Constructions and Applied Examinations of a Kind of Square–Conservative Schemes in High Precision in the Time Direction

Ji Zhongzhen (季仲贞) and *Wang Bin* (王 斌)
LASG, Institute of Atmospheric Physics, Chinese Academy of Sciences, Beijing 100080
Received September 11, 1992; revised November 3, 1992

ABSTRACT

In order to meet the needs of work in numerical weather forecast and in numerical simulations for climate change and ocean current, a kind of difference scheme in high precision in the time direction developed from the completely square–conservative difference scheme in explicit way is built by means of the Taylor expansion. A numerical test with 4–wave Rossby–Haurwitz waves on them and an application of them on the monthly mean current the of South China Sea are carried out, from which, it is found that not only do the new schemes have high harmony and approximate precision but also can the time step of the schemes be lengthened and can much computational time be saved. Therefore, they are worth generalizing and applying.

Key words: Completely square–conservative, Explicit scheme, *High precision in the time direction*, *Harmonious* dissipative operator

I. INTRODUCTION

Numerical weather forecast and numerical simulations for climate change and ocean current belong to problems of low–speed revolving fluid dynamics, which need long–time numerical integrations. To meet their special character, implicit and explicit schemes completely keeping the square conservatism have been developed successively and many successful results obtained. In real work, however, it is found that there might be computational chaos sometimes in some parts of the integration region, with which a long–time integration might be broken off, although the scheme used is completely square– conservative and the nonlinear computational instability can be avoided. Smoothing and filtering are not good ways to solve the problem because with them the complete square conservatism of the scheme may be broken up to some extent, although the computational results may be improved. Therefore, further analyses and tests on the aforesaid schemes are carried out and the direct relation between the local computational chaos and the approximate precision of the schemes is found. Usually, the time–direction precision of traditional difference schemes which are completely square–conservative two–order. Can a difference scheme not only having a higher precision in the time direction but also completley keeping the square conservatism be constructed? The answer is affirmative. In this paper, we try to analyze and discuss this problem in detail. Also, the new schemes constructed in this paper are tested numerically and the results are satisfactory, which indicate a bright future for the further developments and applications of them.

II. COMPLETELY SQUARE–CONSERVATIVE DIFFERENCE SCHEMES IN EXPLICIT WAY

A time evolutionary equation in operator form

$$\begin{cases} \dfrac{\partial F}{\partial t} + \bar{L}F = 0, \\ \lim\limits_{t \to 0} F = F^{(0)}(x) \ , \end{cases} \tag{1}$$

is considered, where $\bar{L} = \bar{L}(F,x,t)$ is a nonlinear or a linear operator, $F \equiv F(x,t)$ is the undetermined function.

It is supposed that the two–dimensional coordinate space $(x, \ t)$ is scattered into a grid space and the discrete function is $F(jh,n\tau) = (F)_j^n$, of which the inner product and the norm are defined in the similar way of Zeng et al. (1981).

The discrete function $(F)_j^{n+1}$ can be expanded into the Taylor's series

$$(F)_j^{n+1} = (F)_j^n + \tau\left(\frac{\partial F}{\partial t}\right)_j^n + \frac{\tau^2}{2!}\left(\frac{\partial^2 F}{\partial t^2}\right)_j^n + \cdots, \tag{2}$$

and the finite terms of Eq.(2) are selected to form the following equation

$$\frac{F_j^{n+1} - F_j^n}{\tau} = \left(\frac{\partial F}{\partial t}\right)_j^n + \varepsilon\tau\left(\frac{\partial^2 F}{\partial t^2}\right)_j^n, \tag{3}$$

in which ε is an undetermined parameter. The two equalities

$$\frac{\partial F}{\partial t} = -\bar{L}F, \qquad \frac{\partial^2 F}{\partial t^2} = -\frac{\partial \bar{L}F}{\partial t}, \tag{4}$$

which can be derived from Eq.(1) easily, are substituted for (3), and the following equation

