A Problem Related to Nonlinear Stability Criteria for Multi–layer Quasi–geostrophic Flow①

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Received June 1, 1991

ABSTRACT

The second author studied the nonlinear stability of N–layer quasi–geostrophic flow subject to perturbations of parameters and initial data, and established the stability criteria for the flow in question, which involve finding out the lowest eigenvalue of an elliptic boundary value problem.

In this paper when the domain is a periodic zonal channel, a formula of the lowest eigenvalue is established, which is useful for further studies and practical applications.

I. INTRODUCTION

In 1965 and 1966, using the variational principle and *a priori* estimates, Arnold (1965,1966) presented a method in studying the nonlinear stability of planar ideal incompressible flow, and obtained two criteria so–called Arnold's first and second theorem. The method has been applied, particularly in recent years, to geophysical fluid dynamics, and fruitful results were obtained.

Generally speaking, the criteria parallel to Arnold's first theorem are not difficult to find for various fluid systems, since the corresponding second derivative of the functional near the equilibrium (that is, stationary state) is sign definite. However, for the criteria parallel to Arnold's second theorem, in order to guarantee the second derivative of functional near the equilibrium to be sign definite, it is usually necessary to find the lowest eigenvalue of an elliptic boundary value problem, see McIntyre and Shepherd (1987), Mu and Zeng (1989), and Mu (1991).

The second author (1991) established the nonlinear stability criteria parallel to Arnold's second theorem for N–layer quasi–geostrophic flow. The criteria involve the lowest eigenvalue of a boundary problem, and the calculation of this eigenvalue is the purpose of this paper. This is not easy to tackle it in general, so we restrict the domain of the flow to be a periodic zonal channel usually considered in geophysical fluid dynamics, and obtain a formula of the lowest eigenvalue. And a lower bound of this value is also given for practical applications.

①Supported by National Natural Science Foundation of China and LASG, Institute of Atmospheric Physics, Chinese Academy of Sciences.

II. STATEMENT OF THE PROBLEM

Consider a stratified fluid of N superimposed layers of constant densities $\rho_1 < \rho_2 < \ldots < \rho_N$, the layers being stacked according to increasing density, such that the density of the upper layer is ρ_1. The motion obeys a system of coupled elliptic equations (cf. Holm (1985, p.32)).

$$\frac{\partial \overline{\omega}}{\partial t} + J(\overline{\psi},\overline{\omega}) = 0 \ , \tag{2.1}$$

$$\overline{\omega} = \nabla^2 \overline{\psi} + AT\overline{\psi} + \overline{f} \ , \tag{2.2}$$

where $\nabla^2 = \partial^2 / \partial x^2 + \partial^2 / \partial y^2$ is the Laplacian, A is an $N \times N$ diagonal matrix

$$A = \mathrm{diag}(\alpha_1, \ldots, \alpha_N) \ , \quad f_0 = 2\Omega \sin\varphi_0 \ ,$$

$$\alpha_k = \frac{f_0^2}{g\left(\dfrac{\rho_{k+1} - \rho_k}{\rho_0}\right) D_i} > 0 \ , \quad k = 1, \ldots, N \ . \tag{2.3}$$

Ω is the Earth's angular velocity, $\varphi_0 \neq 0$ is the reference latitude, g is the gravitational acceleration, $\rho_0 = (\rho_1 + \ldots + \rho_N)/N$ is the mean density, D_k is the mean thickness of the k–th layer. T is a symmetric real tridiagonal matrix

$$T = (T_{jk}) = \begin{bmatrix} -1 & 1 & 0 & & & \\ 1 & -2 & 1 & & 0 & \\ & \cdot & \cdot & \cdot & & \\ & & \cdot & \cdot & \cdot & \\ & 0 & & 1 & -2 & 1 \\ & & & & 1 & -1 \end{bmatrix} . \tag{2.4}$$

A letter with an upper bar indicates an N–dimensional column vector:

$\overline{\psi} = \mathrm{col}(\psi_k)$, $\overline{\omega} = \mathrm{col}(\omega_k)$, $\overline{f} = \mathrm{col}(f_k)$, etc., where ψ_k is the stream function of the k–th layer, ω_k is the generalized vorticity of the k–th layer, and

$$f_k = f_0 + \beta y \ , \quad k = 1, \ldots, N-1,$$

$$f_N = f_0 + \beta y + \frac{f_0 d(x,y)}{D_N} \ ,$$

$$\beta = \frac{2\Omega \cos\varphi_0}{R} \ ,$$

R is the Earth's radius, $d(x,y)$ is the orography.

