The Theory Study of the Influence of the Topography on the Cold Frontal Motion

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ABSTRACT

In order to study the characteristics of cold frontal motion over the arbitrary topography, the velocity of cold frontal movement is derived by using the one layer shallow–water model. The results show that there exist the retardation in upwind side and rapid descent in the lee slope when the cold front crosses the topography.

I. INTRODUCTION

Clearly, the movement of front will be significantly influenced by the topography. It has been an interesting research topic by many scientists in recent years. However, it is very difficult to give a complete view or quantitative expression on the topographic effects because the topographic profile is very complex. Davies(1984) has ever made a research for a special orography by a semi–geostrophic shallow–water model and some results have been obtained, but the mathematics treatments used in his paper are only fit for this special orography. For the other arbitrary topography,what are the results of this model? In this paper, a method to compute frontal speed over an arbitrary topography is developed, thus we can understand the contributions of the topography more clearly.

Although the one layer semi–geostrophic shallow–water model used in this paper is very simple, it is still accepted by many scientists. Because the model can provide the basic frontal characteristics relatively well and make the mathematics treatments more easy. What is resulted can also explain the main features of real cold frontal motion.

II. BRIEF DESCRIPTIONS OF THE PHYSICAL MODEL

After Davies (1984), the governing equations for the cold front can be written as:

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}-fv=-g'\frac{\partial(h+\eta)}{\partial x}-\alpha fV_g, \tag{1}$$

$$\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+fu=\alpha fU_g, \tag{2}$$

$$\frac{\partial h}{\partial t}+\frac{\partial(hu)}{\partial x}=0, \tag{3}$$

where $(\alpha U_g,\ \alpha V_g)$ which are constants in time and space are geostrophic velocities in the cold air respectively along the $(x,\ y)$ direction. α is defined as $\frac{(\theta-\Delta\theta)}{\theta}$. $\Delta\theta$ is the difference of potential temperature between the warm and cold air. θ is the potential temperature of warm air. The other symbols are the same as the ones in Davies (1984).

Let

$$u=\alpha U_g u^*,\ v=\alpha V_g+\sqrt{g'H}\,v^*,\ h=Hh^*,$$

$$\frac{d}{dt}=\frac{\alpha U_g}{L_f}\frac{d}{dt^*},\ \frac{\partial h}{\partial x}=\frac{H}{L_f}\frac{\partial h^*}{\partial x^*},\ \frac{\partial \eta}{\partial x}=\frac{\eta_{max}}{L}\frac{\partial \eta^*}{\partial x^*}, \tag{4}$$

where H is the depth of the cold air at $X\to-\infty$, (η_{max}, L) are the maximum height and the horizontal scale of the orography respectively. $L_f=\frac{\sqrt{g'H}}{f}$ is the horizontal scale of the front. Thus the nondimensional equations of (1), (2) and (3) can be finally written as

$$F_r^2\left(\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}\right)-v=-\frac{\partial h}{\partial x}-\frac{1}{\epsilon\overline{H}}\frac{\partial \eta}{\partial x}, \tag{5}$$

$$\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}=1-u, \tag{6}$$

$$\frac{\partial h}{\partial t}+\frac{\partial (hu)}{\partial x}=0, \tag{7}$$

where $F_r=\frac{\alpha U_g}{\sqrt{g'H}}$ is the gravitational Froude number, $\epsilon=\frac{fL}{\sqrt{g'H}}$ is the rotational Froude number and $\overline{H}=\frac{H}{\eta_{max}}$. So $\epsilon\overline{H}$ represents the relative magnitude of cold frontal slope and topographic slope.

