Energetics Constraint for Linear Disturbances Development

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ABSTRACT

In development of baroclinic disturbances, baroclinity of basic temperature field varies with conversion of available potential energy. The growth rate which depends on the baroclinity varies as well. However, in previous linear theories, the growth rate was considered constant, so development of disturbances was not constrained by energy sources in the linear theories. In terms of energy conservation and conversion in an isolated atmosphere, we may study the variations in the baroclinity and growth rate and draw the corresponding pictures of perturbation developments in the varying environments. The amplification for the most unstable Eady wave is discussed as an example. It will be found that growth of baroclinic perturbations constrained by energy conservation is significantly different from the growth at the initial constant rate after mature stage.

Key words: Linear disturbance development, Eddy mean–field interaction, APE conversion, Energetics constraint, Eady waves

I. INTRODUCTION

The development of baroclinic disturbances depends on conversion of available potential energy, which is associated with changes in baroclinity and stratification of basic temperature field. This process manifests an interaction between perturbations and basic circulation fields, and is affected by nonlinearities. Due to the mathematical difficulties, studies of nonlinear processes were achieved mostly by numerical procedures.

In analytical studies, while, disturbance development was discussed with linear theories originated by Charney (1947) and Eady (1949). The linear theories are of significance because (Simmons and Hoskins, 1978) the application for explaining disturbance growth may extend over most of the intensification period, though nonlinear effects are important in the mature and decay phases of cyclone life cycles.

In a linear theory disturbances grow exponentially with time. The growth rate is usually taken as a constant. It means that the linear perturbations will grow infinitely with time and so there must exist infinite available potential energy in the atmosphere. This is generally not true. As the basic temperature gradient is reduced by conversion of available potential energy in the real atmosphere, the growth rate which depends on baroclinity varies with time, and development of baroclinic disturbance will be restricted eventually. Therefore, although the asymptotic behavior of disturbance development can be illustrated approximately by linear theory, the energy conservation may be violated in the simplified theory.

To consider the energy conservation in linear theory, it is necessary to be able to evaluate the changes of basic fields as perturbations grow up. The variation in growth rate related to the changes of basic fields was considered by Hart (1971). For the mathematical difficulties, the study was confined to a simplified Eady type disturbance. Also, the changes of basic fields were independent of the disturbance development in his study. So the feedback of perturbations to the basic fields was not considered.

When the primitive equations are divided into the basic state and perturbation equations, interactions between perturbations and basic fields cannot be revealed by solving the two equation sets separately. The connection between disturbance development and variations of basic fields may be revealed in terms of energy conversion in the atmosphere. It is because growth of baroclinic waves depends on conversion of available potential energy which reduces baroclinity of basic temperature field. The basic circulation is then dominated by the mean state equations related to the change of basic thermal structure. If it is constrained to be geostrophic balance, for example, its variation may be evaluated from the thermal wind equation. This energetics constraint on baroclinic disturbance development has never been studied fully and is discussed in the present study.

II. ENERGY CONSERVATION EQUATION

The energy equation for per unit air mass derived from momentum equations is given by

$$\frac{d}{dt}(k+\varphi) = -\alpha\nabla\cdot(p\mathbf{v}) + \alpha p\nabla\cdot\mathbf{v} + \mathbf{v}\cdot\mathbf{F}_r ,$$

where, $k=\mathbf{v}\cdot\mathbf{v}/2$ and $\varphi=gz$ are the specific kinetic and potential energies respectively; $\mathbf{F}_r$ measures frictional force which is in the opposite direction of large scale air motion. Applying the adiabatic thermodynamic energy equation

$$C_v\frac{dT}{dt} + p\frac{d\alpha}{dt} = 0$$

together with continuity equation

$$\frac{d\alpha}{dt} = \alpha\nabla\cdot\mathbf{v}$$

and integrating with respect to time yields

$$d(k+\varphi+C_vT) = -\int_t (\alpha\nabla\cdot(p\mathbf{v}) - \mathbf{v}\cdot\mathbf{F}_r)dt.$$

The whole variation in an isolated atmosphere V reads

$$\int_V \rho d(k+\varphi+C_vT)dV = -\int_V\int_t (\nabla\cdot(p\mathbf{v}) - \rho\mathbf{v}\cdot\mathbf{F}_r)dt\ dV.$$