$$\frac{F_j^{n+1} - F_j^n}{\tau} = -(\bar{L}F)_j^n - \varepsilon\tau\left(\frac{\partial \bar{L}F}{\partial t}\right)_j^n \tag{5}$$

is obtained. If $(\bar{L}F)_j^n$ is replaced by the corresponding difference form $(LF)_j^n$, the difference scheme

$$\frac{F_j^{n+1} - F_j^n}{\tau} = -(LF)_j^n - \varepsilon_n\tau\left(\frac{\delta LF}{\delta t}\right)_j^n \tag{6}$$

can be constructed based on Eq.(5). Furthermore, Eq.(6) can be rewritten into a general form

$$\frac{F_j^{n+1} - F_j^n}{\tau} = -(LF)_j^n - \varepsilon_n\tau(BF)_j^n \tag{7}$$

or into an equal and simple form

$$\frac{F_j^{n+1} - F_j^n}{\tau} = -(AF)_j^n \ , \tag{8}$$

where $(AF)_j^n = (LF)_j^n + \varepsilon_n\tau(BF)_j^n$ and B is a dissipative operator.

The following theorems proved by Ji et al. (1991) and Wang et al. (1990) show how to construct a completely square–conservative difference scheme in explicit way.

Theorem 1: If the expression:

$$\tau\|A_n F^n\|^2 - 2(A_n F^n, F^n) = 0 \tag{9}$$

is true, Scheme (8) is a completely square–conservative difference scheme.

Theorem 2: If L_n is an a–symmetrical operator, B is a positive definite operator, $(B_n F)^n \leqslant O(1)$ and $2K_3\frac{\tau}{h} < 1$, then Scheme (7) is a completely square–conservative difference scheme in explicit way with a constant time step when

$$\varepsilon_n = K_1 / \left[\left(1 - \frac{\tau}{h}K_2\right) + \sqrt{\left(1 - \frac{\tau}{h}K_2\right)^2 - \left(\frac{\tau}{h}K_3\right)^2}\ \right] , \tag{10}$$

where

$$\begin{cases} K_1 = \|L_n F^n\|^2 / (B_n F^n, F^n) , \\ K_2 = (B_n F^n, L_n F^n)h / (B_n F^n, F^n) , \\ K_3 = \|B_n F^n\| \cdot \|L_n F^n\| h / (B_n F^n, F^n) . \end{cases} \tag{11}$$

III. THE HARMONIOUS DISSIPATIVE OPERATORS AND THE PRECISION OF A DIFFERENCE SCHEME

Various difference schemes compatible with Eq.(1) can be established if more terms of Series (2) are chosen. The operator B relative to each of the schemes is called a harmonious dissipative operator. B is called one–order harmonious dissipative operator if determined by

$$(BF)^n = -\left(\frac{\partial^2 F}{\partial t^2}\right)^n + O(\tau) . \tag{12}$$

Generally, B is called an m–order harmonious dissipative operator if it satisfies

$$(BF)^n = -2\left[\frac{1}{2!}\frac{\partial^2 F}{\partial t^2} + \frac{\tau}{3!}\frac{\partial^3 F}{\partial t^3} + \cdots + \frac{\tau^{m-1}}{(m+1)!}\frac{\partial^{m+1} F}{\partial t^{m+1}}\right]^n + O(\tau^m) . \tag{13}$$

From the definition above, the following theorems can be easily testified.