We might as well make a rough estimation of the aforementioned parameters. For the real cold front, the range of $\triangle\theta$ is usually from 4K to 10K, and H is from 4Km to 10Km. If $\alpha U_g=10\text{m}/\text{s}$, then $g'=0.13$—$0.33\text{m}/\text{s}^2$, $F_r^2=0.03$—0.18. Hence semi–geostrophic approximation can be used and (5) can be further simplified as

$$v=\frac{\partial h}{\partial x}+\frac{1}{\epsilon\overline{H}}\frac{\partial \eta}{\partial x}, \tag{8}$$

Eqs.(6), (7) and (8) are the basic equations of this physical model. Note that the flow velocity on the downslope can reach the very large values for steep orography and then the order of F_r^2 cannot be neglected compared with other parameters. If it is the case, the semi–geostrophic approximation will not be valid and too large down–shift velocity will be produced. Thus we will focus our discussion on the situations in which the topographic slope is not too large. At this time the effect of topography is not so strong that the cold front can cross over the topography.

Usually, the movement of the cold front is characterized by that of the ground front line. Thus it is enough to study the velocity of the frontal line.

According to Davies(1984), the following two equations can be obtained from the Eqs. (6)~(8) under the assumption of initial uniform potential vorticity

$$\frac{\partial^2 (hu)}{\partial x^2}-hu=-1, \tag{9}$$

$$\frac{\partial^2 h}{\partial x^2}-h+1+\frac{1}{\epsilon\overline{H}}\frac{d^2\eta}{dx^2}=0. \tag{10}$$

The equations (9) and (10) are what we want to discuss further. They are in principle the same as that of Davies. However, Davies solved the problem with a special orography. In this paper, we try to develop the general approach to solve the problem with an arbitrary orography. The method is described as follows.

Assuming the position of cold frontal line is $x_0(t)$ at time t, where the depth of the cold air is zero, then we have the following boundary conditions

$$h=0; \qquad x=x_0(t)$$
$$h\to 1,\ u\to 1; \quad x\to -\infty. \tag{11}$$

Thus the solution to (9) with boundary conditions (11) can be obtained as

$$hu=1-e^{x-x_0}; \quad x\leqslant x_0(t), \tag{12}$$

$$\frac{\partial(hu)}{\partial x}=-e^{x-x_0}; \quad x\leqslant x_0(t). \tag{13}$$

If η is the smooth function, the solution to (10) will have the form of

$$h=A(t)e^{x}+B(t)e^{-x}+D(x), \tag{14}$$

where $D(x)$ is an arbitrary function which satisfies

$$\frac{d^2D(x)}{dx^2}-D(x)+1+\frac{1}{\epsilon\overline{H}}\frac{d^2\eta}{dx^2}=0. \tag{15}$$

For an arbitrary topography, if η, $\frac{d\eta}{dx}$ and $\frac{d^2\eta}{dx^2}\to 0$ as $x\to -\infty$, then $D(x)$ at the negative infinite is finite and (14) can be reduced to

$$h=-D(x_0)e^{x-x_0}+D(x). \tag{16}$$

Thus

$$\frac{\partial h}{\partial t}=\left[-\frac{dD(x_0)}{dx_0}+D(x_0)\right]\frac{dx_0}{dt}e^{x-x_0}. \tag{17}$$

Substituting (13), (17) into (7), we can obtain

$$-\frac{dD(x_0)}{dx_0}+D(x_0)=1\Big/\frac{dx_0}{dt}. \tag{18}$$

Let C represent $\frac{dx_0}{dt}$. Clearly, C is the moving speed of the front line. Differentiating (18) with respect to t, (18) becomes

$$-\frac{d^2D(x_0)}{dx_0^2}+\frac{dD(x_0)}{dx_0}=-\frac{1}{C^3}\frac{dC}{dt}. \tag{19}$$

Eliminating $\frac{dD(x_0)}{dx_0}$ from (18) and (19), we can obtain

$$-\frac{d^2D(x_0)}{dx_0^2}+D(x_0)=\frac{1}{C}-\frac{1}{C^3}\frac{dC}{dt}. \tag{20}$$