It follows, in an air column of horizontal section area A that

$$\frac{1}{g}\int_A\int_{p_t}^{p_s} d(k+\varphi+C_vT)dpdA = -\int_t\int_\Omega pv_n\cdot d\Omega\ dt + \frac{1}{g}\int_A\int_{p_t}^{p_s}\int_t \mathbf{v}\cdot\mathbf{F}_r dt\ dp\ dA, \tag{1}$$

where, subscripts s and t denote the variables on the surface and top of the air column, respectively, and v_n the velocity component normal to the outline surface Ω of the air column. In the atmosphere with free boundaries, the first integration on the right hand side does not vanish. For a constant section are A and $w_s=0$ at the lower boundary $z=0$, we have

$$\int_t \int_\Omega p v_n \cdot d\Omega \ dt = \int_t \int_A p_t w_t \ dA \ dt$$

$$= p_t A \int_t [w_t] \ dt = p_t A[\delta z] = W_v ,$$

in which, [ ] signifies the horizontal mean at a given time defined as

$$[\ ] = \frac{1}{A} \int_A \quad dA,$$

and W_v measures the volume work done by the system. Moreover, there is (McHall, 1990a, referred to as M hereafter)

$$\frac{1}{g} \int_{p_t}^{p_s} \int_A d\varphi dA dp = -W_v + \frac{R}{g} \int_{p_t}^{p_s} \int_A dT \ dA dp.$$

Inserting the above two equations into (1) gains

$$dK = -dH + \frac{1}{g} \int_A \int_{p_t}^{p_s} \int_t \mathbf{v} \cdot \mathbf{F}_r dt \ dp dA \tag{2}$$

with

$$dH = \frac{1}{g} \int_A \int_{p_t}^{p_s} C_p dT \ dp dA,$$

in which, K indicates the kinetic energy of air parcels integrated over the whole isolated atmosphere. This equation tells that kinetic energy variation in a dry isolated atmosphere is equivalent to the total enthalpy variation $-\Delta H$, which is generally not identical to the variation of total potential energy

$$-\Delta P = -\frac{1}{g} \int_A \int_{p_t}^{p_s} \Delta(\varphi + C_v T) dp \ dA$$

in an air column with limited height. Thus, the available potential energy may be defined more precisely as the convertible total enthalpy called the available enthalpy.

III. CONVERSION OF AVAILABLE ENTHALPY

The baroclinic potential temperature fields may be represented generally by

$$\theta_1 = [\theta_1](p) + \check{\theta}_1 , \quad \theta_2 = [\theta_2](p) + \check{\theta}_2 ,$$

in which, $\vee$ denotes the departures from the horizontal mean defined previously, which will vanish if temperature field becomes barotropic. The mean fields may be devided further into

$$[\theta_1](p) = \langle \theta_1 \rangle + \theta_1^* (p), \quad [\theta_2](p) = \langle \theta_1 \rangle + \theta_2^* (p),$$

where

$$\langle \theta_1 \rangle = \frac{1}{p_s - p_t} \int_{p_t}^{p_s} [\theta_1] \ dp$$

gives the mean potential temperature of the whole isolated atmosphere. Using these expressions gives

$$-\Delta H = \frac{C_p}{g p_s^{\kappa}} \int_A \int_{p_t}^{p_s} p^{\kappa}(\theta_1 - \theta_2) dp dA$$
$$= \frac{A C_p}{g p_s^{\kappa}} \int_{p_t}^{p_s} p^{\kappa}(\theta_1^* - \theta_2^*) dp \quad (\kappa = \frac{R}{C_p}). \tag{3}$$

It was noted in M that in conversion of available enthalpy in an isolated atmosphere, potential enthalpy is conserved so that

$$\int_A \int_{p_t}^{p_s} (\theta_2 - \theta_1) dp \ dA = 0.$$

It gives

$$\int_{p_t}^{p_s} \theta_2^* \ dp = 0. \tag{4}$$

Moreover, the total entropy production in the isolated atmosphere is evaluated from

$$\Delta S = \frac{C_p}{g} \int_A \int_{p_t}^{p_s} \ln \frac{\theta_2}{\theta_1} dp \ dA.$$

If we write

$$\ln \frac{\theta_2}{\theta_1} = \ln\left(1 + \frac{\theta_2^* + \check{\theta}_2}{\langle \theta_1 \rangle}\right) - \ln\left(1 + \frac{\theta_1^* + \check{\theta}_1}{\langle \theta_1 \rangle}\right)$$
$$\approx \frac{1}{\langle \theta_1 \rangle}(\theta_2^* - \theta_1^* + \check{\theta}_2 - \check{\theta}_1) - \frac{1}{2\langle \theta_1 \rangle^2}(\theta_2^{*2} - \theta_1^{*2} + \check{\theta}_2^2 - \check{\theta}_1^2 + 2\theta_2^* \check{\theta}_2 - 2\theta_1^* \check{\theta}_1).$$

there is

$$\Delta S \approx \frac{C_p}{2g\langle \theta_1 \rangle^2} \int_{p_t}^{p_s} \left(A(\theta_1^{*2} - \theta_2^{*2}) + \int_A (\check{\theta}_1^2 - \check{\theta}_2^2) dA \right) dp. \tag{5}$$

The entropy production cannot be negative in an isolated atmosphere.

