Group Velocity of Anisotropic Waves—Part I: Mathematical Expression

Yong L. McHall

Department of Earth, Atmospheric and Planetary Sciences, MIT, MA 02139, USA

Received December 17, 1992; revised February 4, 1993

ABSTRACT

The group velocity used in meteorology in the last 30 years was derived in terms of conservation of wave energy or crests in wave propagation. The conservation principle is a necessary but not a sufficient condition for deriving the mathematical form of group velocity, because it cannot specify a unique direction in which wave energy or crests propagate. The derived mathematical expression is available only for isotropic waves. But for anisotropic waves, the traditional group velocity may have no a definite direction, because it varies with rotation of coordinates. For these reasons, it cannot be considered as a general expression of group velocity. A ray defined by using this group velocity may not be the trajectory of a reference point in an anisotropic wave train. The more general and precise expression of group velocity which is applicable for both isotropic and anisotropic waves and is independent of coordinates will be derived following the displacement of not only a wave envelope phase but also a wave reference point on the phase.

Key words: Isotropic group velocity, General group velocity, Oblique group velocity

I. INTRODUCTION

For the wave group composed of plane waves, group velocity is the velocity at which a wave envelope propagates. If the plane waves have phase planes φ_j and similar wavenumber vectors $k_j = \nabla \varphi_j$, the mathematical expression of group velocity is given by (Brillouin, 1960; Panchev, 1985)

$$c_g = \lim_{\delta k \to 0} \frac{\delta \omega}{\delta k} = \frac{\partial \omega}{\partial k}, \tag{1}$$

where

$$\omega = -\frac{\partial \varphi}{\partial t}$$

indicates angular frequency of the waves, and k, called the wavenumber, is the length of wavenumber vector **k**. It is important to note that δk in the previous expression represents the difference of wavenumbers between these waves, but not the variation with time or space. For the wave group produced by plane waves propagating along a constant direction in a homogeneous medium, wavenumbers of these waves do not change with time and space, but $\delta k \neq 0$. So group velocity may still be defined. While, for the waves with phase surfaces

$$\varphi_j = \mathbf{k} \cdot \mathbf{x} - \omega_j t \tag{2}$$

and the same wavenumber vector, group velocity cannot be defined since $\delta k \equiv 0$, though the wave frequency may depend on wavenumber. Analogously, the $\delta \omega$ in (1) gives the difference of wave frequencies between the waves. For the waves possessing phase surfaces

$$\varphi_j = \mathbf{k}_j \cdot \mathbf{x} - \omega t,$$

group velocity of these waves is always zero since $\delta\omega \equiv 0$, even if their frequency depends on wavenumber and is a function of time and space. In this sense, wave dispersion is a property of wave train itself, which is independent of medium. Another important point is that for the plane waves of which the phase planes are not sufficiently large, a classical wave group may be produced only if they propagate in the same wave path. For group velocity is a physical subject, its physical meaning should not be ignored when we deal with it in mathematics.

In a wavenumber space, wavenumber vector is a Cartesian vector given by

$$\mathbf{k} = k_1 \hat{\mathbf{k}}_1 + k_2 \hat{\mathbf{k}}_2 + k_3 \hat{\mathbf{k}}_3, \qquad k = \sqrt{k_1^2 + k_2^2 + k_3^2},$$

where $\hat{\mathbf{k}}_i$ $(i = 1,2,3)$ indicate unit vectors along coordinates $\mathbf{k}_i$. According to classical mathematics, (1) represents a directional derivative

$$c_g = \frac{\partial \omega}{\partial k_1} \cos \alpha_1 + \frac{\partial \omega}{\partial k_2} \cos \alpha_2 + \frac{\partial \omega}{\partial k_3} \cos \alpha_3 \tag{3}$$

along the direction of $\mathbf{k}$. Here,

$$\cos \alpha_i = \frac{k_i}{k} \tag{4}$$

are the direction cosines of the wavenumber vector in a wavenumber space. Usually, we set the Cartesian coordinates in a physical space with axes x_i towards the directions of $\mathbf{k}_i$, so that the wavenumber vector may be represented equivalently by

$$\mathbf{k} = k_1 \hat{\mathbf{x}}_1 + k_2 \hat{\mathbf{x}}_2 + k_3 \hat{\mathbf{x}}_3,$$

where, $\hat{\mathbf{x}}_i$ are unit vectors of the Cartesian coordinates in a three dimensional physical space. In this case, $\cos \alpha_i$ are also the direction cosines of wavenumber vector in the Cartesian coordinates.

