.2 as long as $N > 2000$ (Fig.5). This is consistent with Eckmann's estimation.

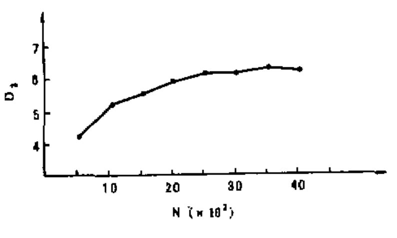


Fig.5. A case of the dependence of the correlation dimension D_2 on the quantity of data N.

IV. THE CHAOTIC BEHAVIOR AND PREDICTABILITY OF THE REAL ATMOSPHERE

It is well known that atmospheric motions are highly irregular. However, it is only in the recent years that people have begun to investigate the dynamical characteristics of those irregular motions by going deep into observations. Back to 10 years ago, it was very difficult to find the structure of a system from complex meteorological data. In early 1980's, the improvement of theories and methods of reconstructing dynamical systems made it possible for people to apply some concepts and methods used in theoretical models for studying chaos to systems determined by observations. Some characteristic quantities describing the behavior of attractors, such as dimensions, entropies and Lyapunov exponents etc. can be estimated from the time series of observations. Therefore, the above difficult has been overcome quite well. This achievement is significant. First of all, the study of nonlinear problems has entered a vast world from the field of pure theoretical models. It is possible now to study the dynamical behavior of the investigated atmospheric systems by means of direct observations. Secondly, it has shortened significantly the distance between the chaotic theory and its applications, which begins a new phase of the chaotic theory. In atmospheric sciences, some significant applied studies have caught people's attention, especially in the area of predictability theory. The chaotic phenomena which were found in the study of predictability theory (Lorenz, 1963) have benefited the problem of predictability itself after 20–year continuous theoretical study.

Some very interesting work has appeared in the area of applying the reconstruction theory and methods to study the dynamical behavior of the weather and climate systems described by observations. They are representing a tendency that some chaotic–marked data analysis methods will become "conventional weapons". The work may be divided into two aspects:

a. The dimension of some weather and climate attractors has been studied. It is demonstrated that atmospheric motions are chaotic. Some of the results are given in Table 3.

It is shown by those results that except the first two long–term climate attractors, the dimensionalities of most atmospheric motions, including the microscale turbulence and short–term climate processes such as ENSO, are in the range from 5 to 8. This demonstrates that the real atmospheric motions are a chaotic system with finite degree of freedom. Although they do not point out which variables are controlling the systems concretely, this conclusion qualitatively tells us that it is possible to establish deterministic atmospheric dynamical models, and points out that the number of basic variables is from 5 to 15.

b. The problem of predictability of some processes in the real atmosphere has been studied, and their predictable time scales have been found. Especially , the predictability of short–term climate systems such as ENSO has been discussed earlier than methods of prediction. Those discussions are valuable to selecting predictors and prolonging the period of validity. As mentioned previously, the characteristic quantities describing the predictability in the theory of chaos are the Lyapunov exponent and entropy. The reciprocal of the maximal Lyapunov exponent $1/\lambda_1$ is usually used to represent the maximal predictable time scale Tu, and the reciprocal of the order–2 entropy $1/K_2$ is used to represent the average predictable time scale Ta. Some published results are shown in Table 4.

It can be found from Table 4 that values of Ta calculated from daily surface pressures in Shanghai and Berlin, respectively, are close to each other, which is consistent with the predictable time scale found by previous statistical methods. However, two predictable time scales Tu and Ta calculated from the monthly mean surface pressure anomalies at Darwin differ greatly. The former is about 3 times the latter.

Table 3. Dimensions of Some Weather and Climate Attractors

Source	Sample capacity	Sampling interval	Correlation dimension	Saturated embedding dimension
Oxygen Isotope Record from V28–238 (The Pacific Ocean) (Nicolis, 1984)	about 500	2000yr	3.1	4
Oxygen Isotopes $\delta^{18}O$ (the Atlantic Ocean) (Fraedrich, 1986)	182	2000–4000yr	4.4	about 12
Daily surface pressure of Berlin (Fraedrich,1987)	1680–1800	1 day	6.8–7.1	
Daily surface pressure of Shanghai (Yan Shaojin et al., to be published)	5000	1 day	7.7–7.9	18
Wind speed and temperature in the atmospheric boundary layer of Beijing (Yang Peicai et al., 1988A)	4x4000	1 sec	5.5–7.2	12–14
Mean horizontal wind speed in the mid–atmosphere of kyoto (Yang Peicai et al., 1988B)	2x1500	150 sec	5.5	14
Monthly mean anomaly of pressure at the sea surface level (Yang Peicai et al., 1990)	1260	1 month	5.5–6.8	15

Table 4. The Predictable Time Scales of Some Atmospheric Systems *

Source	λ_1	T_λ	K_2	T_k
Isotopes of Oxygen $\delta^{18}O$ of the Atlantic Ocean (about 1 million years) (Fraedrich,1987)				7–10 Thousand Yr
Monthly mean anomaly of pressure at Darwin (1260 months) (Yang Peicai et al., 1990)	0.031	32 month	0.089	11 month
Daily surface pressure of Berlin (1800 days) (Fraedrich, 1987)				9–12 day
Daily surface pressure of Shanghai (5000 days) (Yan Shaojin et al., to be published)			0.10	10 day
Mean horizontal wind speed in the lower stratosphere of kyoto (1600 minutes) (Yang Peicai et al., 1988B)			0.14	70 min
Wind speed and temperature in the atmospheric boundary layer of Beijing (4000 seconds) (Yang Peicai et al., 1988A)	0.043	24 sec		

* The predictable time scale is the time needed for doubling errors

Another problem caused concerns in the predictability study is whether there is a way to prolong the period of validity of the system. The value of the predictable time scale depends on the sensitivity of the system to initial values. The latter is the intrinsic property of the system. Therefore, in this sense, there is no room for improving. However, a predicted object usually has more than one indices. Better predictability might be found from other indices. Besides, it is possible to improve the predictability of the system by some filter programs to eliminate external noise. Those methods are equivalent to constructing a new dynamical system which has relatively low degree of chaos.

Now, we take the time series DP of the monthly mean surface pressure anomalies at Darwin which is a good index for ENSO activity as an example (Yang Peicai et al., 1990) to demonstrate how to improve the predictability of ENSO by means of filtering. Perform 3–point, 6–point and 9–point moving average on DP and mark the new indices as DP3, DP6 and DP9, respectively. The statistical analysis of those new indices shows (Table 5) that they are almost equivalent to DP in terms of representing the ENSO activity. However, the dimension and predictable time scale of attractors constructed by those new indices have changed *significantly (Table 6). It is clear that new attractors keep the chaotic characteristics, but their* correlation dimension decreases significantly, and the corresponding predictable time scale increases. This shows that the indices have improved the predictability of ENSO.

Table 5. The Correlation between DP and the Moving–Averaged Index

Index 1	Index 2	Index 1→Index 2			Delay(month)		Index 2→Index1	
		3	2	1	0	1	2	3
DP	DP3	0.54	0.60	0.82	0.84	0.82	0.62	0.54
DP	DP6	0.71	0.76	0.78	0.79	0.76	0.71	0.56
DP	DP9	0.65	0.72	0.78	0.81	0.77	0.72	0.65

Table 6. The Correlation Dimensions and Predictable Time Scales of DP and the Moving–Averaged Index

Index	Correlation dimension	Tu (month)	Ta (month)
DP	6.8	32	11
DP3	6.2	40	28
DP6	5.7	50	33
DP9	5.5	62	21

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