$$J(\overline{\psi},\overline{\omega}) = \mathrm{col}\left(\frac{\partial \psi_k}{\partial x}\frac{\partial \omega_k}{\partial y} - \frac{\partial \psi_k}{\partial y}\frac{\partial \omega_k}{\partial x}\right) .$$

The domain D is bounded and multiply (or simply) connected with smooth boundary

$$\partial D = \bigcup_{j=0}^{I} \partial D_j \ ,$$

where each ∂D_j is a simple closed curve, and ∂D_0 is the outer boundary.

The boundary conditions are

$$\left.\frac{\partial \overline{\Psi}}{\partial s}\right|_{\partial D}=0\ , \quad \frac{d}{dt}\oint_{\partial D_j}\left(\frac{\partial \overline{\Psi}}{\partial n}\right)ds=0\ , \quad j=0,\ldots,J\ , \tag{2.5}$$

where $\partial\overline{\Psi}/\partial s$ and $\partial\overline{\Psi}/\partial n$ are the tangential and normal derivative of the vector function $\overline{\Psi}$ along ∂D respectively, and s is the arclength along ∂D.

Now let $\overline{\Psi}(x,y)$, $\vec{\omega}(x,y)$ be a stationary vector solution of the unperturbed problem (2.1), (2.2), (2.5), hence

$$J(\overline{\Psi},\ \vec{\omega})=0\ .$$

Then we suppose ψ_k is a function of ω_k , $k=1,\ldots,N$.

$$\psi_k=Q_k(\omega_k)\ ,\quad k=1,\ldots,N,$$

where $Q_k(\xi)$ is a continuously differentiable function of ξ, and

$$\xi\in[\min_D\omega_k,\ \max_D\omega_k]\ .$$

Suppose

$$\min\left(-\frac{\nabla\psi_k}{\nabla\omega_k}\right)=\tilde{C}_{1k}>0\ ,\quad K=1,\ldots,N\ ,$$

$$\max\left(-\frac{\nabla\psi_k}{\nabla\omega_k}\right)=\tilde{C}_{2k}<\infty\ ,\quad K=1,\ldots,N\ ,$$

where

$$\frac{\nabla\psi_k}{\nabla\omega_k}=\frac{\partial\psi_k/\partial x}{\partial\omega_k/\partial x}=\frac{\partial\psi_k/\partial y}{\partial\omega_k/\partial y}\ .$$

Mu (1991) established the following criteria:

The stationary solution $\overline{\Psi},\vec{\omega}$ is Liapunov stable to perturbations of parameters and initial data if

$$\tilde{C}_{1k}\bar{\lambda}_1>1\ ,\quad k=1,\ldots,N\ , \tag{2.6}$$

where $\bar{\lambda}_1$ is the lowest eigenvalue of the following boundary value problem

$$\nabla^2\vec{\varphi}+AT\vec{\varphi}+\lambda\vec{\varphi}=0 \tag{2.7}$$

$$\left.\frac{\partial\vec{\varphi}}{\partial s}\right|_{\partial D}=0\ ,\quad \oint_{\partial D_j}\left(\frac{\partial\vec{\varphi}}{\partial n}\right)ds=0\ ,\quad j=1,\ldots,J\ , \tag{2.8}$$

$$\varphi_1\big|_{\partial D_0}=0\ ,\quad \oint_{\partial D_1}\left(\frac{\partial\vec{\varphi}}{\partial n}\right)ds=\mathrm{col}(c,0,\ldots,0)\ ,$$

where c is an arbitrary constant, λ is the eigenvalue to be determined, and the other notations have been previously mentioned.

However, how to determine $\bar{\lambda}_1$ has not been given in Mu (1991), which is of great importance to theoretical research and practical applications. The aim of this paper is to establish a formula of the lowest eigenvalue $\bar{\lambda}_1$ for periodic zonal channel.

III. PRELIMINARIES

Before embarking on Problems (2.7) and (2.8), let us simplify it by applying the following discussion.

Denote $A^{1/2}=\mathrm{diag}(\alpha_1^{1/2},...,\alpha_N^{1/2})$, a square root of matrix A; $A^{-1/2}=\mathrm{diag}(\alpha_1^{-1/2},...,\alpha_N^{-1/2})$, the inverse of $A^{1/2}$.