For the baroclinic reference field, available enthalpy conversion is computed from (3) under the conditions (4) and (5). To determine the reference field which possesses the least available potential enthalpy, we make the function

$$F = p^{\kappa} \theta_2^* - \lambda_1 \theta_2^{*2} - \lambda_2 \theta_2^*,$$

where, λ_1 and λ_2 are two constant Lagrangian multiplies. Its Euler equation gives

$$\frac{\partial F}{\partial \theta_2^*} - \frac{d}{dp}\left(\frac{\partial F}{\partial \theta_{2_p}^*}\right) = 0, \qquad \theta_{2_p}^* = \frac{\partial \theta_2^*}{\partial p}$$

and follows that

$$\theta_2^* = \frac{p^{\kappa} - \lambda_2}{2\lambda_1}. \tag{6}$$

The magnitudes of λ_1 and λ_2 are now computed by inserting (6) into (4) and (5), producing

$$\lambda_2 = \frac{p_s^{\kappa+1} - p_t^{\kappa+1}}{(\kappa+1)(p_s - p_t)}$$

and

$$\lambda_1 = -\frac{1}{2}\sqrt{F/G}\ ,$$

in which

$$F = \frac{p_s^{2\kappa+1} - p_t^{2\kappa+1}}{2\kappa+1} - \frac{(p_s^{\kappa+1} - p_t^{\kappa+1})^2}{(\kappa+1)^2(p_s - p_t)} \tag{7}$$

and

$$G = \int_{p_t}^{p_s} \left(\theta_1^{*\,2} + \frac{1}{A}\int_A (\check{\theta}_1^2 - \check{\theta}_2^2) dA \right) dp - 2\langle\theta_1\rangle^2 (p_s - p_t)\frac{\Delta s}{C_p}. \tag{8}$$

Here, Δs is the entropy production per unit mass. The evaluation of G may be further referred to the study of McHall (1990b). Consequently, the reference field is rewritten as

$$\theta_2 = \langle\theta_1\rangle + \sqrt{G/F}\left(\frac{p_s^{\kappa+1} - p_t^{\kappa+1}}{(\kappa+1)(p_s - p_t)} - p^{\kappa} \right) + \check{\theta}_2 .$$

When $\check{\theta}_2 = 0$, the barotropic reference field becomes the final or the lowest state discussed in M. The extreme conversion of available enthalpy with respect to this reference field reads

$$-\Delta H = \frac{AC_p}{gp_s^{\kappa}}\left(\sqrt{FG} + \int_{p_t}^{p_s} p^{\kappa}\theta_1^{*}\, dp \right). \tag{9}$$

This equation gives the dependence of available enthalpy conversion or kinetic energy generation on baroclinity and stratification of the reference state.

To give an example, we assume an initial mean temperature field possessing constant lapse rate Γ. The mean potential temperature is given by

$$[\theta_1] = T_s\left(\frac{p_s}{p}\right)^b, \qquad b = \kappa - \frac{R}{g}\Gamma,$$

where, T_s measures surface temperature. The baroclinities of the initial and reference fields are represented by the root–mean–square of potential temperature deviation $\theta_R = \sqrt{[\check{\theta}^2]}$. For $T_s = 280$ K and $\theta_{R_1} = 6$ K, the variation of available enthalpy conversion with reducing baroclinity in an isolated dry atmosphere from 1000 to 200 hPa is displayed in Fig.1. In this diagram, reduction of baroclinity is represented by $\Delta\theta_R = \theta_{R_1} - \theta_{R_2}$, and the amount of conversion is transferred to the mean wind speed (m / s) by

$$\langle v\rangle = \sqrt{\frac{-2g\Delta H}{A(p_s - p_t)}}.$$

The solid and dashed lines are drawn for $\Delta s^{*} = 0$, 0.035 J / (deg^2 • kg), respectively. Here, Δs^{*} measures the entropy variation per unit mass produced in the process of reducing one degree of θ_R.

Fig. 1 shows that baroclinity of basic field is weakened by conversion of available enthalpy. More kinetic energy will be produced in the atmosphere with higher baroclinity and

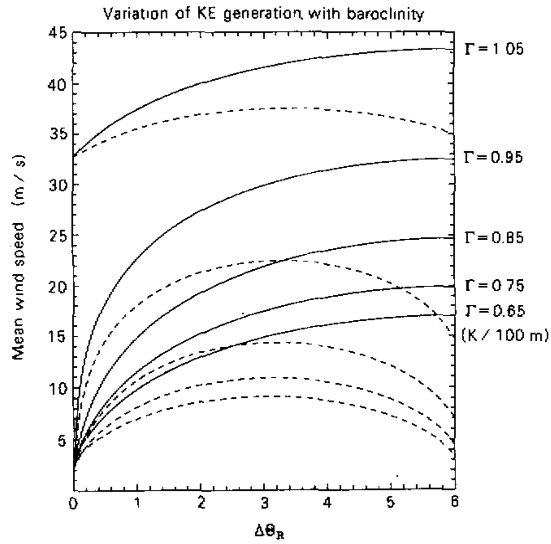

Fig. 1. Variations of the largest available enthalpy conversion with reducing baroclinity of an isolated dry atmosphere. Solid and dashed lines are drawn for $\Delta s^* = 0$, 0.035 J / (deg^2 · kg), respectively.

