An Easy Algorithm for Solving Radiative Transfer Equation in Clear Atmosphere

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ABSTRACT

An accurate and rapid method for solving radiative transfer equation is presented in this paper. According to the fact that the multiple scattering component of radiance is less sensitive to the error of phase function than the single scattering component is,we calculate the multiple scattering component by using delta-Eddington approximation and the single scattering component by solving radiative transfer equation. On the ground, when multiple sattering component is small, for example, when the total optical depth τ is small, the accurate radiance can be obtained with this method. For the need of the space remote sensing, the upward radiance at the top of the atmosphere is mainly studied, and an approximate expression is presented to correct the multiple scattering component. Compared with the more precise Gauss-Seidel method,the results from this method show an accuracy of better than 10% when zenith angle $\theta < 50°$ and $\tau \leqslant 1$. The computational speed of this method is, however, much faster than that of Gauss-Seidel method.

I. INTRODUCTION

The upward radiance at the top of the atmosphere is important in many aspects in space sensing. For example, the background radiance is main source of the light noise in lidar remote sensing from space (Sun, 1986). On the other hand, the essential atmospheric and aerosol parameters on a global scale can be obtained from satellites by measuring the emergent outward intensities at the top of the atmosphere.

There are many methods for solving the radiative transfer equation. The methods which are usually used are Spherical Harmonics method (Dave, 1975), Discrete-Ordinates method (Liou, 1980), and Gauss- Seidel method (Box and deepak, 1979). In these methods, the intensity is expanded in a Fourier cosine series in terms of azimuth angle, and phase function is expanded in a Legendre polynomial series. Because of the highly asymmetric of the phase functions, the Legendre expansion of phase function has to be expanded in many terms (Wiscombe, 1977; Wang et al, 1990). This may result in many-term expansion of the intensity. While the solutions yield accurate results, they often take a large amount of computer time. Therefore, some authors presented some simple methods, for instance: two-stream method, Eddington method, and delta-Eddington method. But these methods are only suitable for calculating flux densities, but not intensities.

Qiu (1986) studied the effect of Legendre expansion of scattering phase function with a finite number of terms and suggested an improved algorithm for solving radiative transfer equation. He pointed out "through correcting the single scattering component by the exact phase function, an accurate radiance solution can be obtained under Legendre expansion of phase function with less terms". According to his result that the error of the radiance solution is mainly caused by inaccuracy of the single scattering component as expansion terms of phase function are small, this paper tries to find a simpler way to obtain the accurate radiance solution.

In this paper, the radiance solution is obtained by adding the multiple scattering component calculated by delta–Eddington method and the accurate single scattering component. When we calculate the upward radiance at the top of the atmosphere, we also confirm a simple approximate expression to correct the multiple scattering component. The result we get is accurate enough with the error of a few percent when $\theta < 50^\circ$.

II. THEORY AND METHOD

The transfer equation for diffuse solar radiance $I(\tau,\mu,\varphi)$ in a plane–parallel atmosphere is

$$\mu\frac{dI(\tau,\mu,\varphi)}{d\tau} = I(\tau,\mu,\varphi) - \frac{\tilde{\omega}}{4\pi}\int_0^{2\pi}\int_{-1}^{+1} I(\tau,u',\varphi')P(\mu,\varphi,\mu',\varphi')d\mu' d\varphi'$$

$$-\frac{\tilde{\omega}}{4\pi}\pi F_0 P(\mu,\varphi,-\mu_0,\varphi_0)\exp(-\tau/\mu_0) \ , \tag{1}$$

where τ is the optical depth, μ is the cosine of zenith angle, φ is the azimuth angle, $\tilde{\omega}$ is the single scattering albedo, πF_0 is the solar irradiance perpendicular to the direction of incidence, $P(\mu,\varphi,\mu',\varphi')$ is phase function defining the light incident at μ',φ' which is scattered in the direction μ,φ. The second and third terms in the right side are source functions respectively due to multiple scattering and single scattering.

It should be noted that in Eq.(1),positive μ denotes the upward radiation,and negative μ denotes the downward radiation.