Let $M=-A^{1/2}TA^{1/2}$, then we have

Lemma 3.1. There exists an orthogonal matrix L, such that

(1) $L'ML=B=\mathrm{diag}(\lambda_1,...,\lambda_N)$,

(2) $0=\lambda_1<...\lambda_N\leqslant\max\limits_{2\leqslant j\leqslant N-1}(\alpha_{j-1}+2\alpha_j+\alpha_{j+1})$, $N\geqslant 3$,

(3) $\lambda_1=0$, $\lambda_2=\alpha_1+\alpha_2$, $N=2$,

(4) Any element in the first row of L is positive, where L' means the transpose of L.

Proof. By the definitions of T and M , we can see that $M=(M_{jk})$ is a real symmetric tridiagonal matrix with

$$M_{k+1,k}M_{k,k+1}=\alpha_k\alpha_{k+1}>0\ ,\quad k=1,...,N-1\ ,$$

by (2.3), hence, M is a Jacobian matrix. It is proved in matrix theory that an $N\times N$ Jacobian matrix has N distinct real eigenvalues (Franklin, 1968, Sec. 7.11). Thus, let the eigenvalues of M (say, in increasing order) be λ_k , $k=1,...,N$, and the corresponding normalized real eigenvectors be $\vec{L}_k=\mathrm{col}(L_{1k},...,L_{Nk})$, $k=1,...N$. That is,

$$M\vec{L}_k=\vec{L}_k\lambda_k\ ,\quad(\lambda_1<\lambda_2<...<\lambda_n)$$

$$\vec{L}_k^2=\sum_{j=1}^{N}L_{jk}^2=1\ ,\quad k=1,...,N\ . \tag{3.1}$$

Without loss of generality, we may assume that $L_{1k}>0$ if $L_{1k}\neq 0$ (we will show that $L_{1k}\neq 0$, $k=1,...,N$, later).

Put $L=(L_{jk})=(\vec{L}_1,...,\vec{L}_N)$, that is, L is a matrix whose k–th column is $\vec{L}_k$, $k=1,...,N$

By (3.1), $\vec{L},...,\vec{L}_k$ form an orthonormal set:

$$\vec{L}_i'\vec{L}_j=\delta_{ij}=\begin{cases}1\ , & i=j\ ,\\ 0\ , & i\neq j\ ,\end{cases}$$

hence, L is an orthogonal matrix, i.e.,

$$L'L=E(\text{the unit matrix})\ . \tag{3.2}$$

By (3.1), we have

$$ML=LB\ . \tag{3.3}$$

Multiplying both sides of (3.3) on the left by L' and using (3.2), we obtain (1).

Next, we consider matrix $-AT$, which is similar to M , since $-AT=A^{1/2}(-A^{1/2}TA^{1/2})A^{-1/2}=A^{1/2}MA^{-1/2}$.

First note that zero is an eigenvalue of matrix $-AT$, since zero corresponds to an eigenvector col (1,...,1). Then applying Gershgorin's theorems (Franklin, 1968) to $-AT$, (the theorems say, the eigenvalues of an $N\times N$ matrix $S=(S_{jk})$ are in the union of disks centered at S_{jj} with radius $\sum\limits_{k\neq j}|S_{jk}|$, $j=1,...,N$, and also in the union of disks centered at

S_{kk} with radius $\sum_{j\neq k}|S_{jk}|$, $k=1,...,N$.) we obtain (2). While (3) is derived by simple calculation.

Finally, it remains to show that $L_{1k}\neq 0,\ k=1,...,N$, we prove it by contradiction.

Assuming that there were k and $j\geqslant 2$, such that

$$L_{1k}=L_{2k}=...=L_{j-1k}=0\ ,\quad L_{jk}\neq 0\ ,$$

then the $(j-1)$–th component of the right member of (3.1) would be zero by assumption, while the $(j-1)$–th component of the left would be $-(\alpha_{j-1}\ \alpha_j)^{1/2}L_{jk}\neq 0$, which is a contradiction, and this completes the proof of Lemma 3.1.