lower static stability. For a given initial state, kinetic energy increases stably as baroclinity is reduced in a reversible process. However, it may not be the case in irreversible processes with the assumed entropy production. Because dissipation and diffusion destroy available potential energy and so reduce kinetic energy generation, especially when baroclinity has decreased to some extent in disturbance development. The dashed lines show that total generation of kinetic energy in the whole system may be less than the conversion into potential energy in the irreversible process when baroclinity becomes low.

IV. GROWTH OF LINEAR DISTURBANCES

Equation (2) gives an energetics consideration of baroclinic disturbance development. For the linear perturbations derived from linearized equations, the exact energy conservation equation offers a necessary test and constraint for the simplified theory, as discussed in the first section and following.

When the three-dimensional velocity is resolved into a zonal mean (denoted by an overbar) and perturbation portion (denoted by a prime), variation of kinetic energy of per unit air mass is written as

$$\Delta k = \frac{\bar{v}_2^2 - \bar{v}_1^2}{2} + \frac{v'^2_2 - v'^2_1}{2} + \bar{v}_2 v'_2 - \bar{v}_1 v'_1 ,$$

where, v is three-dimensional velocity; subscripts 1 and 2 indicate initial and final states respectively. Integrating it over the whole mass domain produces

$$\Delta K = \Delta \overline{K} + \Delta K' \tag{10}$$

with

$$\Delta \overline{K} = \frac{1}{g} \int_A \int_{p_t}^{p_s} \frac{\bar{v}_2^2 - \bar{v}_1^2}{2} dp \; dA$$

and

$$\Delta K' = \frac{1}{g}\int_A \int_{p_t}^{p_s} \frac{v_2'^2 - v_1'^2}{2} dp\ dA. \tag{11}$$

Consequently, (2) is replaced by

$$\Delta \overline{K} + \Delta K' = -\Delta H + \frac{1}{g}\int_A \int_{p_t}^{p_s} \int_t \mathbf{v} \cdot \mathbf{F}_r dt\ dp dA. \tag{12}$$

In the linear theory, growth of perturbations is represented usually by

$$v' = v_0 e^{\pm \mu t},$$

where, $v_0 = v'|_{t=0}$ and

$$\mu = \mu(\check{\theta}_2,\ a_i) \tag{13}$$

measures growth rate which depends on baroclinity usually (a_i are parameters involved). In the previous studies, it has not been discussed how to choose a sign of the growth rate. In this study, the positive or negative sign will be chosen by the positive or negative conversion of available enthalpy to kinetic energy.

The growth rate is usually considered as a constant, although it may depend on baroclinity which varies as wave grows. To consider the change of growth rate with basic fields, we divide the growth time into n intervals each of which is sufficiently small so that the growth rate in a varying environment may be considered approximately as constant at each time interval. Consequently, the growth of perturbation velocity is rewritten as

$$v'_n = v_0 \exp\left(\pm \sum_{i=1}^{n} \mu_i \Delta t_i \right). \tag{14}$$

Here, subscript i denotes the time mean in a time interval $\Delta t_i = t_i - t_{i-1}$. Inserting (14) into (11) yields

$$\Delta K' = \frac{1}{2g}\int_A \int_{p_t}^{p_s} v_0^2 \left[\exp\left(\pm 2\sum_{i=1}^{n} \mu_i \Delta t_i \right) - 1 \right] dp\ dA. \tag{15}$$

While, variation in kinetic energy of basic field is estimated from (Lorenz, 1955)

$$\frac{d\overline{K}}{dt} = \{\overline{H} \cdot \overline{K}\} + \{K' \cdot \overline{K}\} - \overline{D}, \tag{16}$$

where, $\{\overline{H} \cdot \overline{K}\}$ indicates the conversion of mean available enthalpy to mean kinetic energy; $\{K' \cdot \overline{K}\}$ the conversion of eddy kinetic energy to mean kinetic energy, and

$$\overline{D} = -\frac{1}{g}\int_A \int_{p_t}^{p_s} \overline{\mathbf{v}} \cdot \overline{\mathbf{F}}_r dp\ dA$$

represents zonal mean dissipation. If $\bar{v}_y = \bar{v}_z = 0$, there is $\{\overline{H} \cdot \overline{K}\} = 0$ and so

$$\Delta \overline{K}_i = \frac{\Delta t_i}{g}\int_A \int_{p_t}^{p_s} \left(\overline{v'_{x_i} v'_{y_i}} \frac{\partial \bar{v}_{x_i}}{\partial y} + \overline{v'_{x_i} v'_{z_i}} \frac{\partial \bar{v}_{x_i}}{\partial p} \right) dp\ dA - \overline{D}_i \Delta t_i\ .$$