We see from the transfer equation that the source function of the single scattering is directly proportional to the phase function, and it is quite sensitive to the expansion error of phase function; the source function of multiple scattering is the product of multiple scattering radiance and multiple scattering phase function to be integrated in turn over 4π steradians with respect to $d\mu d\varphi$, and it is not so sensitive to the error of phase function.Therefore, when phase function is expanded with less terms, the less accurate radiance solution is mainly caused by the error of the single scattering component. Therefore, by adding the exact single scattering radiance to the multiple scattering component obtained under Legendre expansion of phase function with less terms, we may obtain an accurate radiance solution. In this paper, we use delta–Eddington method to calculate the multiple scattering radiance.

Eddington's approximation assumes that the radiance can be given by

$$I(\tau,\mu,\varphi) = I_0 + v\cos\varphi I_x + v\sin\varphi I_y + \mu I_z \ , \tag{2}$$

where $v = \sin\theta$, and I_0, I_x, I_y and I_z are all functions of τ only. The phase function may therefore be approximated by

$$P(\Theta) = 1 + 3g\cos(\Theta) \ , \tag{3}$$

$$\cos(\Theta) = \mu\mu' + (1-\mu^2)^{1/2}(1-\mu'^2)^{1/2}\cos(\varphi - \varphi') \ , \tag{4}$$

where Θ is the scattering angle.

Using (2),(3)and (1), and setting $\frac{\partial I_x}{\partial x} = \frac{\partial I_y}{\partial y} = 0$, we have

$$\frac{dI_0}{d\tau}=(1-\tilde{\omega}g)I_z-\frac{3}{4}\tilde{\omega}g\mu_0F_0\exp(-\tau/\mu_0)\ , \tag{5a}$$

$$\frac{dI_z}{d\tau}=3(1-\tilde{\omega})I_0-\frac{3}{4}\tilde{\omega}F_0\exp(-\tau/\mu_0)\ , \tag{5b}$$

$$I_x=\frac{3\tilde{\omega}g}{4\pi(1-\tilde{\omega}g)}F_0\sin\theta_0\cos\varphi_0\exp(-\tau/\mu_0)\ , \tag{6a}$$

$$I_y=\frac{3\tilde{\omega}g}{4\pi(1-\tilde{\omega}g)}F_0\sin\theta_0\sin\varphi_0\exp(-\tau/\mu_0)\ , \tag{6b}$$

We use delta-Eddington approximation to redefine the asymmetry factor,the single-scattering albedo,and the optical depth in the forms:

$$g'=g/(1+g)\ , \tag{7a}$$

$$\tilde{\omega}'=(1-g^2)\tilde{\omega}/(1-g^2\tilde{\omega})\ , \tag{7b}$$

$$\tau'=(1-\tilde{\omega}g^2)\tau\ . \tag{7c}$$

Assume that there is no diffuse downward radiation at the top of the atmosphere, $I(0,-\mu,\varphi)=0$. The upward radiation at the bottom of the atmosphere can be determined according to the practical conditions.

We may obtain I_0 and I_z from (5a),(5b) and the boundary conditions, and then we may obtain I from (6a), (6b) and (2). In this paper, we use I_E to indicate the radiance calculated by delta-Eddington method.

When only single scattering contribution is considered, the radiative transfer equation is

$$\mu\frac{dI(\tau,\mu,\varphi)}{d\tau}=I(\tau,\mu,\varphi)-\frac{\tilde{\omega}}{4\pi}\pi F_0P(\mu,\varphi,-\mu_0,\varphi_0)\exp(-\tau/\mu_0)\ . \tag{8}$$

The boundary conditions at the top and the bottom of the atmosphere are

$$I(0,-\mu,\varphi)=0\ , \tag{9a}$$

$$I(\tau,\mu,\varphi)=0\ . \tag{9b}$$

We obtain an equation for I (we define it as I_E^s, where "s" stands for single scattering) by using delta-Eddington method:

$$I_E^s=I_0^S+v\cos\varphi I_x^s+v\sin\varphi I_y^s+\mu I_z^s\ , \tag{10}$$

where

$$I_0^s(\tau)=C_1\exp(-\sqrt{3}\tau)+C_2\exp(\sqrt{3}\tau)-\alpha\exp(-\tau/\mu_0)\ , \tag{11a}$$

$$I_z^s(\tau)=-\sqrt{3}[C_1\exp(-\sqrt{3}\tau)-C_2\exp(\sqrt{3}\tau)]+\beta\exp(-\tau/\mu_0)\ , \tag{11b}$$