Denote the first row of matrix L in Lemma 3.1 by

$$\vec{V}=(l_{11},...,L_{1N})\ , \tag{3.4}$$

and let us introduce a transform

$$\vec{\varphi}=A^{1/2}L\vec{p}\ , \tag{3.5}$$

and its inverse

$$\vec{p}=L'A^{-1/2}\vec{\varphi}\ , \tag{3.6}$$

then we have

Lemma 3.2. Problem (2.7), (2.8) is equivalent to the following boundary value problem for the uncoupled equations

$$\nabla^2\vec{p}-B\vec{p}+\lambda\vec{p}=0 \tag{3.7}$$

with boundary conditions

$$\left.\frac{\partial\vec{p}}{\partial s}\right|_{\partial D}=0\ ,\qquad \vec{V}\vec{p}|_{\partial D_0}=0\ ,$$

$$\oint_{\partial D_1}\left(\frac{\partial\vec{p}}{\partial n}\right)ds=c\alpha_1^{-1/2}\vec{V}'\ ,$$

$$\oint_{\partial D_j}\left(\frac{\partial\vec{p}}{\partial n}\right)ds=0\ ,\quad j=1,...,J\ . \tag{3.8}$$

by the transform (3.5), where $\vec{V}$ is defined by (3.4), $\vec{V}'$ is the transpose of $\vec{V}$, and B is the diagonal matrix in Lemma 3.1.

Proof. Substituting (3.5) into (2.7) and multiplying the resulting equality on the left by matrix $L'A^{-1/2}$, then we derive the uncoupled Eqs. (3.7) by Lemma 3.1. The boundary conditions (3.8) are obtained from (2.8) simply by Eqs. (3.5) and (3.6).

Since the transform (3.5) is nonsingular, hence the two problems are equivalent under the transform (3.5). This completes the proof.

IV. THE MAIN RESULTS

In this section, we restrict the domain D to be a periodic zonal channel.

D is periodic in x direction with period 2l, and closed in y direction, $0\leqslant y\leqslant Y$. ∂D_0 is the line $y=0$, and ∂D_1 is the line $y=Y$. (4.1)

We now consider Problem (2.7), (2.8), or equivalently Problem (3.7), (3.8), with $J=1$

and domain (4.1):

$$\nabla^2 \vec{p} - B\vec{p} + \lambda\vec{p} = 0 \ , \quad \left.\frac{\partial \vec{p}}{\partial s}\right|_{y=0} = 0 \ , \quad \left.\frac{\partial \vec{p}}{\partial s}\right|_{y=Y} = 0 \ , \tag{4.2}$$

$$\int_{-l}^{l} \left.\frac{\partial \vec{p}}{\partial n}\right|_{y=Y} ds = 0 \ , \quad \vec{V}\vec{p}|_{y=0} = 0 \ , \quad \int_{-l}^{l} \left.\frac{\partial \vec{p}}{\partial n}\right|_{y=0} ds = c\alpha_1^{-1/2}\vec{V}' \ . \tag{4.3}$$

Let $\vec{p}$ be an eigenfunction of Problem (4.2), (4.3), we write $\vec{p}$ in the form of Fourier series since D is periodic in x direction,

$$\vec{p} = \sum_{k=-\infty}^{\infty} \vec{p}_k(y) e_k(x) \ , \tag{4.4}$$

where $\vec{p}_k(y) = \mathrm{col}(p_{1k}(y), p_{2k}(y), \ldots, p_{Nk}(y))$, and

$$e_k(x) = (2l)^{-1/2} \exp\left(\frac{ik\pi x}{l}\right) , \quad k = 0, \pm 1, \pm 2, \ldots$$

By using (4.4), we change Problem (4.2), (4.3) to a boundary value problem for a system of ordinary differential equations. We state the result as

Lemma 4.1. Problem (4.2), (4.3) is equivalent to the following boundary value problem

$$\vec{p}_k''(y) - B\vec{p}_k(y) + \left(\lambda - \frac{k^2\pi^2}{l^2}\right)\vec{p}_k(y) = 0 \tag{4.5}$$

$$\vec{p}_k(0) = \vec{p}_k(Y) = 0 \ , \quad k \neq 0 \tag{4.6}$$

$$\vec{V}\vec{p}_0(0) = 0 \ , \tag{4.7}$$

$$\vec{p}_0'(0) = c(2l\alpha_1)^{-1/2}\vec{V}' \ ; \quad \vec{p}_0'(Y) = 0 \ , \tag{4.8}$$

by the expansion (4.4), where $\vec{p}_k' = d\vec{p}_k / dy$, $\vec{p}_k'' = d^2\vec{p}_k / dy^2$.