Here, v_x, v_y and v_z denote velocity components along x, y and z directions respectively. Inserting it together with (15) into (12) yields

$$\int_A \int_{p_i}^{p_r} \left[\left(\overline{v'_{x_i} v'_{y_i}} \frac{\partial \bar{v}_{x_i}}{\partial y} + \overline{v'_{x_i} v'_{z_i}} \frac{\partial \bar{v}_{x_i}}{\partial p} \right) \Delta t_i + \frac{v'^2_{i-1}}{2} (e^{\pm 2\mu_i \Delta t_i} - 1) \right] dp dA$$

$$= g\Delta H_i - gD'_i \Delta t_i \tag{17}$$

in which

$$D'_i = -\frac{1}{g} \int_A \int_{p_i}^{p_s} \overline{\mathbf{v}'_i \cdot \mathbf{F}'_{r_i}} \, dp \; dA$$

represents perturbation dissipation, and from (9)

$$\Delta H_i = \frac{AC_p}{gp_s^{\kappa}} \sqrt{F} \left(\sqrt{G_i} - \sqrt{G_{i-1}} \right). \tag{18}$$

Here, F and G are evaluated from (7) and (8) respectively. It can be proved that $\Delta H_i = H_i - H_{i-1}$. If we know the perturbation velocity components and temperature field in a given area, growth of perturbation which depends on variation of basic field may be evaluated from (17).

Equation (17) includes the variables of perturbation velocities. When growth of perturbation velocities is constrained by energy conversion, the development may be discussed, from energetics of view, by consider the conversion of available enthalpy which depends on perturbation temperature field. Thus, baroclinic disturbance development may eventually be evaluated from variation of temperature perturbations only as shown below.

The eddy kinetic energy generation in linear theory is estimated from (Lorenz, 1955)

$$\frac{dK'}{dt} = -\{K' \cdot \bar{K}\} + \{H' \cdot K'\} - D'. \tag{19}$$

Here, $\{H' \cdot K'\}$ denotes conversion of eddy available enthalpy to eddy kinetic energy. When $\{H' \cdot K'\} \neq 0$, we define the energy conversion ratios as

$$\eta_1 = \frac{\{K' \cdot \bar{K}\}}{\{H' \cdot K'\}}, \quad \eta_2 = \frac{\{\bar{H} \cdot \bar{K}\}}{\{H' \cdot K'\}}, \quad \eta_3 = \frac{D'}{\{H' \cdot K'\}}.$$

Applying them for (19) produces

$$\{H' \cdot K'\} = \frac{1}{1 - \eta_1 - \eta_3} \frac{dK'}{dt}.$$

Thus,

$$\{\bar{H} \cdot \bar{K}\} = \frac{\eta_2}{1 - \eta_1 - \eta_3} \frac{dK'}{dt}.$$

While, conversion of available enthalpy may be divided into (Lorenz, 1955)

$$\frac{dH}{dt} = -\{\bar{H} \cdot \bar{K}\} - \{H' \cdot K'\}$$

in an isolated atmosphere. It may be obtained also by substituting (10), (19) and (16) into (2), giving

$$\frac{dK'}{dt} = -\eta \frac{dH}{dt}, \qquad \eta = \frac{1 - \eta_1 - \eta_3}{1 + \eta_2},$$

or, from (3) and (15),

$$\int_A \int_{p_t}^{p_s} \left\{ \frac{v_0^2}{2} \left[\exp\left(\pm 2 \sum_{i=1}^{n} \mu_i \Delta t_i \right) - 1 \right] - C_p \frac{p^\kappa}{p_s^\kappa} \sum_{i=1}^{n} \eta_i (\theta_{i-1}^* - \theta_i^*) \right\} dp \; dA = 0. \tag{20}$$

If $\{H' \cdot K'\} = 0$, (19) becomes

$$\int_A \int_{p_t}^{p_s} \left(\frac{v_{i-1}'^2}{2g} (e^{\pm 2\mu_i \Delta t_i} - 1) + (\{K' \cdot \overline{K}\}_i + D'_i)\Delta t_i \right) dp \; dA = 0. \tag{21}$$

The consumption of perturbation kinetic energy is then caused by dissipation and conversion into mean kinetic energy.

Equation (17) or (20) is the energetic constraint relationship for development of baroclinic disturbances. The latter depends on temperature perturbation field only. Inserting the baroclinity parameter $\check{\theta}_{2_i}$ or $(\partial\bar{\theta} / \partial y)_i$ into (20), we may solve numerically the Δt_i in a process with specified entropy production. Applying the results for (9), (13) and (14) acquires further the successive values of available enthalpy, growth rate and perturbation velocity at times $t_i = t_{i-1} + \Delta t_i$, respectively. Comparing the obtained results with observations allows us to examine the used theoretical growth rate. An example of using this energetic constraint relationship is shown in next section.

