An Elegant Coupled Model

Chanchal Guha-Roy and *Dilip Kumar Sinha*
Center for Atmospheric Sciences, University of Calcutta 92, Acharya Prafulla Chandra Road, Calcutta-700009, India
Received July 14, 1992; revised October 9, 1992

ABSTRACT

This paper deals with a new family of coupled wave equations which are basically nonlinear in nature. An analytical study enables us to show that these equations exhibit solitary wave profiles. Finally some remarks are drawn from the standpoint of atmospheric problem.

I. INTRODUCTION

Most of the recent observations reveal that mathematical models are indispensable tools in the field of atmospheric sciences. Nowadays, the study of dispersive effects in coupled nonlinear equations is an important topic in the theory of atmospheric flows (Zeytounian, 1990). Moreover, various coupled systems have been proposed to describe the interaction of long waves with short wave packets in nonlinear dispersive media (Nihoul, 1985). After the remarkable works of Redekopp (1977) and Redekopp & Weidman (1978), it has been established that one can develop KdV equation or modified KdV equation in the atmosphere under certain conditions. These equations, in turn, give rise to the solitary wave solutions. Meteorologists have now investigated blocking high (which generally forms in high or middle latitude zone, and is accompanied by meridional shear of basic wind field in the atmosphere) as solitary wave for a long time (Xu, 1989).

Some time ago, one of us introduced (Guha-Roy, 1987) a generic system of coupled equations and found some interesting wave like solutions. A few extentions of this work have been done by Huibin & Kelin (1990) and Krishnan (1990). The motivation of this paper is to seek a more generalized family of coupled equations which represents the fifth degree KdV-modified KdV equation in the absence of one of the variables. Finally we make some remarks which may be useful from the modelling point of view.

II. MODEL

The set of coupled equations we consider here are

$$w_t + \alpha h^2 h_x + \beta w^2 w_x + \lambda w w_x + \gamma w_{3x} = \mu w_{5x} \ , \tag{1a}$$

$$h_t + \delta (wh)_x + \varepsilon h h_x = 0, \tag{1b}$$

where $w_{kx} = [\partial^k w / \partial x^k]$ and $\alpha, \beta, \lambda, \gamma, \mu, \delta, \varepsilon$ are arbitrary parameters. It should be noted that in Guha-Roy (1987), the effect of the term (w_{5x}) has not been taken into account. It is thus interesting to see how the nonlinearity competes with dispersion effect in the present situation.

As the most interesting feature of the coupled equations (Guha-Roy et al., 1986 & 1990) is its ability to produce steady progressive wave solutions, for analyzing Eq. (1) we assume a travelling wave variable $z=x-\sigma t$ such that

$$w=w(z), \quad h=h(z), \tag{2}$$

where σ is the constant anticipated velocity of the wave.

By the use of Eq. (2), we can write Eq. (1) as

$$-\sigma w_z+\frac{1}{3}\alpha(h^3)_z+\frac{1}{3}\beta(w^3)_z+\frac{1}{2}\lambda(w^2)_z+\gamma w_{3z}=\mu w_{5z}\ , \tag{3a}$$

$$-\sigma h_z+\delta(wh)_z+\frac{1}{2}\varepsilon(h^2)_z=0. \tag{3b}$$

Integrating Eq. (3b) yields

$$h=\frac{2(\sigma-\delta w)}{\varepsilon}, \tag{4}$$

where the constant of integration is set to zero on account of the smoothness condition that w does not blow up as $h\to 0$.

Now eliminating h from Eq. (4) and Eq. (3a) and applying the boundary conditions (Kawamoto, 1984) that w and its higher derivatives with respect to z tend to vanish when $|z|$ approaches to infinity, we arrive at the expression

$$w_{4z}-aw_{2z}-bw^3-cw^2+dw=0, \tag{5}$$

Where a, b, c and d are given by

$$a=(\gamma/\mu),\ b=\frac{1}{3\mu\varepsilon^3}(\beta\varepsilon^3-8\alpha\delta^3),$$

$$c=\frac{1}{2\mu\varepsilon^3}(\lambda\varepsilon^3+16\alpha\sigma\delta^2),$$

$$d=\frac{1}{\mu\varepsilon^3}(\sigma\varepsilon^3+8\alpha\delta\sigma^2).$$

It may be remarked that by employing Krishnan (1990), one can modify our equation (1) as

$$w_t+\alpha h^m h_x+\beta w^m w_x+\lambda w w_x+\gamma w_{3x}=\mu w_{5x}, \tag{6a}$$

$$h_t+\delta(wh)_x+\varepsilon h^{m+1}h_x=0, \tag{6b}$$

where m is a positive integer. It is important to notice that by making use of the arguments described earlier, we can always extract an equation from (6) which is similar to that of Eq. (5). Needless to mention that in such cases the values of the parameters will vary depending on the nonlinear term 'm'.

Let us solve the Eq. (5). Without loss of generality one can seek a solution of w as

$$w(z)=A\,\text{sech}^p(Bz),$$

where A, B and p are to be determined.

Inserting the above ansatz into Eq. (5), we have the following relations:

$$bA^2-120B^4=0, \tag{7}$$

$$cA+120B^4-6aB^2=0, \tag{8}$$

$$4aB^2 - 16B^4 - d = 0, \tag{9}$$

$$p = 2. \tag{10}$$

These relations lead to

$$A = [(3aN - c) / b], \tag{11}$$

$$B = [N(3aN - c) / 2b]^{1/2}, \tag{12}$$

and

$$d = (2N / b^2)[(3aN - c)\ (ab - 6aN^2 + 2cN)] \tag{13}$$

$$\text{with} \quad N = \sqrt{(b/30)}.$$

Hence the exact solitary wave solution of Eq. (1) can be written as

$$w(z) = [A / \cosh^2(Bz)], \tag{14}$$

where A and B are expressed by Eq. (11) and (12) respectively. In Figs. 1–4, we have depicted the wave profiles for various parametric values of a, b and c.