However, the multidimensional group velocity of plane waves used in meteorological studies was given by

$$\mathbf{U} = \sum_{i=1}^{N} U_i \hat{\mathbf{x}}_i, \quad U_i = \frac{\partial \omega}{\partial k_i}. \tag{5}$$

It gives

$$U = \sqrt{\sum_{i=1}^{N} \left(\frac{\partial \omega}{\partial k_i} \right)^2}.$$

As showed previously, (3) is derived exactly from the classical mathematics. While, we will find in the next section that the derivation of (5) is not perfect. These two expressions of group velocities are identical only for isotropic waves as discussed in Section III, but are different for anisotropic waves. Thus, it is necessary to choose from one of them the more general one which is applicable for both isotropic and anisotropic waves. The choice will be made by the basic fact that group velocity is the phase velocity of wave envelope, which has a specific direction normal to the phase surface and is independent of coordinates.

II. HISTORICAL REVIEW

Initially, a mathematical expression of a multidimensional group velocity was represented by (Landau and Lifshitz, 1959; Bretherton, 1970)

$$\mathbf{c}_g = \frac{\partial \omega}{\partial \mathbf{k}}. \tag{6}$$

This equation is generally not equivalent to (5). The formula

$$\partial / \partial \mathbf{k} \equiv \left(\frac{\partial}{\partial k_1}, \cdots, \frac{\partial}{\partial k_N} \right) \tag{7}$$

is only a definition given by some authors (e.g., Milne-Thomson, 1968) and used for writing mathematical notations conveniently. In addition, there are some other definitions also. According to calculus of vectors (Williamson et al., 1968; Baxandall and Liebeck, 1981)

$$\begin{aligned} \frac{\partial \omega}{\partial \mathbf{k}} &= \frac{1}{\sqrt{k_1^2 + k_2^2 + k_3^2}} \left(k_1 \frac{\partial \omega}{\partial k_1} + k_2 \frac{\partial \omega}{\partial k_2} + k_3 \frac{\partial \omega}{\partial k_3} \right) \\ &= \frac{\partial \omega}{\partial k_1} \cos \alpha_1 + \frac{\partial \omega}{\partial k_2} \cos \alpha_2 + \frac{\partial \omega}{\partial k_3} \cos \alpha_3, \end{aligned}$$

where $\mathbf{k}$ is a unit wavenumber vector. This equation does not represent a vector again. In this case, the left hand side of (7) is evaluated in terms of the directional derivative in the direction of wavenumber vector, and is not identical to the right hand side.

In fact, the traditional group velocity (5) used in meteorology was not derived from (6). The detailed derivation of (5) may be found in the studies of Whitham (1960, 1961, 1974b) and Eckart (1960). The only physical principle which they applied was that group velocity is a velocity at which wave crests are conserved when waves propagate in a homogeneous medium. These studies were concentrated mostly on isotropic waves, of which their frequency and phase speed are independent of propagation direction.

For multidimensional waves, that wave crests propagate conservatively at a velocity is not a sufficient reason for us to regard this velocity as group velocity, because this velocity is not unique for the same waves. In physics, group velocity is the velocity at which a constant phase surface of wave envelope moves. Thus, group velocity is in the direction normal to the phase surfaces, and is parallel to wavenumber vector. This group velocity is illustrated by the vector $\mathbf{c}_g$ in Fig. 1, which shows the propagation of a wave envelope of plane waves. Apart from this group velocity $\mathbf{c}_g$, the phase displacement may also be described by some other velocities in the directions different from the group velocity, such as the velocity $\mathbf{U}$ shown in Fig. 1. If the scale of wave phase is much greater than the wavelength, it may be considered that wave energy is also transported by the velocity $\mathbf{U}$ along its direction. Following the velocity $\mathbf{U}$, wave crests are also conserved. In derivation of the traditional group velocity (5), only the movement of wave crest was considered. It has not been proved that (5) represents the velocity in the direction of wave group propagation. Thus, it is possible that the velocity (5) is the velocity $\mathbf{U}$ but not the $\mathbf{c}_g$ in Fig. 1. For example, Whitham (1960) showed that his group velocity was generally not in the direction of wave propagation. This will be discussed further in Section V.

If group velocity is considered as the phase velocity of a wave envelope in the direction of wave propagation, the discussion may be understood more easily. It is well known that phase speed of plane waves in the direction of wavenumber vector is given by

$$c = \frac{\omega}{k},$$

which is not a Cartesian vector, because

$$c_i = \frac{\omega}{k_i} \neq c \cos \alpha_i$$

and

$$c \neq \sqrt{c_1^2 + c_2^2 + c_3^2} .$$

For a wave group composed of two plane waves with wavenumber vectors $\mathbf{k}_a$ and $\mathbf{k}_b$ and angular frequencies ω_a *and* ω_b, respectively, phase surface of the wave envelope may be represented by

$$\varphi_e = \Delta k_1 x_1 + \Delta k_2 x_2 + \Delta k_3 x_3 - \Delta\omega t,$$

where

$$\Delta k_i = k_{b_i} - k_{a_i} \quad , \qquad \Delta\omega = \omega_b - \omega_a \quad .$$