Theorem 3: Scheme (7) constructed according to Theorem 2 is of two–order precision in the time direction when B is a one–order harmonious dissipative operator, i.e.,

$$(BF)^n = -\left(\frac{\partial^2 F}{\partial t^2}\right)^n + O(\tau) . \tag{14}$$

Theorem 4: Scheme (7) derived from Theorem 2 is of three–order precision in the time direction when B is a two–order harmonious dissipative operator, i.e.,

$$(BF)^n = -\left(\frac{\partial^2 F}{\partial t^2} + \frac{\tau}{3}\frac{\partial^3 F}{\partial t^3}\right)^n + O(\tau^2) . \tag{15}$$

Corollary 1: If B is defined by the following expression:

$$(BF)^n \approx -\left(\frac{\partial^2 F}{\partial t^2}\right)^n \approx \left(\frac{\partial LF}{\partial t}\right)^n \approx \frac{LF^{n+1} - LF^n}{\tau} = -\left(\frac{\partial^2 F}{\partial t^2} + \frac{\tau}{2}\frac{\partial^3 F}{\partial t^3}\right)^n + O(\tau^2) \ , \tag{16}$$

then Scheme (7) is of two–order precision in the time direction and B is a one–order harmonious dissipative operator.

Corollary 2: If B is determined by

$$(BF)^n \approx -\left(\frac{\partial^2 F}{\partial t^2}\right)^n \approx \left(\frac{\partial LF}{\partial t}\right)^n \approx \frac{LF^n - LF^{n-1}}{\tau} - \left(\frac{\partial^2 F}{\partial t^2} - \frac{\tau}{2}\frac{\partial^3 F}{\partial t^3}\right)^n + O(\tau^2) \ , \tag{17}$$

then Scheme (7) is of two–order precision in the time direction and B is a one–order harmonious dissipative operator.

Corollary 3: If B is set to be

$$(BF)^n \approx -\left(\frac{\partial^2 F}{\partial t^2}\right)^n \approx \left(\frac{\partial LF}{\partial t}\right)^n \approx \frac{LF^{n+1} - LF^{n-1}}{2\tau} - \left(\frac{\partial^2 F}{\partial t^2}\right)^n + O(\tau^2) \ , \tag{18}$$

then Scheme (7) is of two–order precision in the time direction and B is a one–order harmonious dissipative operator.

Corollary 4: Scheme (7) is of three–order precision in the time direction and B is a two–order harmonious dissipative operator when it satisfies

$$(BF)^n \approx -\left(\frac{\partial^2 F}{\partial t^2}\right)^n \approx \left(\frac{\partial LF}{\partial t}\right)^n \approx \frac{5}{6}\left(\frac{LF^{n+1} - LF^n}{\tau}\right) + \frac{1}{6}\left(\frac{LF^n - LF^{n-1}}{\tau}\right)$$
$$= \frac{5LF^{n+1} - 4LF^n - LF^{n-1}}{6\tau} = -\left(\frac{\partial^2 F}{\partial t^2} + \frac{\tau}{3}\frac{\partial^3 F}{\partial t^3}\right)^n + O(\tau^2) \ . \tag{19}$$

It is worth noting that the term LF^{n-1} in the corollaries above is difficult to calculate directly in real computations. Therefore, an approximate way to calculate it is adopted here

$$\begin{cases} F^* = F^n - \tau LF^n \\ \tilde{F}^{n-1} = F^n - \tau L\left(\dfrac{F^n + F^*}{2}\right) . \end{cases} \tag{20}$$

It is found in real computations that there is little effect on the precision of the scheme and on the order of harmonious dissipation of the operaor B although F^{n+1} is replaced by $\tilde{F}^{n+1}$.

IV. THE CONSTRUCTION OF HARMONIOUS DISSIPATIVE OPERATOR

In this secton, an effective method, Runge–Kutta method, is cited and generalized to construct a kind of practical and useful harmonious dissipative operator. The details are shown in the following theorems and corollaries which can be easily proved.