$$I_x^s=\frac{3\tilde{\omega}g}{4\pi}F_0\sin\theta_0\cos\varphi_0\exp(-\tau/\mu_0)\ , \tag{12a}$$

$$I_y^s=\frac{3\tilde{\omega}g}{4\pi}F_0\sin\theta_0\sin\varphi_0\exp(-\tau/\mu_0)\ , \tag{12b}$$

and

$$\alpha=3\tilde{\omega}F_0\mu_0^2(1+g)/4(1-3\mu_0^2)\ ,$$

$$\beta=3\tilde{\omega}F_0\mu_0(1+3g\mu_0^2)/4(1-3\mu_0^2)\ ,$$

C_1, C_2 are determined by boundary conditions (9a), (9b).

The exact solutions of Eq.(8) at the top and the bottom of the atmosphere are

$$I^s(0,\mu,\varphi)=\frac{\tilde{\omega}\mu_0 F_0}{4(\mu+\mu_0)}P(\mu,\varphi,-\mu_0,\varphi_0)\{1-\exp[-\tau(1/\mu+1/\mu_0)]\} \ . \tag{13a}$$

$$I^s(\tau,-\mu,\varphi)=\begin{cases}\dfrac{\tilde{\omega}\mu_0 F_0}{4(\mu-\mu_0)}P(-\mu,\varphi,-\mu_0,\varphi_0)[\exp(-\dfrac{\tau}{\mu})-\exp(-\dfrac{\tau}{\mu_0})] \ , \ (\mu\neq\mu_0)\\ \dfrac{\tilde{\omega}\tau F_0}{4\mu_0}P(-\mu_0,\varphi_0,-\mu_0,\varphi_0)\exp(-\dfrac{\tau}{\mu_0}) \ , \ (\mu=\mu_0)\end{cases} \tag{13b}$$

From all these, we obtain the intensity solution:

$$I=I_E-I_E^s+I^s \ . \tag{14}$$

In the delta-Eddington method, the phase function is only expanded one term, the error of phase function will influence the accuracy of the multiple scattering result. When the upward radiance at the top of the atmosphere is calculated, the proportion of the multiple scattering component is greater than the single scattering component, and the influence of the calculating error of the multiple scattering radiance can not be neglected. In this paper, we confirm a simple approximate expression to correct the multiple scattering radiance.

The approximate expression is

$$I^m=I_E^m(\mu^*/\mu)^{3.0-4.8\sqrt{\frac{A}{1+A}}} \ , \tag{15}$$

where

$$I_E^m=I_E-I_E^s \ ,$$

$$\mu^*=0.707+(1-\mu_0)\times 0.15\tau \ ,$$

A is the albedo of the ground.

The intensity solution is

$$I=I^m+I^s \ . \tag{16}$$

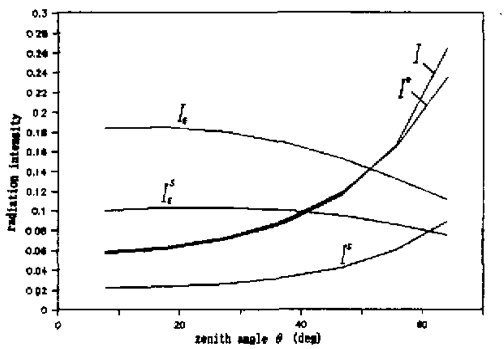

Fig.1. Comparisons of the radiation intensities and their corresponding single scattering components from Gauss-Seidel method I^*, I^s and delta-Eddington method I_E, I_E^s. Also shown is the intensity solution I from Eq.(16).

$F_0 = 1, \quad \tilde{\omega} = 1, \quad \tau = 0.6(\tau_a = 0.5, \tau_m = 0.1), \quad A = 0, \quad \mu_0 = 0.6139, \quad \varphi = 0^0.$

Fig.1 gives an example. It shows the radiation intensities and their corresponding single scattering components, on the top of the atmosphere, calculated by Gauss–Seidel method and delta–Eddington method as a function of zenith angle θ (I^* and I^s are from the Gauss–Seidel method, and I_E and I_E^s are from delta–Eddington method). In addition, it also shows the intensity solution derived from Eq.(16). The optical depth $\tau = 0.6$, the solar zenith angle cosine $\mu_0 = 0.6139, \varphi = 0^0$.