Proof. It is simple and omitted.

By the theory of ordinary differential equations, we have

Lemma 4.2. The eigenvalues of Problem (4.5), (4.6) with $k \neq 0$ are

$$\lambda = \lambda_j + \pi^2\left(\frac{k^2}{l^2} + \frac{n^2}{Y^2}\right) , \quad k,n = 1,2,\ldots, \ ; \ j = 1,\ldots,N \ , \tag{4.9}$$

where λ_j is the j-th diagonal element in matrix $\boldsymbol{B}$.

Now we can give our

Theorem 4.1. The lowest eigenvalue of Problem (2.7), (2.8) with domain (4.1) is equal to the unique root of equation

$$K(\lambda) = 0 \ , \quad \lambda \in (0,m) \ , \tag{4.10}$$

where

$$K(\lambda) = \frac{L_{11}^2 \cot(Y\lambda^{1/2})}{\lambda^{1/2}} - \sum_{j=2}^{N} \frac{L_{1j}^2 \mathrm{cth}(Y(\lambda_j - \lambda)^{1/2})}{(\lambda_j - \lambda)^{1/2}} \ , \tag{4.11}$$

$$m = \min\left(\lambda_2 \ , \frac{\pi^2}{4Y^2}\right) , \quad (L_{11}, \ldots, L_{1N}) = \vec{V} \tag{4.12}$$

Proof. Since any eigenvalue of Problem (4.5), (4.6) with $k \neq 0$ is no less than m by comparison between (4.9) and (4.12). We need only to find the eigenvalue of Problem (4.5), (4.7), (4.8), $(k=0)$ for $\lambda < m$.

First, we solve Problem (4.5), (4.8) $(k=0)$ for the moment, and obtain

$$P_{j0}(y) = -\frac{cL_{1j}(2l\alpha_1(\lambda_j-\lambda))^{-1/2}\mathrm{ch}((Y-y)(\lambda_j-\lambda)^{1/2})}{\mathrm{sh}(Y(\lambda_j-\lambda)^{1/2})}\ , \quad \text{if} \quad \lambda < \lambda_j\ ;$$

$$P_{j0}(y) = \frac{cL_{1j}(2l\alpha_1(\lambda-\lambda_j))^{-1/2}\cos((Y-y)(\lambda-\lambda_j)^{1/2})}{\sin(Y(\lambda-\lambda_j)^{1/2})}\ , \quad \text{if} \quad \lambda > \lambda_j\ . \tag{4.13}$$

$p_{j0}(y) = \text{constant if } \lambda = \lambda_j$ and $c = 0$.

Though we can show that the eigenvalues of Problem (4.5), (4.7), (4.8) $(k=0)$ can not be zero or negative, we need not to do so since the eigenvalues of Problem (2.7), (2.8) are all positive (Mu, 1991). Therefore, let $0 = \lambda_1 < \lambda < m$, $(m \leqslant \lambda_2 < \lambda_3 < \ldots < \lambda_N)$ and let the corresponding solutions of (4.13) satisfy (4.7), we obtain (4.10).

Finally, we shall prove the existence and uniqueness of the root of (4.10).

By Lemma 3.1, L_{11} and L_{12} are positive, then $K(\lambda)$ of (4.11) is strictly decreasing in $\lambda \in (0,m)$, and $K(\lambda)$ tends to positive infinity as λ tends to zero. On the other hand, if $m = \lambda_2 \leqslant \pi^2/(2Y)^2$ then $K(\lambda)$ tends to negative infinity as λ tends to m ; if $m = \pi^2/(2Y)^2 < \lambda_2$, then $K(\lambda)$ tends to negative as λ tends to m . Hence there is one and only one root of (4.10) in $(0,m)$. This completes the proof.

Remark. We can find the matrices B and L of Lemma 3.1 simply by computer (Franklin, 1968; Dubrulle, et al, 1968).