Theorem 5: The operator B

$$BF = -\frac{2[\varphi(F,\tau) - R_1]}{\tau} \tag{21}$$

is a one–order harmonious dissipative operator if

$$\varphi(F,\tau) = C_1 R_1 + C_2 R_2 \ , \tag{22}$$

where

$$R_1 = -LF, \quad R_2 = -L(F + b_{21}\tau R_1) \tag{23}$$

and the coefficients C_1, C_2 and b_{21} obey

$$C_1 + C_2 = 1, \quad C_2 b_{21} = \frac{1}{2} . \tag{24}$$

Corollary 1: In Theorem 5, if set $C_1 = 0$, $C_2 = 1$ and $b_{21} = \frac{1}{2}$, then the operator B satisfies:

$$BF = \frac{2(LF^* - LF)}{\tau}, \quad F^* = F - \frac{1}{2}\tau LF, \tag{25}$$

and is called a centred Euler operator.

Corollary 2: In Theorem 5, if set $C_1 = \frac{1}{2}$, $C_2 = \frac{1}{2}$ and $b_{21} = 1$, then the operator B satisfies

$$BF = \frac{(L\tilde{F} - LF)}{\tau}, \quad \tilde{F} = F - \tau LF . \tag{26}$$

and is called an improved Euler operator.

Theorem 6: If set

$$\varphi(F,\tau) = C_1 R_1 + C_2 R_2 + C_3 R_3 , \tag{27}$$

where

$$\begin{cases} R_1 = -LF , \\ R_2 = -L(F + b_{21}\tau R_1) , \\ R_3 = -L(F + b_{31}\tau R_1 + b_{32}\tau R_2) , \end{cases} \tag{28}$$

and the coefficients C_1, C_2, C_3 and b_{21}, b_{31}, b_{32} satisfy

$$\begin{cases} C_1 + C_2 + C_3 = 1 , \\ C_2 b_{21} + C_3(b_{31} + b_{32}) = \frac{1}{2} , \\ C_2 b_{21}^2 + C_3(b_{31} + b_{32})^2 = \frac{1}{3} , \\ C_3 b_{21} b_{32} = \frac{1}{6} , \end{cases} \tag{29}$$

then the operator B defined by (21) is a two–order harmonious dissipative operator.

Corollary 3: In Theorem 6, if set $C_1 = \frac{1}{4}$, $C_2 = 0$, $C_3 = \frac{3}{4}$, $b_{21} = \frac{1}{3}$, $b_{31} = 0$, and $b_{32} = \frac{2}{3}$, then the operator B satisfies

$$BF = \frac{3}{2} \cdot \frac{L\left(F + \frac{2}{3}\tau R_2\right) - LF}{\tau} , \tag{30}$$

where

$$R_2 = -L\left(F - \frac{1}{3}\tau LF\right) , \tag{31}$$

and is called a Heun operator.

Corollary 4: In Theorem 6, if set $C_1 = \frac{1}{6}$, $C_2 = \frac{2}{3}$, $C_3 = \frac{1}{6}$, $b_{21} = \frac{1}{2}$, $b_{31} = -1$, and $b_{32} = 2$, then the operator B satisfies

$$BF = -\frac{4R_2 + R_3 - 5R_1}{3\tau} , \tag{32}$$

where

$$\begin{cases} R_1 = -LF , \\ R_2 = -L(F + \frac{1}{2}\tau R_1) , \\ R_3 = -L(F - \tau R_1 + 2\tau R_2) , \end{cases} \tag{33}$$

and is called a Kutta operator.

Theorem 7: if set

$$\varphi(F,\tau) = C_1 R_1 + C_2 R_2 + C_3 R_3 + C_4 R_4 , \tag{34}$$

where

$$\begin{cases} R_1 = -LF , \\ R_2 = -L(F + b_{21}\tau R_1) , \\ R_3 = -L(F + b_{31}\tau R_1 + b_{32}\tau R_2) , \\ R_4 = -L(F + b_{41}\tau R_1 + b_{42}\tau R_2 + b_{43}\tau R_3) , \end{cases} \tag{35}$$

and the coefficients C_1, C_2, C_3, C_4 and b_{21}, b_{31}, b_{32}, b_{41}, b_{42}, b_{43} satisfy