V. EADY WAVE DEVELOPMENT

The growth rate of the most unstable Eady wave is given approximately by (Lindzen and Farrell, 1980; Hoskins, 1983)

$$\mu = 0.31 \frac{g}{N\bar{\theta}} \left| \frac{\partial\bar{\theta}}{\partial y} \right|, \qquad N = \sqrt{\frac{g}{\bar{\theta}} \frac{\partial\bar{\theta}}{\partial z}}.$$

Here,

$$\bar{\theta} = \langle \theta_1 \rangle + \theta^* (p) + \bar{\check{\theta}},$$

in which, $\bar{\check{\theta}}$ is represented by θ_R. For simplicity, we consider an idealized example for which θ_R does not vary with height. From (6), we have

$$\frac{\partial\bar{\theta}}{\partial z} = -\frac{gp}{R\overline{T}} \frac{\partial\bar{\theta}}{\partial p}$$

with

$$\overline{T} = \bar{\theta} \left(\frac{p}{p_s} \right)^\kappa.$$

Thus, we obtain

$$\frac{\partial\bar{\theta}}{\partial z} = -\frac{g p_s^\kappa}{C_p (2\lambda_1 (\langle \theta_1 \rangle + \bar{\check{\theta}}) - \lambda_2 + p^\kappa)}.$$

Since $\bar{\check{\theta}} \ll \langle \theta_1 \rangle$, the previous equation may be replaced by

$$\frac{\partial\bar{\theta}}{\partial z} = -\frac{g p_s^\kappa}{C_p (2\lambda_1 \langle \theta_1 \rangle - \lambda_2 + p^\kappa)}.$$

Furthermore,

$$\mu = 0.31\sqrt{-2C_p\frac{\lambda_1}{p_s^{\kappa}}\left|\frac{\partial\bar{\theta}}{\partial y}\right|}\ , \tag{22}$$

which depends on baroclinity of basic circulation. It is assumed usually that the functional expression of growth rate is unchanged in the period when linear theory is available, though the magnitude may vary.

Supposing further that the half width of disturbance in the direction of mean temperature gradient is Y, we define

$$\theta_R^2 = \frac{1}{2Y}\int_{-Y}^{Y}\left(\frac{\partial\bar{\theta}}{\partial y}y\right)^2 dy = \frac{Y^2}{3}\left(\frac{\partial\bar{\theta}}{\partial y}\right)^2 . \tag{23}$$

It follows that

$$\left|\frac{\partial\bar{\theta}}{\partial y}\right| = \sqrt{3}\frac{\theta_R}{Y} .$$

For Y=1000 km and θ_R=6 K, it gives $|\partial\bar{T}/(\partial y)| \approx 10.3$ K / 1000 km on the surface. This mean temperature gradient over the width of 2000 km is typical for synoptic baroclinic systems.

In general, the conversion ratios vary with time and space. In previous studies, however, only the mean values averaged in an area were evaluated (Oort, 1964; Saltzman, 1970; Sheng and Hayashi, 1990). If we are interested in the gross picture of wave growth, we may use the mean values to discuss the general development over a disturbance domain. In this case, with (3), (20) is replaced by

$$\int_A\int_{p_t}^{p_s} v'^2_{i-1}(e^{\pm 2\mu_i\Delta t_i} - 1)dp\ dA = -2g\langle\eta_i\rangle\Delta H_i . \tag{24}$$

Here, ΔH_i is calculated from (18). When $-\Delta H_i > 0$, we choose the positive sign, otherwise the negative sign is used. Also, v'_{i-1} is calculated from (14), so (24) does not require the particular information about velocity perturbation fields after initial time.

The values of conversion ratios may be obtained from previous studies. Song (1971) studied numerically the energetics of unstable disturbances in variant zonal currents which containing both vertical and horizontal shears with a linear quasi-geostrophic model. The $\langle\eta_1\rangle$ shown by his energy cycle integrated over the entire mass for the baroclinic disturbances of wavelength 4000 km was generally higher than the climatological values (Oort, 1964; Saltzman, 1970; Oort and Peixoto, 1974; Tenenbaum, 1976; Tomatsu, 1979). In the barotropic disturbances, this ratio may be negative (Song, 1971; Brennan and Vincent, 1990). The ratio η_2 was zero in his model. The dissipation of eddy kinetic energy was not provided explicitly in the study. However, it may be estimated by using (19) for his numerical results The mean $\langle\eta\rangle$ calculated from his three baroclinic experiments BC, AS and PB is about 0.498.