For $N=2$. we calculate B and L explicitly and have

Corollary 4.1. The lowest eigenvalue of Problem (2.7), (2.8) with domain (4.1) is the only root of

$$\frac{\alpha_2\cot(Y\lambda^{1/2})}{\lambda^{1/2}} - \frac{\alpha_1\mathrm{cth}(Y(\alpha_1+\alpha_2-\lambda)^{1/2})}{(\alpha_1+\alpha_2-\lambda)^{1/2}} = 0\ ,$$

$$\lambda \in (0,m)\ , \quad m = \min\left(\alpha_1+\alpha_2\ ,\ \frac{\pi^2}{(2Y)^2}\right).$$

The Formula (4.11) is rather complicated and (4.10) can be only solved by computer. Therefore, it is useful to give a positive lower bound of the root of (4.10) both for computational and application purposes. To do this, we first give

Lemma 4.3. The following inequalities hold;

(1) $\dfrac{\cot x}{x} > x^{-2} - \dfrac{4}{\pi^2}\ , \quad x \in \left(0, \dfrac{\pi}{2}\right)$,

(2) $\dfrac{\coth}{x} < x^{-2} + \dfrac{1}{3}\ , \quad x > 0$,

Proof. (1) Represent $\sin x$ and $\cos x$ in the form of infinite products, we have (Gradshteyn et al., 1980)

$$\sin x = x\prod_{k=1}^{\infty}\left(1-\frac{x^2}{(k\pi)^2}\right),$$

$$\cos x = \prod_{k=0}^{\infty}\left(1-\frac{x^2}{((k+\frac{1}{2})\pi)^2}\right),$$

hence,

$$\frac{\cot x}{x}=\left(\frac{1-4x^2/\pi^2}{x^2}\right)\prod_{k=1}^{\infty}\left(\frac{(1-x^2/(k+\frac{1}{2})\pi)^2}{(1-x^2/(k\pi)^2}\right)>\frac{(1-4x^2/\pi^2)}{x^2}=x^{-2}-\frac{4}{\pi^2}\ ,$$
$$x\in(0,\frac{\pi}{2})\ ,$$

since every factor in the above infinite product is great than one.

(2) Since $8x^2\exp(2x)/3>0$, $(x>0)$, integrating both sides of above inequality from 0 to x successively, we have

$$\frac{2(1-2x+2x^2)\exp(2x)}{3}-\frac{2}{3}>0\ ,$$

$$\left(1-\frac{4x}{3}+\frac{2x^2}{3}\right)\exp(2x)-1-\frac{2x}{3}>0\ ,$$

$$\left(1-x+\frac{x^2}{3}\right)\exp(2x)-1-x-\frac{x^2}{3}>0\ ,$$

hence,

$$\exp(2x)-1>\frac{2x}{\left(1-x+\frac{x^2}{3}\right)}\ ,$$

consequently,

$$\frac{\mathrm{cth}x}{x}=\frac{1+\dfrac{2}{exp(2x)-1}}{x}<x^{-2}+\frac{1}{3}\ ,$$

as required.

Now, note that

$$\frac{\mathrm{cth}(Y(\lambda_j-\lambda)^{1/2})}{Y(\lambda_j-\lambda)^{1/2}}<\frac{\mathrm{cth}(Y(\lambda_j-m)^{1/2})}{Y(\lambda_j-m)^{1/2}}\ ,\quad \text{if}\quad \lambda_j<\lambda<m\ ,$$

then by Lemma 4.3, we have

$$\frac{K(\lambda)}{Y}>L_{11}^2\left(\frac{1}{\lambda Y^2}-\frac{4}{\pi^2}\right)-L_{12}^2\left(\frac{1}{(\lambda_2-\lambda)Y^2}+\frac{1}{3}\right)$$
$$-\sum_{j=3}^{N}L_{1j}^2\frac{\mathrm{cth}(Y(\lambda_j-m)^{1/2})}{Y(\lambda_j-m)^{1/2}}=R(\lambda)\ . \tag{4.14}$$

It is clear that $R(\lambda)=0$ is essentially a quadratic equation of λ , and there is a unique root in $(0,m)$,

$$R(h)=0\ ,\quad h\in(0,m)\ .$$

Hence, $K(h)>0$ by (4.14), therefore h is a lower bound of the root of (4.10), since $K(\lambda)$ is strictly decreasing by Theorem 4.1.

V. CONCLUSION

We have found a formula of the lowest eigenvalue with a lower bound of Problem (2.7), (2.8) in a periodic zonal channel (4.1). There fore the nonlinear stability of the multilayer quasi–geostrophic flow in the zone can be studied by Criterion (2.6) and Formula (4.10).

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