$$\begin{cases} C_1 + C_2 + C_3 + C_4 = 1 , \\ C_2 a_2 + C_3 a_3 + C_4 a_4 = \frac{1}{2} , \\ C_2 a_2^2 + C_3 a_3^2 + C_4 a_4^2 = \frac{1}{3} , \\ C_2 a_2^3 + C_3 a_3^3 + C_4 a_4^3 = \frac{1}{3} , \\ C_3 b_{32} a_2 + C_4 [b_{42} a_2 + b_{43} a_3] = \frac{1}{6} , \\ C_3 a_3 b_{32} a_2 + C_4 a_4 [b_{42} a_2 + b_{43} a_3] = \frac{1}{8} , \\ C_3 b_{32} a_2^2 + C_4 [b_{42} a_2^2 + b_{43} a_3^2] = \frac{1}{12} , \\ C_4 b_{43} b_{32} b_{21} = \frac{1}{24} , \end{cases} \tag{36}$$

and

$$\begin{cases} a_2 = b_{21} \ , \\ a_3 = b_{31} + b_{32} \ , \\ a_4 = b_{41} + b_{42} + b_{43} \ , \end{cases} \tag{37}$$

then the operator B defined by (21) is a three–order harmonious dissipative operator.

Corollary 5: In Theorem 7, if set $C_1 = \frac{1}{6}$, $C_2 = \frac{1}{3}$, $C_3 = \frac{1}{3}$, $C_4 = \frac{1}{6}$, $b_{21} = \frac{1}{2}$, $b_{31} = 0$, $b_{32} = \frac{1}{2}$, $b_{41} = 0$, $b_{42} = 0$ and $b_{43} = 1$, then a classical Runge–Kutta operator which satisfies:

$$BF = -\frac{2(R_2 + R_3) + R_4 - 5R_1}{3\tau} \ , \tag{38}$$

where

$$\begin{cases} R_1 = -LF \\ R_2 = -L(F + \frac{1}{2}\tau R_1) \\ R_3 = -L(F + \frac{1}{2}\tau R_2) \\ R_4 = -L(F + \tau R_3) \ , \end{cases} \tag{39}$$

can be obtained.

Corollary 6: In Theorem 7, if set

$$\begin{cases} C_1 = \frac{1}{8}, \ C_2 = \frac{3}{8}, \ C_3 = \frac{3}{8}, \ C_4 = \frac{1}{8} \\ b_{21} = \frac{1}{3}, \ b_{31} = -\frac{1}{3}, \ b_{32} = 1, \ b_{41} = 1, \ b_{42} = -1, \ b_{43} = 1 \ , \end{cases}$$

then another three–order harmonious dissipative operator B which satisfies

$$BF = -\frac{3(R_2 + R_3) + R_4 - 7R_1}{4\tau} \ , \tag{40}$$

where

$$\begin{cases} R_1 = -LF \ , \\ R_2 = -L(F + \frac{1}{3}\tau R_1) \ , \\ R_3 = -L(F - \frac{1}{3}\tau R_1 + \tau R_2) \ , \\ R_4 = -L(F + \tau R_1 - \tau R_2 + \tau R_3) \ , \end{cases} \tag{41}$$

can be obtained.

V. TESTS AND APPLICATIONS

In Section 4, three species of harmonious dissipative operators with different order are constructed by using the Runge–Kutta method. According to the definition of the harmonious dissipative operatore, it is commonly known that, the higher the order of harmony of

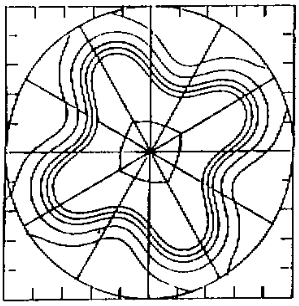

Fig. 1.1 R–H waves (40 days).