From this figure, we can see that although I_E and I_E^s diverge much from the accurate I^* and I^s, respectively, if we subtract the single scattering components from their corresponding radiation intensities ($I_E^m = I_E - I_E^s$, $I^m = I^* - I^s$), the differences, which are multiple scattering components, are very similar. Through correcting this multiple scattering component I_E^m, we can derive an accurate intensity solution I. When the zenith angle θ is less than 50°, the relative error is less than 3.1%.

III. .COMPARATIVE RESULTS

Table 1 presents the comparison of downward radiation intensities calculated by Eq.(14) and the results presented in the paper of Weinman and Twitty (1975). The problem is assumed to be characterized by $\pi F_0 = 1$, $\tau_a = 0.5$, $\tau_m = 0.145$, $\mu = \mu_0 = 0.966$, the phase functions are obtained from their Table 1 and Eqs.(2a), (2b). From our Table 1, we can see that our results agree well with the exact numerical solutions, the overall accuracy is better than 4%. If we calculated the radiation intensities with delta–Eddington method, the relative error of the results could be 95%. From Table 1,we can also see that the intensity is mainly composed of its single scattering contribution. Therefore, the error of the multiple scattering contribution has little influence on the calculating result of the intensity. We can derive the accurate intensity solution by adding the accurate single scattering component and the multiple scattering component calculated by delta–Eddington method.

Table 1. Comparison of the Exact Intensities and the Ones Calculated from Eq.(14). $\mu = \mu_0 = 0.966$, $\tau_a = 0.5$, $\tau_m = 0.145$

Almucantar azimuth φ(deg)	Scattering angle Θ(deg)	I Exact numerical	I Calculated from Eq.(14)	I Single scatter	ER(%) Relative error
0	0.0	2.3590	2.3271	2.2885	1.4
1	0.3	2.3390	2.3001	2.2615	1.7
2	0.5	2.2820	2.2532	2.2146	1.3
4	1.0	2.0730	2.0514	2.0127	1.0
6	1.6	1.7810	1.7121	1.6735	3.9
8	2.1	1.4700	1.4172	1.3786	3.6
10	2.6	1.1910	1.1577	1.1191	2.8
12	3.1	0.9704	0.9525	0.9139	1.8
14	3.6	0.8112	0.8006	0.7620	1.3
16	4.1	0.7001	0.6907	0.6522	1.3
18	4.6	0.6198	0.6096	0.5711	1.6
20	5.2	0.5574	0.5361	0.4976	3.8
25	6.4	0.4444	0.4329	0.3943	2.6
30	7.7	0.3734	0.3611	0.3226	3.3
35	8.9	0.3239	0.3182	0.2797	1.8
40	10.2	0.2875	0.2859	0.2474	0.6
50	12.6	0.2376	0.2427	0.2043	2.1
60	14.9	0.2044	0.2105	0.1722	3.0
70	17.1	0.1809	0.1861	0.1479	2.9
80	19.2	0.1629	0.1677	0.1296	3.0
90	21.1	0.1487	0.1545	0.1165	3.9

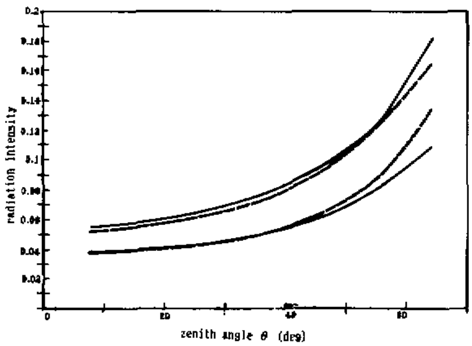

Fig.2a. Comparison of I calculated from Eq.(16) and I^* derived from Gauss-Seidel method for different τ. $F_0 = 1, \tilde{\omega} = 1, \tau = 0.35(\tau_a = 0.25,\ \tau_m = 0.1), 0.6(\tau_a = 0.5, \tau_m = 0.1), A = 0, \mu_0 = 0.4617,\ \varphi = 60°$. Dashed lines indicate the calculating result of Eq.(16), Solid lines represent the Gauss-Seidel solution.