To examine the development of linear disturbances, we may select a representation of the growth rate near, for example, the central area of disturbances. Also, we may discuss the growth of mean perturbation velocity defined as $\langle v'\rangle = \sqrt{2\langle k'\rangle}$, where k' measures specific perturbation kinetic energy. For this case, (24) is rewritten as

$$\langle v'_{i-1}\rangle^2(e^{\pm 2\mu_i\Delta t_i} - 1)A(p_s - p_t) = -2g\langle\eta_i\rangle\Delta H_i \tag{25}$$

and follows that

$$e^{\pm \mu_i \Delta t_i} = \sqrt{1 - \frac{2g\langle \eta_i \rangle \Delta H_i}{A(p_s - p_t)\langle v'_{i-1} \rangle^2}}\,.$$

Thus we obtain, from (14),

$$\langle v'_i \rangle = \langle v'_{i-1} \rangle \sqrt{1 - \frac{2g\langle \eta_i \rangle \Delta H_i}{A(p_s - p_t)\langle v'_{i-1} \rangle^2}}\,. \tag{26}$$

Inserting the baroclinity parameter θ_{2_i} into this equation gives the perturbation velocity varying with baroclinity of basic field. From

$$t_i = t_{i-1} \pm \frac{1}{\mu_i} \ln \frac{\langle v'_i \rangle}{\langle v'_{i-1} \rangle},$$

we obtain the time variation in baroclinity of the basic field, or $\check{\theta}_{2_i} = \check{\theta}_2(t_i)$.

Thus, (26) tells the time dependence of the perturbation. Furthermore, (13) and (18) give, respectively, the time variations of growth rate and available enthalpy, namely $\mu_i = \mu(t_i)$ and $\Delta H_i = \Delta H(t_i)$. When the available enthalpy is exhausted, (21) gives

$$e^{\pm \mu_i \Delta t_i} = \sqrt{1 - 2g\Delta t_i \frac{\langle \{K' \cdot \overline{K}\}_i \rangle + \langle D'_i \rangle}{A(p_s - p_t)\langle v^2_{i-1} \rangle}}\,.$$

Thus,

$$\langle v_i \rangle = \langle v_{i-1} \rangle \sqrt{1 - 2g\Delta t_i \frac{\langle \{K' \cdot \overline{K}\}_i \rangle + \langle D'_i \rangle}{A(p_s - p_t)\langle v^2_{i-1} \rangle}}\,. \tag{27}$$

For the initial field and $\langle \eta_i \rangle$ assumed previously with $\Gamma = 0.65$ K / 100 m, the calculated time variations of mean horizontal temperature gradient in the reversible process of $\Delta s^* = 0.035$ J / (deg^2 • kg) are displayed by the dashed lines in Fig. 2. The solid lines present the e-folding time $\gamma = 1 / \mu$. It is shown that the basic baroclinity decreases at an

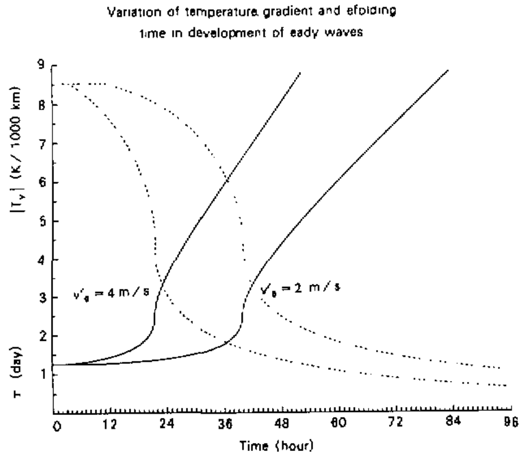

Fig. 2. Variations of mean temperature gradient (dashed) and e–folding time (solid) in an irreversible process ($\Delta s^* = 0.035$ J / (deg^2 • kg)) of Eady wave development.

increasing rate in the early stage of wave development. The decrease is slowed down in the late stage. For the same entropy production, conversion of available enthalpy is more efficient when the perturbation has greater initial velocity. The e–folding time increases considerably with decreasing baroclinity, especially in enhanced perturbations.

The growths of the mean perturbation velocity calculated from (26) with different initial values are depicted in Fig. 3. Since kinetic energy generation evaluated from (9) is the largest in the assumed irreversible process, this figure shows the fastest growths of unstable perturbations in the process. As suggested also by Fig. 1, wave development in an irreversible process is damped by diffusion and dissipation, so that perturbations will decay after baroclinity has been greatly reduced. In the decaying process, kinetic energy generation is less

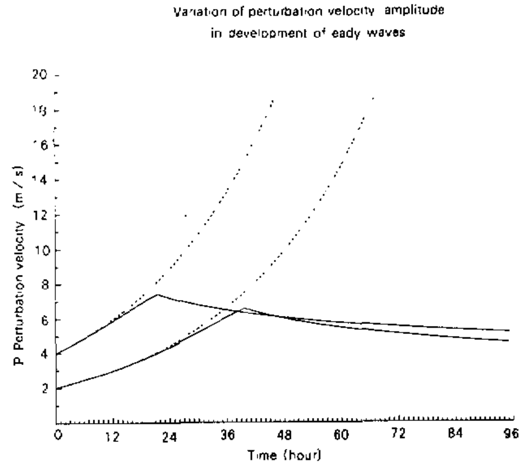

Fig. 3. Growths of the mean perturbation velocity of Eady wave in an irreversible processes of ($\Delta s^* = 0.035$ J / (deg^2 · kg)). The dashed lines represent the growth at constant initial rate.