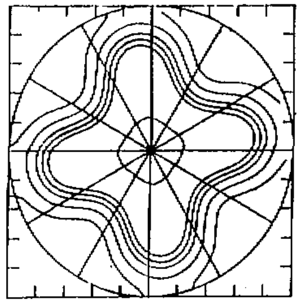

Fig. 1.2 R–H waves (70 days).

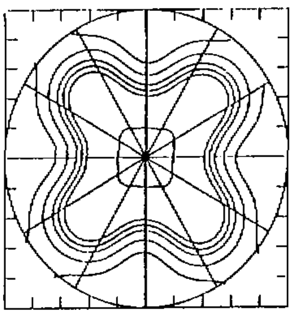

Fig. 1.3 R–H waves (100 days).

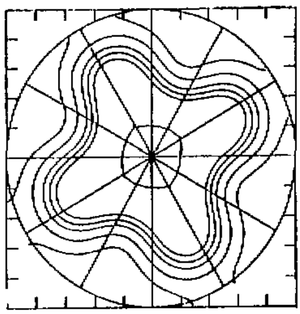

Fig. 1.4 R–H waves (120 days).

Fig.1. The figures for numerical tests on explicit complete square conservative difference schemes whose time step is constant (with 4–wave Rossby–Haurwitz waves).

operator B is, the higher the time–direction precision of the corresponding difference scheme is got and the longer the time step can be taken. Therefore, among the harmonious dissipative operators constructed in Section 4, the operator in Theorem 7 has the highest precision in the time direction and the one in Theorem 5 has the lowest precision.

Table 1. Comparison of the Results of the Four Schemes

	Original implicit scheme	one–order explicit scheme	simplified explicit scheme	two–order explicit scheme
Integration days	30	30	30	30
CPU Time(min)	9.80	2.60	2.12	2.92
$\lvert \vec{V} \rvert_{max}$ (cm / s)	25.75	25.62	25.64	25.88
Location (I, J)	(26, 23)	(26, 23)	(26, 23)	(26, 23)
H_{max} (cm)	18.06	17.98	17.99	18.04
Location (I, J)	(5, 47)	(5, 47)	(5, 47)	(5, 47)
H_{min} (cm)	−41.09	−40.92	−40.94	−41.05
Location (I, J)	(47, 4)	(47, 4)	(47, 4)	(47, 4)

Note: $\vec{V}$ is the velocity of the sea currents, H is the elevation of the sea surface.