Since phase speeds in different directions represent also the displacement of a wave phase, the speeds along individual coordinates may be derived from the relation of phase displacement

$$\Delta k_1 x_1 + \Delta k_2 x_2 + \Delta k_3 x_3 - \Delta\omega t = \Delta k_1 x_{1_0} + \Delta k_2 x_{2_0} + \Delta k_3 x_{3_0} - \Delta\omega t_0$$

by setting $x_j = x_{j_0}$ $(j \neq i)$, giving

$$U_i = \frac{x_i - x_{i_0}}{t - t_0} = \frac{\Delta\omega}{\Delta k_i} .$$

Again, $\Delta\omega$ and Δk_i are the difference of wave frequencies and wavenumbers between different waves respectively. When Δk_i is sufficiently small, it gives the components shown in (5). As to a single wave, these components are not Cartesian components of the phase velocity of wave envelope. However, in some previous studies (e.g., Eckart, 1960), group velocity (5) was given directly by taking the non-Cartesian components as Cartesian components.

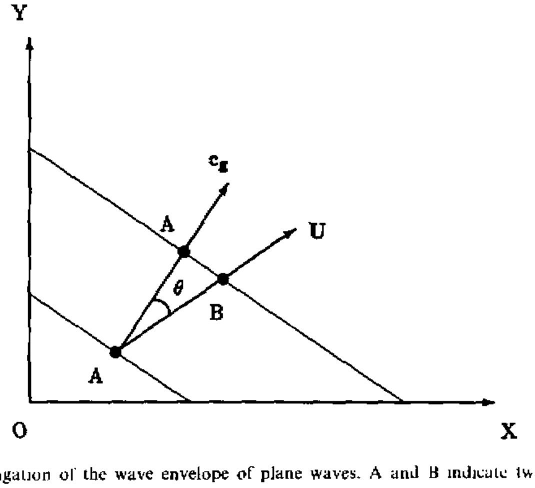

Fig. 1. Propagation of the wave envelope of plane waves. A and B indicate two different reference wave points.

III. DEPENDENCE ON COORDINATES

1. *General Proof*

The traditional group velocity (5) may be not in the direction of wave propagation, and so cannot be chosen uniquely (referring to Fig. 1). It will be difficult to use this velocity for the study of wave mechanics, because it may vary with rotation of coordinates. Suppose that a wave group is made by the three dimensional plane waves which possess wave phase

$$\varphi = k_1 x_1 + k_2 x_2 + k_3 x_3 - \omega t, \tag{8}$$

and propagate in a constant direction; the group velocity calculated from (5) is not in the direction of the wave propagation. Now, we rotate the original Cartesian coordinates, so that the waves in the new coordinates propagate along x_l^* direction. In this new coordinate frame, the wave phase may be rewritten as

$$\varphi = k_l^* x_l^* - \omega t. \tag{9}$$

So, the group velocity evaluated from either (3) or (5) is

$$c_g^* = \frac{\partial \omega}{\partial k_l^*}. \tag{10}$$

According to (5), the direction of the group velocity in the new coordinates is along the direction of wave propagation. So the group velocity is changed by coordinate transfer. This dependence of the traditional group velocity on coordinates can be proved generally in the following.

The functional notation of a vector represented in Cartesian coordinates may be changed if the coordinates are transferred into a non-Cartesian coordinates (Braae, 1968). Thus, the discussion here is confined to the transform between Cartesian coordinates. If the angles between original coordinates x_i and new coordinates x_j^* are given by α_{ij}, transform of coordinates are represented by

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \mathbf{A} \begin{bmatrix} x_1^* \\ x_2^* \\ x_3^* \end{bmatrix}, \tag{11}$$

where,

$$\mathbf{A} = \begin{bmatrix} \cos\alpha_{11} & \cos\alpha_{12} & \cos\alpha_{13} \\ \cos\alpha_{21} & \cos\alpha_{22} & \cos\alpha_{23} \\ \cos\alpha_{31} & \cos\alpha_{32} & \cos\alpha_{33} \end{bmatrix}$$

is an orthogonal matrix. The corresponding transform of wavenumber vector is determined by

$$\begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = \begin{bmatrix} \dfrac{\partial \varphi}{\partial x_1} \\ \dfrac{\partial \varphi}{\partial x_2} \\ \dfrac{\partial \varphi}{\partial x_3} \end{bmatrix} = \mathbf{A} \begin{bmatrix} \dfrac{\partial \varphi}{\partial x_1^*} \\ \dfrac{\partial \varphi}{\partial x_2^*} \\ \dfrac{\partial \varphi}{\partial x_3^*} \end{bmatrix} = \mathbf{A} \begin{bmatrix} k_1^* \\ k_2^* \\ k_3^* \end{bmatrix}. \tag{12}$$

The derivation is referred to Braae (1968). Since

$$(k_1\ k_2\ k_3)\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=(k_1^*\ k_2^*\ k_3^*)\mathbf{A}^T\mathbf{A}\begin{bmatrix}x_1^*\\x_2^*\\x_3^*\end{bmatrix}$$

$$=k_1^*x_1^*+k_2^*x_2^*+k_3^*x_3^*,$$

where $\mathbf{A}^T$ is a transpose matrix of $\mathbf{A}$, phase surface (8) becomes

$$\varphi=k_1^*x_1^*+k_2^*x_2^*+k_3^*x_3^*-\omega t$$

in the new coordinates.