Using relation (15), (20) becomes

$$\frac{dC}{dt}+\left(1+\frac{1}{\in\overline{H}}\frac{d^2\eta(x_0)}{dx_0^2}\right)C^3-C^2=0. \tag{21}$$

or

$$\frac{dC}{dx_0}+\left(1+\frac{1}{\in\overline{H}}\frac{d^2\eta(x_0)}{dx_0^2}\right)C^2-C=0. \tag{22}$$

Eq. (22) is the equation which describes the movement of the cold frontal line. It is the typical type of Bernoulli's equation. Let $C_1=\frac{1}{C}$, then (22) becomes

$$\frac{dC_1}{dx_0}+C_1=1+\frac{1}{\in\overline{H}}\frac{d^2\eta(x_0)}{dx_0^2}. \tag{23}$$

With the initial condition: $t=0$, $C=1$ the solution to (23) can be written as

$$C_1(\bar{x})=1+\frac{1}{\in\overline{H}}e^{-\bar{x}}\int_{\bar{x}_0}^{\bar{x}}e^{x_0}\frac{d^2\eta(x_0)}{dx_0^2}dx_0 \tag{24}$$

or

$$C(\bar{x})=\frac{1}{\left[1+\frac{1}{\in\overline{H}}e^{-\bar{x}}\int_{\bar{x}_0}^{\bar{x}}e^{x_0}\frac{d^2\eta(x_0)}{dx_0^2}dx_0\right]}, \tag{25}$$

where $\bar{x}_0$ is the value of x_0 at $t=0$, $\bar{x}$ is the position of cold front at present time. If the initial position of frontal line is in [illegible]stream of the topography, then $\eta\to 0$, $\frac{d\eta}{dx}\to 0$ at $x=\bar{x}_0$. (25) can be written as

$$C(\bar{x})=\frac{1}{\left[1+\frac{1}{\in\overline{H}}\left(\frac{d\eta(\bar{x})}{d\bar{x}}-\eta(\bar{x})+e^{-\bar{x}}\int_{\bar{x}_0}^{\bar{x}}\eta(x_0)e^{x_0}dx_0\right)\right]}. \tag{26}$$

Eq.(26) determines the movement of cold front. It depends on the following two factors: One is the distribution of the topography, namely, $\frac{d\eta(\bar{x})}{d\bar{x}}-\eta(\bar{x})+e^{-\bar{x}}\int_{\bar{x}_0}^{\bar{x}}\eta(x_0)e^{x_0}dx_0$ (represented by σ hereafter) in which the first term represents the topographic slope, the second represents the topographic height and the third can be viewed as the interaction of the cold frontal motion and topography. The other is dependent on the magnitude of $(\in\overline{H})^{-1}$. The larger the $\in\overline{H}$ is, the lesser modification of the moving speed is.

If η is the piecewise continuous function, then frontal speed can be obtained piecewisely from (25). For example, we take the orographic function as

$$\eta(x)=\begin{cases}\frac{1}{2}\left[1-\cos(2\pi x)\right] & ; \quad x\in[0,1]\\ 0 & ; \quad x\notin\left[0,1\right]\end{cases} \tag{27}$$

which is once used in Davies (1984), then we can find

$$C(\bar{x})=\begin{cases}1 \quad ; \quad \bar{x}<0 & (28)\\ \dfrac{1}{\left[1+\dfrac{2\pi^2}{\epsilon\bar{H}(1+4\pi^2)}(\cos 2\pi\bar{x}+2\pi\sin 2\pi\bar{x}-e^{-\bar{x}})\right]} \quad ; \quad \bar{x}\in[0,1] & (29)\\ \dfrac{1}{\left[1+\dfrac{2\pi^2}{\epsilon\bar{H}(1+4\pi^2)}(e^{-(\bar{x}-1)}-e^{-\bar{x}})\right]} \quad ; \quad \bar{x}>1 & (30)\end{cases}$$

In fact, (29) is the same as that of Davies.