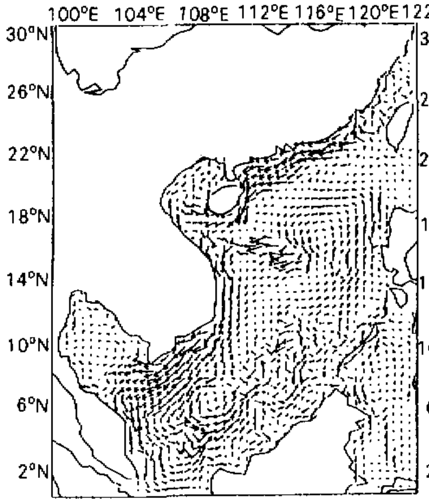

Fig. 2.1 Original implicit scheme

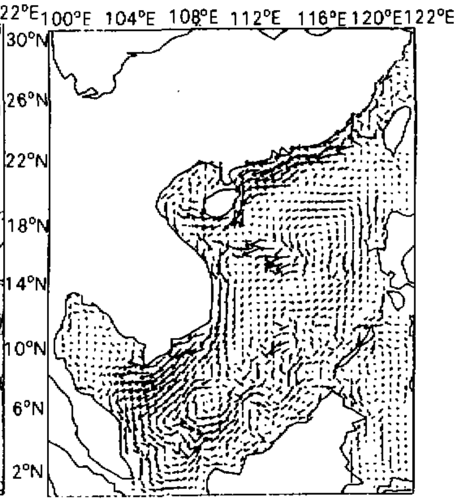

Fig. 2.2 1–order explicit scheme

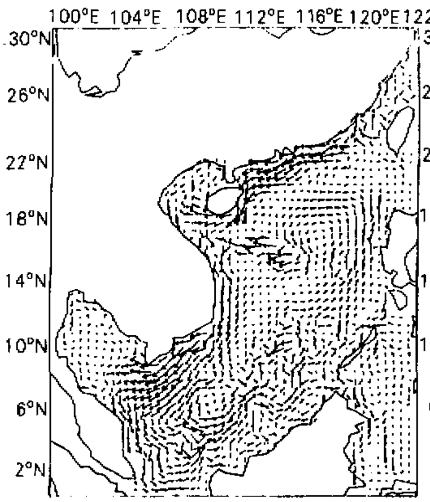

Fig. 2.3 Simplified explicit scheme

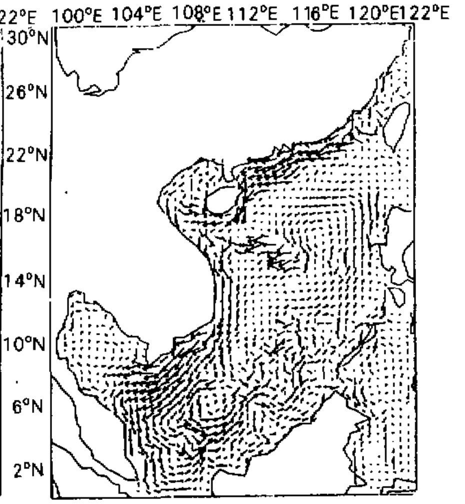

Fig. 2.4 2–order explicit scheme

Fig.2. The figures for the comparison of monthly mean sea currents in January.

However, it is necessary to remark that the computational effect of a harmonious dissipative operator is hardly influenced by its order of harmony because its space–direction precision is not hightened yet, only the time benefits may be affected by the order obviously. Thus, to increase the space–direction precision of a scheme with a high time–direction precision (or with a high order harmonious dissipative operator), it may be an important way to improve the computational results.

Numerical tests on harmonious dissipative operators and the corresponding difference schemes completely keeping the square conservatism, based on the barotropic shallow water equation set describing the atmospheric motion in the Northern Hemisphere covered by 80 × 20 grids with 4–wave Rossby–Haurwitz waves, are carried out, and satisfactory results are obtained (Fig.1). The harmonious dissipative operator used here is of three–order and takes 18 seconds for a one–day integration with a 390–second time step. However, it takes more time for a one–day integration if a lower order harmonious dissipative operator is used, 50 seconds for the one–order one and 21 seconds for the two–order one.

Moreover, an implicit scheme and three explicit schemes, which are all completely square–conservative, are used to simulate the monthly mean currents of the South China Sea in January, respectively, and the computational effects which can be seen from Fig.2 are almost the same (Zeng et al.,1989). Additionally, the CPU time that each scheme takes is given in Table 1, from which, it is easy to know that the explicit schemes have much better time benefits than the implicit scheme and the CPU time of the explicit schemes is only 1 / 5 to 1 / 3 of that of the implicit scheme.

The authors are grateful to Ms. Li Rongfeng for her kind help.

REFERENCES

Ji Zhongzhen and Wang Bin (1991), A Further Discussion on the Construction and Application of Difference scheme of Evolution Equations, *Scientia Atmospherica Sinica*, **15(1)**: 72–81.

Wang Bin and Ji Zhongzhen (1990), The construction and preliminary test of explicit complete square conservative difference schemes, *Chinese Science Bulletin*, **35(20)**: 1724–1728.

Zeng Qingcun and Ji zhongzhen (1981), On the computational stability of evolution equation, *Mathematica Numerica Sinica*, **3(1)**: 79–86.

Zeng Qingcun, Li Rongfeng and Ji Zhongzhen, et al. (1989), A numerical modeling for the monthly mean circulations in the South China Sea, *Scientia Atmospherica Sinica*, **13(2)**: 127–138.