It has been noted that group velocity may also be defined for the wave group propagating in a constant direction. For this wave group, it is always possible to choose a new coordinate system with x_1^* -axis towards the propagation direction. In this case, α_{i1} in the first column of $\mathbf{A}$ are the direction angles of the wavenumber vector in the original coordinates. Thus,

$$\begin{bmatrix}k_2^*\\k_3^*\end{bmatrix}=\begin{bmatrix}\cos\alpha_{12} & \cos\alpha_{22} & \cos\alpha_{32}\\ \cos\alpha_{13} & \cos\alpha_{23} & \cos\alpha_{33}\end{bmatrix}\begin{bmatrix}k_1\\k_2\\k_3\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix}.$$

This gives

$$\begin{bmatrix}k_1\\k_2\\k_3\end{bmatrix}=\mathbf{A}\begin{bmatrix}k_1^*\\0\\0\end{bmatrix}=\begin{bmatrix}k_1^*\cos\alpha_{11}\\k_1^*\cos\alpha_{21}\\k_1^*\cos\alpha_{31}\end{bmatrix}.$$

The geometric meaning of this relationship is obvious.

After this rotation, phase plane (8) becomes (9), and the wave frequency is rewritten as

$$\omega(k_1^*)=\omega[k_1(k_1^*),\ k_2(k_1^*),\ k_3(k_1^*)].$$

The traditional group velocity (5) gives

$$U^*=\frac{\partial\omega}{\partial k_1^*}=\sum_{i=1}^{3}\frac{\partial\omega}{\partial k_i}\cos\alpha_{i1}\ . \tag{13}$$

Now, let us consider firstly the isotropic waves, of which frequency can be represented by $\omega=\omega(a_j,k)$, where $k=\sqrt{k_1^2+k_2^2+k_3^2}$, and a_j are other parameters. For these waves,

$$\sum_{i=1}^{3}\frac{\partial\omega}{\partial k_i}\cos\alpha_{i1}=\frac{\partial\omega}{\partial k}=\sqrt{\sum_{i=1}^{3}\left(\frac{\partial\omega}{\partial k_i}\right)^2}.$$

So, U^* is just the group velocity given by (1), and is unchanged by the coordinate transform. However, for anisotropic waves,

$$\sum_{i=1}^{3}\frac{\partial\omega}{\partial k_i}\cos\alpha_{il} \neq \sqrt{\sum_{i=1}^{3}\left(\frac{\partial\omega}{\partial k_i}\right)^2}$$

in general. In this case, group velocity represented by the traditional expression may be altered after rotation of coordinates as shown by the example below, and is generally different from the new group velocity.

2. *Example of Rossby Wave*

In the coordinates with x-axis eastward and y-axis northward, the classical Rossby wave on a β-plane is dominated by the conservation equation of absolute vorticity

$$\left(\frac{\partial}{\partial t}+\bar{u}\frac{\partial}{\partial x}\right)\nabla^2\psi'+\beta+\frac{\partial\psi'}{\partial x}=0, \tag{14}$$

where, $\bar{u}$ and β are both constant representing the mean zonal flow of atmosphere and the meridional variation of horizontal Coriolis force, respectively. If the horizontal coordinates are rotated on the horizontal plane by an angle θ, we have

$$\begin{bmatrix} x \\ y \end{bmatrix}=\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} x^* \\ y^* \end{bmatrix}, \tag{15}$$

and

$$\begin{bmatrix} \dfrac{\partial}{\partial x} \\ \dfrac{\partial}{\partial y} \end{bmatrix}=\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} \dfrac{\partial}{\partial x^*} \\ \dfrac{\partial}{\partial y^*} \end{bmatrix}.$$

Applying this relationships for (14) yields the conservation equation in the new coordinates

$$\left[\frac{\partial}{\partial t}+\bar{u}\left(\frac{\partial}{\partial x^*}\cos\theta-\frac{\partial}{\partial y^*}\sin\theta\right)\right]\nabla^{*2}\psi^*+\beta\left(\frac{\partial\psi^*}{\partial x^*}\cos\theta-\frac{\partial\psi^*}{\partial y^*}\sin\theta\right)=0, \tag{16}$$

where,

$$u'^* = -\frac{\partial\psi^*}{\partial y^*}, \qquad v'^* = \frac{\partial\psi^*}{\partial x^*}$$

and

$$\nabla^{*2} = \frac{\partial^2}{\partial x^{*2}}+\frac{\partial^2}{\partial y^{*2}}.$$

Here, u'^* is in the direction of positive x^* which departs from the eastward direction by the angle θ, and v'^* is in the direction of positive y^*. Eq. (16) is a more general form of absolute vorticity conservation equation, while (14) is an example of $\theta=0$. In the following discussions, the $*$ in the previous equations will be omitted. In this case, x-axis may not point to the east.