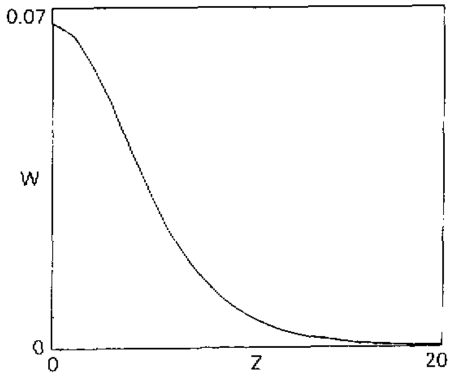

Fig. 1. W versus Z for a = 1, b = 30, c = 1.

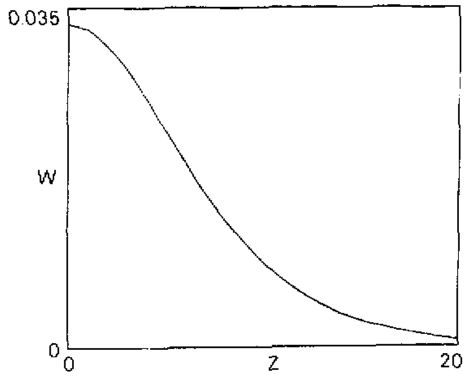

Fig. 2. W versus Z for a = 1, b = 30, c = 2.

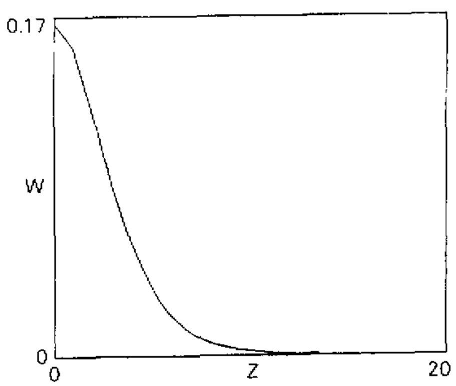

Fig. 3. W versus Z for a = 2, b = 30, c = 1.

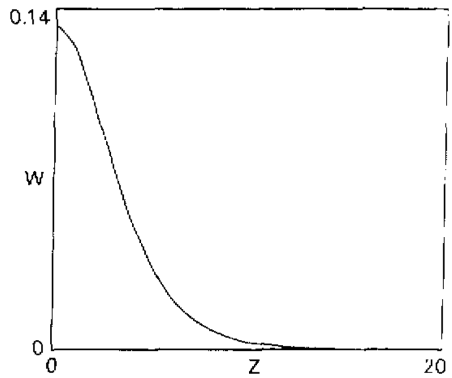

Fig. 4. W versus Z for a = 2, b = 30, c = 2.

III. CLOSING REMARKS

In this paper, we have investigated a class of coupled nonlinear equations that yield solitary wave solutions. Such a result is found to be similar to that obtained by Benney (1979) in the context of an atmospheric model. It may be mentioned that Benney had considered a system of equations concerning the large amplitude Rossby waves and studied the solutions by removing the restriction of weak nonlinearity. For a detailed information, one can go through the paper of Benney (1979). It may be remarked that our equations involve the fifth degree KdV –modified KdV equation which has now some importance in the meteorological problems in relation to the onset of chaos. In addition, it is worthwhile to point out that an increase in nonlinearity in the variable h into Eq. (1a) with a corresponding increase in nonlinearity in the same variable into Eq. (1b) does not distort the existence of the solitary wave pattern. To summarize, it may be emphasized that as we meet coupled wave equations so often in the atmospheric physics, our prescription will be helpful to know the characteristics of the given physical system.

We thank Prof. P.K. Das and Dr. B. Bagchi for useful discussions. One of us (C. Guha–Roy) is also thankful to the Department of Ocean Development, Govt. of India for the financial support.

REFERENCES

Benney, D.J. (1979), *Stud. Appl. Math.*, **60**: 1–10.

Guha–Roy, C. (1987), *J. Math. Phys.*, **28**: 2087–2088.

Guha–Roy, C. & Sinha, D.K. (1990), *Physica Scripta*, **42**: 643–645.

Guha–Roy, C., Bagchi, B. & Sinha, D.K. (1986), *J. Math. Phys.*, **27**: 2558–2560.

Huibin, L. & Kelin, W. (1990), *J. Phys. A: Math. Gen.*, **23**: 4097–4105.

Kawamoto, S. (1984), *J. Phys. Soc. Jpn.*, **53**: 1203–1205.

Krishnan, E.V. (1990), *J. Math. Phys.*, **31**: 1155–1156.

Nihoul, J.C.J. (Ed.) (1985), Coupled Ocean–Atmosphere Models, (Elsevier, Amsterdam / Oxford / New York) 767 pp.

Redekopp. L.G. (1977), *J. Fluid Mech.*, **82**: 725–745.

Redekopp., L.G. & Weidman, P.D. (1978), *J. Atmos. Sci.*, **35**: 790–804.

Xu Xihua (1989), *Adv. Atmos. Sci.*, **6**: 457–466.

Zeytounian, R. (1990), Asymptotic Modelling of Atmospheric Flows, (Springer–Verlag) 396 pp.