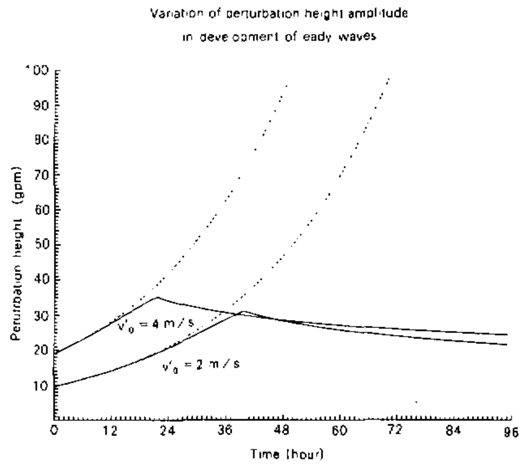

Fig. 4. The same as Fig. 3 but for growths of perturbation geopotential height. The dashed lines represent the growth at constant initial rate.

than the conversion into potential energy. This damping process may be principally similar to the damped Eady edge wave shown by Farrell (1985).

If other conditions are the same, perturbations with larger initial velocity grow more rapidly, and decay at an earlier time. The rate of decay is smaller than that of the fastest development. for comparison, the wave growths at constant initial rate are depicted by the dashed lines in the figure. The main difference is found after mature stage.

For quasi–geostrophic disturbances, we have

$$v'_x \sim -\frac{1}{f_0}\frac{\partial\varphi'}{\partial y} \sim \frac{n_y}{f_0}\varphi', \qquad v'_y \sim -\frac{1}{f_0}\frac{\partial\varphi'}{\partial x} \sim \frac{n_x}{f_0}\varphi', \tag{28}$$

where, n_x and n_y are zonal and meridional wavenumber respectively. Supposing that n_x and n_y are similar in cyclonic disturbances, and perturbation kinetic energy is contributed mainly by horizontal components, we see

$$\varphi' \sim \frac{f_0}{\sqrt{2}n_x} v'.$$

If wavelength of the unstable wave is twice the width of disturbance, there are $2\pi / n_x = 4Y$ and

$$\varphi' \sim \sqrt{2} f_0 Y \frac{v'}{\pi}.$$

The later is equivalent to

$$\varphi' = \varphi_0 \exp\Big(\pm \sum_{i=1}^{\tau} \mu_i \Delta t_i \Big),$$

where, φ_0 denotes the initial perturbation geopotential given by

$$\varphi_0 = \sqrt{2} f_0 Y \frac{v_0}{\pi}.$$

The growth of geopotential perturbation at midlatitude are sketched in Fig. 4. Again, development of disturbance constrained by energy conservation (the solid lines) is significantly different from the growth at a constant initial rate (the dashed lines) after mature stage.

VI. DISCUSSIONS

In the linear theories, unstable disturbances grow exponentially at a constant rate. This is different from the satuation in the real atmosphere, since growth of disturbances is constrained by kinetic energy source. this energetics constraint for baroclinic disturbance development can be studied in terms of conversion of available enthalpy called traditionally the available potential energy. This study gives generally the quantitative linkage between baroclinic perturbation development and energy conversion in an isolated atmosphere. By dividing the growth process into several time intervals, we may consider the time variations in baroclinity of basic temperature field and growth rate of perturbations, and draw the pictures of perturbation development constrained by energy conservation in the processes of variant irreversibilities. This method can be used to examine a baroclinic instability theory by comparing with observations.

We have discussed, as an example, the mean development of a simplified model of Eady wave. The results show that development of disturbance constrained by energy conversion in an isolated atmosphere is restricted by reducing baroclinity and so cannot reach an infinitely large amplitude as predicted by linear theory. In an irreversible process, wave growth is damped by diffusion and dissipation, so that disturbances may decay after baroclinity has been reduced to some extent, but not grow constantly until available enthalpy is entirely exhausted as in a reversible process. The maximum amplitude of disturbance depends not only on the initial perturbation fields and available potential energy, but also on irreversibility of process. The wave growth constrained by energy conversion is greatly different from the growth at a constant rate after mature stage.

The present study does not intend to replace the nonlinear study of disturbance development. It forwards an additional consideration of energetics constraint on wave development. This constraint exists also for nonlinear processes. Since it is generally difficult even in a nonlinear model to evaluate precisely the dissipation and diffusion, the concept of entroy production is particularly useful for us to estimate the statistical effects of dissipation and diffusion without considering the details of irreversible processes.

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