Inserting the wave solution

$$\psi = Ae^{i(k_1x+k_2y-\omega t)} \tag{17}$$

into (16) yields the dispersion relation

$$\omega = \left(\bar{u} - \frac{\beta}{k_1^2 + k_2^2}\right)(k_1 \cos\theta - k_2 \sin\theta). \tag{18}$$

It gives the classical form

$$\omega = \bar{u}k_1 - \frac{\beta k_1}{k_1^2 + k_2^2}$$

in the coordinates with $\theta = 0$. It can be proved that the wave frequency is independent of coordinates.

This Rossby wave is an example of anisotropic waves, because for the same wave phase speed varies as it propagates towards different directions. The traditional group velocity and its components calculated from (18) reads

$$U = \sqrt{\left(\bar{u} - \frac{\beta}{k_1^2 + k_2^2}\right)^2 + \frac{4\beta\bar{u}}{(k_1^2 + k_2^2)^3}(k_1 \cos\theta - k_2 \sin\theta)^2}\ , \tag{19}$$

and

$$U_1 = \left(\bar{u} - \frac{\beta}{k_1^2 + k_2^2}\right)\cos\theta + 2\beta k_1 \frac{k_1 \cos\theta - k_2 \sin\theta}{(k_1^2 + k_2^2)^2}, \tag{20}$$

$$U_2 = \left(\bar{u} - \frac{\beta}{k_1^2 + k_2^2}\right)\sin\theta + 2\beta k_2 \frac{k_1 \cos\theta - k_2 \sin\theta}{(k_1^2 + k_2^2)^2}, \tag{21}$$

respectively. This velocity U intersects with the path of carrier wave when k_2 is different from zero.

Obviously, (16) also dominates the Rossby waves which propagate in a constant direction. Although wavenumbers of the carrier wave do not vary with time and space, group velocity may still be defined when the wavenumbers of individual single waves are different. In this case, the traditional group velocity and its components provided by (19)–(21) are obviously wrong, since the wave group cannot propagate in a direction different from wave propagation.

To give an exact proof, we select the new coordinates with x^* -axis in the direction of wave propagation. In this case, (12) gives

$$\begin{bmatrix} k_1^* \\ k_2^* \end{bmatrix} = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}^T \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = \begin{bmatrix} \sqrt{k_1^2 + k_2^2} \\ 0 \end{bmatrix},$$

where

$$\cos\alpha = \frac{k_1}{\sqrt{k_1^2 + k_2^2}}, \quad \sin\alpha = \frac{k_2}{\sqrt{k_1^2 + k_2^2}}.$$

It tells that the wavenumber vector becomes one dimensional, but its length is unchanged after this coordinate transform. The phase surface of Rossby waves is then replaced by (9), and the anisotropic dispersion relation is rewritten by inserting $k_2^* = 0$ into (18), as

$$\omega = \left(\bar{u}k_1^* - \frac{\beta}{k_1^*}\right)\cos\,\theta^* \quad . \tag{22}$$

where, $\theta^* = \theta+\alpha$ is a constant angle between the new x^*-axis and the eastward direction. The traditional group velocity in the new coordinates gives now

$$U^* = \frac{\partial\omega}{\partial k_1^*} = \left(\bar{u}+\frac{\beta}{k^2}\right)\cos\,\theta^* , \quad k^2 = k_1^{*2} = k_1^2 + k_2^2 \ .$$

It may be obtained also by using (13) for (18). This velocity is always in the direction of wave propagation, its magnitude is different from the U given in (19). Therefore, the velocity represented by (5) is changed by coordinate transform.

As the velocity (5) may vary with rotation of coordinates, it cannot give a general expression of group velocity. One could argue that the traditional group velocity may be applied in meteorology for the study of atmospheric perturbations such as the planetary wave propagation in the slowly varying environments (Hoskins and Karoly, 1981; Karoly and Hoskins, 1982; Wallace and Gutzler, 1981; Dole, 1983). It is true only for isotropic waves. However, most of the atmospheric waves, such as Rossby waves, are anisotropic waves. We have proved that the traditional group velocity may not give a correct expression for these anisotropic waves. So, these applications may be misleading. For example, the observational analysis of Horel and Wallace (1981) and numerical experiments of Simmons (1982) and Navarr (1990) have shown that the remote responses to an isolated tropical heating may extend straightforward to the polar regions, and do not always draw the great circles as predicted by the ray-tracing theory (Hoskins and Karoly, 1981; Karoly and Hoskins, 1982), which was based on the use of traditional group velocity. Also, the deficiency of ray-tracing theory in interpreting the formation of stationary waves at high latitudes has been pointed out by James (1988). There were also the approaches to these meteorological problems without using group velocity. For example, the meridional variations in planetary stationary wave phases and graphical distributions of low frequency anomalies have been explained more realistically according to the meridional asymmetries in orographic and thermal forcings, mean circulation fields and heat and momentum fluxes (McHall, 1990a, b, c).