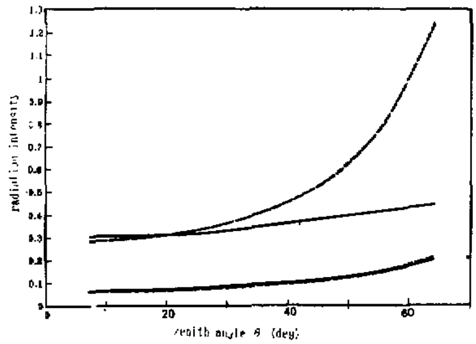

Fig.2b. Comparison of I calculated from Eq.(16) and I^* derived from Gauss-Seidel method for different τ. $F_0 = 1, \tilde{\omega} = 1, \tau = 1.0(\tau_a = 1.0, \tau_m = 0), 4.0(\tau_a = 4.0, \tau_m = 0),\ A = 0,\ \mu_0 = 0.7554, \varphi = 60^0$.

When we calculate the upward radiance at the top of the atmosphere, we apply Eq.(15) to correct the multiple scattering component, and Eq.(16) to calculate the intensity.

The figures below show the upward radiances at the top of the atmosphere.The computations are all made for the analytic Henyey–Greenstein phase function

$$P(\Theta) = \frac{1 - g^2}{(1 + g^2 - 2g\cos\Theta)^{3/2}} \ .$$

The asymmetry factor is given by

$$g = \frac{g_a \tau_a + g_m \tau_m}{\tau_a + \tau_m} \ ,$$

where the aerosol scattering asymmetry factor $g_a = 0.7$, the molecule scattering asymmetry factor $g_m = 0$, τ_a and τ_m are optical depths of aerosol and molecules respectively.

Fig.2a presents the intensity calculated by Gauss–Seidel method and Eq.(16) as the function of zenith angle. The total optical depth $\tau = 0.35, 0.6, \mu_0 = 0.4617$. In this and the following figures, the dashed lines always indicate the calculating results of Eq.(16), whereas the solid lines represent the Gauss–Seidel solutions.

From Fig.2a, we found that the results of Eq.(16) agree very well with the Gauss–Seidel solutions. In the case $\tau = 0.35$, and $\tau = 0.6$, when the zenith angle $\theta < 50^0$, the relative error is less than 5.9% and 5%, respectively.

Fig.2b presents the same case as in Fig.2a, but for $\tau = 1.0, 4.0$, $\mu_0 = 0.7554$. By comparing the results, we can see that the agreement is excellent when $\tau = 1.0$, but the disagreement is significant when $\tau = 4.0$, especially for large zenith angle.

Tables 2a and 2b present the calculating results, as well as the relative errors,corresponding to the cases of Figs.2a and 2b. Here again, we can see that Eq.(16) is feasible when $\tau = 0.35, 0.6$, and 1.0, but when τ is larger, our algorithm needs to be improved.

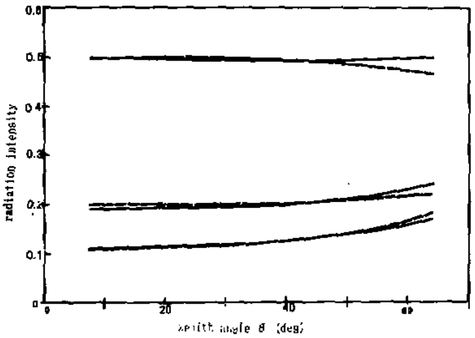

Fig.3. Comparison of I calculated from Eq.(16) and I^* derived from Gauss–Seidel method for different A. $F_0 = 1, \tilde{\omega} = 1, \tau = 0.35(\tau_a = 0.25, \tau_m = 0.1), \mu_0 = 0.4617, \varphi = 60^0$.

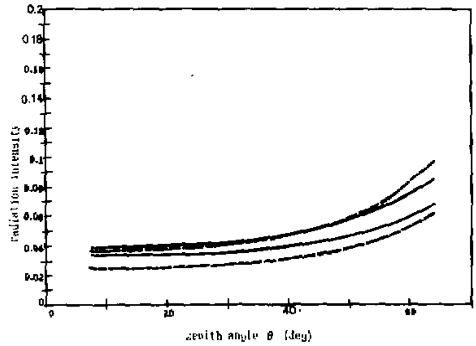

Fig.4. Comparison of I calculated from Eq.(16) and I^* derived from Gauss–Seidel method for different $\tilde{\omega}$. $F_0 = 1, \tilde{\omega} = 1, \tau = 0.35(\tau_a = 0.25, \tau_m = 0.1)$, $A = 0, \mu_0 = 0.7554, \varphi = 60^0$.