In addition, the expression of $\frac{\partial h}{\partial x}$ at ground can be written as

$$\left.\frac{\partial h}{\partial x}\right|_{X=X_0} = -\frac{1}{C(x_0)}. \tag{31}$$

Thus the frontal slope at ground will vary with the variation of the movement of the front, i. e., the frontal slope will be larger as it moves faster and smaller as it moves slower.

III. THE CHARACTERISTICS OF THE COLD FRONTAL MOTION OVER THE TOPOGRAPHY

In order to study the characteristics of the movement of cold frond over the different topography, we might as well take the following two topographic functions:

$$\eta = e^{-12x^2}, \tag{32}$$

$$\eta = \frac{1}{2}(1-\frac{2}{\pi}\text{arctg}10x), \tag{33}$$

where (32) represents typical isolated orography, while (33) simulates the topography from the high in the west to the low in the east.

If $\Delta\theta=6$, $H=7$ Km, $\theta=300$ K and the value of $\frac{\eta_{max}}{L}$ is taken to be 1 / 450, this means that the slope of orography is around 1 / 225, then the value of C at different places over orography expressed by (32) is shown in Fig 1. The speed C is αU_g when the cold front is far away from orography in the upstream side. It decreases gradually as it moves over the upslope of the mountain and reaches minimum at the point of the upslope where the topographic vorticity is zero according to (25). Then it accelerates and reaches maximum at the around point of the lee side where the topographic vorticity is also zero. Afterward, it begins to reduce and recovers to its initial value in the down–stream of the orography. The maximum speed is found at the lee side and minimum at the upwind side. This shows the retardation of the cold front on the upslope and rapid moving at the lee side, and this rather agrees with the experiences of the weather forecaster.

For the topography expressed by (33), if $\Delta\theta=6$ K, $H=7$ Km, $\theta=300$ K, $\frac{\eta_{max}}{L}=\frac{1}{400}$, then the distribution of speed is shown in Fig.2. The cold front rushes faster as it moves down from the high plain to the lower. The same features can be found in Fig.1.

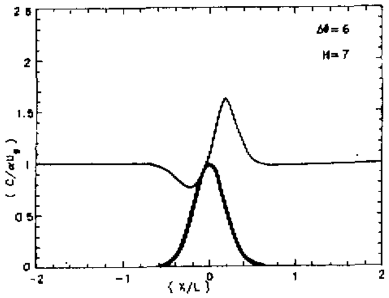

Fig.1. The distribution of nondimensional moving speed. Thick line represents the profile of the topography.

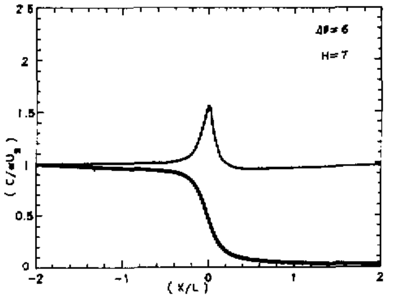

Fig.2. The same as the Fig.1.

Because σ may be less than zero on certain region especially on the downslope. In order to make semi–geostrophic assumption valid, $\epsilon \overline{H}$ must be larger than σ. Thus as far as this model concerned, the cold front can not be too strong or the topographic slope can not be too large. Only in this way, can the solution of semi–geostrophic model be available. Observations (Peng, 1981) show: when a strong cold front approaches a ateep orography, it usually can not cross over the topography and becomes the topographic occluded front because of strong retardation. It is also a reason that we continue to discuss this model.

IV. CONCLUSION

In this paper, a method for calculating the movement of front over the topography has been developed. Two typical topographies, as examples, have been used for study. The results show that there exist the minimum movement and the maximum movement respectively on the upwind slope and downwird slope. They are both located at the point where the topographic vorticity is zero. This provides in some extents the quantitative features on the movement of front over mountain or plateau. Of course, a lot of observational studies are needed to verify the results, which is to be done in the future.

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