IV. GENERAL GROUP VELOCITY

1. *Calculation Formula*

The previous argument gives also the general expression of group velocity. Since group velocity is independent of coordinates, we have $c_g = c_g^*$. Here, c_g^* is evaluated from (10). From (13),

$$c_g = \sum_{i=1}^{3}\frac{\partial\omega}{\partial k_i}\cos\,\alpha_i = \frac{1}{k}\mathbf{k}\cdot\nabla_k\,\omega. \tag{23}$$

This equation is exactly the same as (3). Thus, we have obtained, from different approaches, the more precise mathematical expression of multi-dimensional group velocity. For isotropic waves, it gives the traditional expressions of group velocity as a particular example. Thus, the traditional group velocity may be referred to as isotropic group velocity, while the new expression gives the general group velocity.

For the wave group made by plane waves, wave envelope propagates along the paths of single waves, which is parallel to wavenumber vector.The unit vector of the path is given by

$$\hat{\mathbf{r}} = \sum_{i=1}^{N} \cos \alpha_i \hat{\mathbf{x}}_i \ .$$

Thus, the vectorial expression of group velocity shows,

$$\mathbf{c}_g = \sum_{i=1}^{N} \frac{k_i}{k^2} \mathbf{k} \cdot \nabla_k \omega \hat{\mathbf{x}}_i = \frac{1}{k^2} \mathbf{k} \cdot \nabla_k \omega \mathbf{k} \ . \tag{24}$$

The Cartesian components of group velocity are, then,

$$c_{gi} = \frac{k_i}{k^2} \mathbf{k} \cdot \nabla_k \omega. \tag{25}$$

If the group velocity equals the phase velocity

$$\mathbf{v}_p = \frac{\omega}{k^2} \mathbf{k} \tag{26}$$

defined by Whitham (1974a) and Pedlosky (1979), the waves are nondispersive waves.

The correctness of the new notation can be examined in the following examples. The first one is the nondispersive isotropic sound waves of which group velocity must be equivalent to their phase velocity. Provided that angular frequency of the sound waves in an isothermal atmosphere is given by

$$\omega = c_s \sqrt{k_1^2 + k_2^2 + k_3^2} \ ,$$

where $c_s = \sqrt{c_p RT / c_v}$ measures Laplacian sound speed, equations (23) and (25) show

$$c_g = c_s = v_p \ , \qquad c_{gi} = \frac{c_s}{k} k_i = v_{pi} ,$$

respectively. This group velocity may be obtained also from (1), and is identical to the isotropic group velocity.

Another example is the anisotropic Rossby waves discussed earlier. The group velocity and its components computed for (18) give

$$c_g = \left(\bar{u} + \frac{\beta}{k^2} \right) \cos \ \theta^* \ , \tag{27}$$

and

$$c_{g1} = \frac{k_1}{k} \left(\bar{u} + \frac{\beta}{k^2} \right) \cos \ \theta^* \ , \qquad c_{g2} = \frac{k_2}{k} \left(\bar{u} + \frac{\beta}{k^2} \right) \cos \ \theta^* \ ,$$

respectively. The direction of group velocity is given by

$$\tan \ \alpha_1 = \frac{c_{g1}}{c_{g2}} = \frac{k_1}{k_2} = \frac{v_{p1}}{v_{p2}}$$

Thus, the group velocity is in the direction of wave propagation. After rotation of coordinates, the general group velocity evaluated from (22) is exactly the same as that obtained above, and is also in the direction of wave propagation. It will be proved generally in the following that the general group velocity is independent of coordinate rotation.

2. *Independent of Coordinates*

For the coordinate transform provided by (11), transform of wavenumbers is given by (12). When $k_1^* k_2^* k_3^* \neq 0$, there is

$$\left(\frac{\partial\omega}{\partial k_1^*},\frac{\partial\omega}{\partial k_2^*},\frac{\partial\omega}{\partial k_3^*}\right)=\left(\frac{\partial\omega}{\partial k_1},\frac{\partial\omega}{\partial k_2},\frac{\partial\omega}{\partial k_3}\right)\mathbf{A} \ .$$