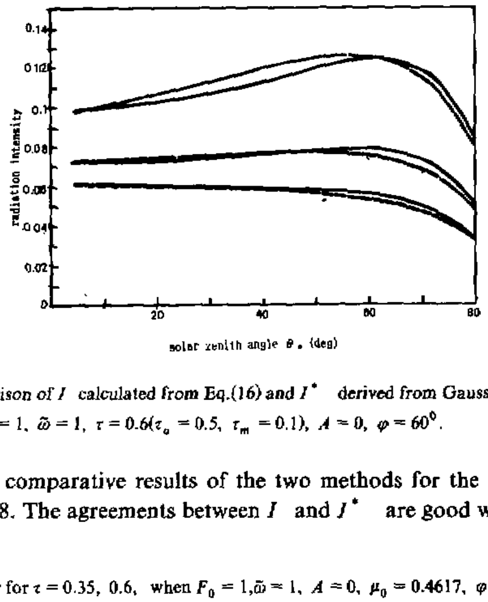

Fig.5. Comparison of I calculated from Eq.(16) and I^* derived from Gauss–Seidel method for different μ. $F_0 = 1$, $\tilde{\omega} = 1$, $\tau = 0.6(\tau_a = 0.5, \tau_m = 0.1)$, $A = 0$, $\varphi = 60^0$.

Fig.3 gives the comparative results of the two methods for the ground surface albedo $A = 0.2, 0.4$, and 0.8. The agreements between I and I^* are good with the errors less than 8% when $\theta < 60^0$.

Table 2a. Relative Error for $\tau = 0.35$, 0.6, when $F_0 = 1, \tilde{\omega} = 1$, $A = 0$, $\mu_0 = 0.4617$, $\varphi = 60^0$

θ (deg)	10	20	30	40	50	60	70
$\tau = 0.35$, ER(%)	2.4	1.2	0.0	1.7	5.9	17.0	52.2
$\tau = 0.60$, ER(%)	5.0	4.9	5.0	4.7	2.4	6.0	34.4

Table 2b. Relative Error for $\tau = 1.0$, 4.0, when $F_0 = 1$, $\tilde{\omega} = 1$, $A = 0$, $\mu_0 = 0.7554$, $\varphi = 60^0$

θ (deg)	10	20	30	40	50	60	70
$\tau = 1.0$, ER(%)	3.6	5.2	7.1	8.4	7.5	0.6	22.4
$\tau = 4.0$, ER(%)	6.0	0.5	9.4	26.8	60.6	135.4	337.5

Fig.4 presents comparisons of I and I^* for the two values of single scattering albedo $\tilde{\omega} = 0.9$, 0.7. It shows that when $\tilde{\omega} = 0.9$, the agreement between I and I^* is again good, the error is less than 8% when $\theta < 50^0$; but when $\tilde{\omega} = 0.7$, the agreement is not as good,the error may reach 27% when $\theta < 50^0$. This indicates that as the absorption of solar radiation increases,the accuracy of our algorithm will decrease.

Fig.5 shows the comparison of the radiation intensities as a function of solar zenith angle for three values of zenith angle cosine $\mu = 0.991$, 0.794, and 0.563. It is seen that the comparison of I and I^* patterns yields rather close agreement. For the three different μ, the errors are less than 7%. This indicates that our method is feasible for almost the full range of solar zenith angles.

IV. SUMMARY AND CONCLUSIONS

In this paper, an easy method for solving radiative transfer equation is presented and numerical tests are given. The agreement of the results show that this method is reasonably correct and feasible.

1. To calculate the downward radiance at the bottom of the atmosphere, when the solar zenith angle is small, and for a small total optical depth,the error of delta-Eddington method solution for multiple scattering component is too small to have much influence on intensity result. We can derive the accurate radiation intensity by using Eq.(14).

2. To calculate the upward radiance at the top of the atmosphere,the error of multiple scattering contribution must be taken into account. We confirm a simple expression to correct the multiple scattering result.By comparing our algorithm results with the Gauss-Seidel ones for several optical depths, surface albedos, single-scattering albedos, and zenith angles, we can see that our results are predicted to an accuracy of better than 10% when $\theta < 50^0$, $\tau \leqslant 1$, and $\tilde{\omega} > 0.7$. This indicates that our method can be effectively utilized in clear atmospheric conditions.

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