It is important to note that this relationship does not hold when the transform changes the dimension of wavenumber vector. Applying this relationship for general group velocity gives

$$c_g^* = \frac{1}{k^*}\mathbf{k}^* \cdot \nabla_{k^*}\omega = \frac{1}{k^*}\left(\frac{\partial\omega}{\partial k_1^*},\frac{\partial\omega}{\partial k_2^*},\frac{\partial\omega}{\partial k_3^*}\right)\begin{bmatrix} k_1^* \\ k_2^* \\ k_3^* \end{bmatrix}$$

$$=\frac{1}{k}\left(\frac{\partial\omega}{\partial k_1},\frac{\partial\omega}{\partial k_2},\frac{\partial\omega}{\partial k_3}\right)\mathbf{A}\begin{bmatrix} k_1^* \\ k_2^* \\ k_3^* \end{bmatrix}$$

$$=\frac{1}{k}\left(\frac{\partial\omega}{\partial k_1},\frac{\partial\omega}{\partial k_2},\frac{\partial\omega}{\partial k_3}\right)\begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix}$$

$$=\frac{1}{k}\mathbf{k}\cdot\nabla_k\omega = c_g \ .$$

These relationships tell that magnitude of the group velocity is independent of the transform. If the group velocity is transferred back to the original coordinates, we may see

$$\mathbf{A}\begin{bmatrix} c_{g1}^* \\ c_{g2}^* \\ c_{g3}^* \end{bmatrix} = \mathbf{A}\begin{bmatrix} k_1^* \\ k_2^* \\ k_3^* \end{bmatrix}\frac{1}{k^{*2}}\mathbf{k}^* \cdot \nabla_{k^*}\omega$$

$$=\begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix}\frac{1}{k^2}\mathbf{k}\cdot\nabla_k\omega = \begin{bmatrix} c_{g1} \\ c_{g2} \\ c_{g3} \end{bmatrix} .$$

Thus, the direction is also independent of the transform. It has been proved in the preceding section that the general group velocity is unchanged also by the particular transform after which the dimension of wavenumber vector is changed.

V. OBLIQUE GROUP VELOCITY

When wave energy is identified with amplitude of wave envelope which moves along with wave envelope phases, wave energy of anisotropic waves moves at velocity U as well as $\mathbf{c}_g$ (referring to Fig. 1). The phase velocity which is not in the normal direction of wave envelope phase may be called the oblique group velocity. It will be proved in the following that the isotropic group velocity is just an oblique group velocity for anisotropic waves.

The direction numbers of general group velocity are (k_1, k_2, k_3), but are (bU_1, bU_2, bU_3) for isotropic group velocity. Where, b is a constant. An angel between the two velocities is calculated from

$$\cos\theta = \frac{\mathbf{k}\cdot\mathbf{U}}{kU} = \frac{k_1U_1 + k_2U_2 + k_3U_3}{\sqrt{(k_1^2+k_2^2+k_3^2)(U_1^2+U_2^2+U_3^2)}} \ . \tag{28}$$

It is obvious in Fig. 1 that

$$U = \frac{c_g}{\cos\theta}. \tag{29}$$

Applying (24) yields

$$U = \sqrt{\left(\frac{\partial\omega}{\partial k_1}\right)^2 + \left(\frac{\partial\omega}{\partial k_2}\right)^2 + \left(\frac{\partial\omega}{\partial k_3}\right)^2} .$$

This is just the magnitude of isotropic group velocity. It is noted that there are infinite oblique group velocities towards different directions, at which wave energy and crest are conserved in propagation. The isotropic group velocity (5) gives only one of them for anisotropic waves.

The relation between the general and isotropic group velocities may be given by (23), that is

$$\mathbf{k}\cdot\mathbf{U} = kc_g \ .$$

It can be obtained also using (28) and (29). Since general group velocity is in the direction of wavenumber vector, we see

$$\mathbf{c_g}\cdot\mathbf{U} = c_g^2 .$$

If $\theta = 0$ as for isotropic waves, we have $U = c_g$. This is just the previous example of sound waves. While, if $\theta \neq 0$ for anisotropic waves, we have $U > c_g$. These relationships between the two group velocities can be confirmed by Rossby waves discussed in the preceding section. From (20), (21) and (27), we have

$$\mathbf{k}\cdot\mathbf{U} = \left(\bar{u} + \frac{\beta}{k_1^2+k_2^2}\right)(k_1\cos\theta - k_2\sin\theta) = kc_g .$$

The angle between these two velocities is calculated from (28). In general, energy of Rossby waves propagates at the speed lower than that of isotropic group velocity used before.

For wave energy moves not only at general group velocity but also at oblique group velocities, a mathematical notation of group velocity should not be derived by considering the conservation of wave energy or wave crests only. Because so obtained group velocity may be an oblique group velocity (Brillouin, 1960; Pedlosky, 1979). Compared with the group velocity (24), oblique group velocity (5) is less significant. It varies with rotation of coordinates and is meaningful merely for the waves with infinitely large phase surfaces. Thus, it is not proper to choose (5) as a general representation of group velocity.

VI. SUMMARIES

The traditional expression of group velocity used in meteorology was derived by considering the conservation of wave energy or crest only. These conservation principles alone cannot specify a unique propagation direction of wave energy or crest, while group velocity has a specfied direction parallel to the normal direction of the phase surface of wave envelope. Therefore, these principles are only a necessary but not a sufficient condition for deriving group velocity. Since the traditional group velocity was derived without considering the direction of wave group propagation, it may represent a moving velocity of wave envelope in an oblique direction for anisotropic waves, which is generally not in the normal direction of the phase surface of wave envelope. Also, the traditional expression of group velocity varies with rotation of Cartesian coordinates in this case. For these reasons, it cannot be considered as a general expression of group velocity, although wave energy or phase surfaces of a wave group moves also at this velocity.

The more general expression of group velocity has been derived in different ways. For a provided dispersion relation, magnitude of group velocity is determined by the directional derivative of wave frequency along the direction of wave propagation in wavenumber space. Its direction is in the normal direction of wave envelope phases. This velocity is also the velocity at which a wave point moves. Thus, the general group velocity represents the propagation velocity of wave particles in quantum physics.

It has been proved generally that both magnitude and direction of general group velocity are independent of transform of Cartesian coordinates. For the waves propagating in varying environments, general group velocity gives a local group velocity at a given time. We shall prove further in Part II that mean wave energy, momentum and wave action propagate at this velocity.

REFERENCES

Baxandall, P.R. and H. Liebeck (1981), *Differential vector calculus*, Longman, 243pp.

Braae, R. (1968), *Vector analysis*, Sir Isaac Pitman & Sons Ltd., 226pp.

Bretherton, F.P. (1970), The general linearized theory of wave propagation, *In Lectures in applied mathematics (H.R. William Ed.)*. **13:** 61–102.

Brillouin, L. (1960), *Wave propagation and group velocity*, Academic Press, 154pp.

Dole, R.M. (1983), Persistent anomalies of the extratropical Northern Hemisphere wintertime circulation. *In Large-Scale Dynamical Processes in the Atmosphere* (Hoskins and Pearce, Eds.), Academic Press, 95–109.

Eckart, C. (1960), *Hydrodynamics of oceans and atmospheres*, Pergamon press, 290pp.

Horel, J.D. and J.M. Wallace (1981), Planetary scale atmospheric phenomenon associated with the Southern Oscillation, *Mon. Wea. Rev.*, **109:** 813–829.

Hoskins, B.J. and D.J. Karoly (1981), The steady linear response of a spherical atmosphere to thermal and orographic forcing, *J. Atmos. Sci.*, **38:** 1179–1196.

James, I.N. (1988), On the forcing of planetary-scale Rossby waves by Antarctica. *Quart. J.R. Met. Soc.*, **114:** 619–637.

Karoly, D.J. and B.J. Hoskins (1982), Three dimensional propagation of planetary waves, *J. Met. Soc.*, Japan, **60:** 109–123.

Landau, L.D. and E.M. Lifshitz (1959), *Fluid Mechanics*, Pergamon Press, 536pp.

Milne-Thomson, L.M. (1968), *Theoretical Hydrodynamics* (Fifth edition), Macmillan & Co. Ltd., 743pp.

McHall, Y.L. (1990a), Planetary stationary waves in the atmosphere--Part I: Orographical stationary waves, *Adv. Atmos. Sci.*, **8:** 211–224.

McHall, Y.L. (1990b) Planetary stationary waves in the atmosphere--Part II: Thermal stationary waves, *Adv. Atmos. Sci.*, **8:** 225–236.

McHall, Y.L. (1990c), Blocking distributions in the atmosphere *Adv. Atmos. Sci.*, **8:** 327–338.

Navarra, A. (1990), Steady linear response to thermal forcing of a anomaly model with an asymmetric climatology, *J. Atmos. Sci.*, **47**: 148–169.

Panchev, S. (1985), *Dynamic meteorology*, D. Reidel Publishing Company, 360pp.

Pedlosky, J. (1979), *Geophysical Fluid Dynamics*, Printed in U.S.A., 33–34.

Simmons, A.J. (1982), The forcing of stationary wave motion by tropical diabatic heating, *Quart. J. Roy. Met. Soc.*, **108**: 503–534.

Wallace, J.M. and D.S. Gutzler (1981), Teleconnections in the geopotential height field during the Northern Hemisphere winter, *Mon. Wea. Rev.*, **109**: 784–812.

Whitham, G.B. (1960), A note on group velocity, *J. Fluid Mech.*, **9**: 347–352.

Whitham, G.B. (1961), Group velocity and energy propagation for three- dimensional waves, *Comm. Pure Appl. Math.*, **14**: 675–691.

Whitham, G.B. (1974a), *Linear and nonlinear waves*, New York, 636pp.

Whitham, G.B. (1974b), Dispersive waves and variational principles, *In Nonlinear Waves*, Cornell Univ. Press, 331pp.

Williamson, R.E., R.H. Crowell and H.F. Trotter (1968), *Calculus of Vector Functions*, Prentice-Hall, Inc., 434pp.