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An Improved Splitting Method[1]

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ABSTRACT

In this paper, an improved splitting method, based on the completely square-conservative explicit difference schemes, is established. Not only can the time-direction precision of this method be higher than that of the traditional splitting methods but also can the physical feature of mutual dependence of the fast and the slow stages that are calculated separately and splittingly be kept as well. Moreover, the method owns an universality, it can be generalized to other square-conservative difference schemes, such as the implicit and complete ones and the explicit and instantaneous ones. Good time benefits can be acquired when it is applied in the numerical simulations of the monthly mean currents of the South China Sea.

Key words: *Improved splitting method, Complete square conservatism, Explicit difference scheme, Second order precision, Economical method*

I. INTRODUCTION

Splitting methods are applied widely in solving the equations of atmospheric motion (Zeng et al., 1980, 1981, 1988 and Yasuo, 1983) due to the separability of the adjustment (fast) stage and the development (slow) stage (Zeng, 1979) and good time effects are obtained. In some traditional splitting methods, however, there exist some defects. First, in the mathematical significance, they are generally of one-order precision in the time direction and the errors that may affect the computational results to some extent are rather great. Although there are two-order precision splitting methods, such as the method of Yasuo (1983), each of them might be only suitable for a fixed scheme. Thus, to heighten their precision and to improve their universalities are two of our major aims to found an improved splitting method. Second, in the physical significance, the two stages mutually exist, depend on and transform in spite of their separability. In a popular sense, the evolution of the fast stage includes the effect of the slow stage and the evolution of the slow stage includes the effect of the fast stage. In the traditional splitting method, however, this important physical feature may not be kept because the two stages are usually absolutely separated to calculate, which may cause the distortion of the results. Therefore, to do our possible to keep this feature is another of our major aims to establish the new method.

II. FUNDAMENTAL PRINCIPLES

First, the improved splitting method to be introduced will be built on the basis of the following differential equation in operator form

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$$\frac{\partial F}{\partial t} + AF = 0 \ . \tag{1}$$

Suppose AF is splitted into:

$$AF = A_1 F + A_2 F \ , \tag{2}$$

where $A_1 F$ denotes the fast stage and its relative time step is written as τ_0, $A_2 F$ denotes the slow stage whose time step is written as τ_s. We set τ_s / τ_0 to be an integer, that is,

$$\tau_s = M\tau_0 \quad (M \geqslant 2, \text{ is an integer}) \ . \tag{3}$$

The time interval $[0, +\infty)$ is divided by τ_s into:

$$0 = t_0 < t_1 < t_2 < \cdots < t_n < \cdots < +\infty \ , \tag{4}$$

where

$$t_n = n\tau_s \quad (n = 0, 1, 2, \cdots) \ . \tag{4'}$$

Then, each small time interval $[t_n, t_{n+1}]$ is divided by τ_0 into:

$$t_n = t_n^{(0)} < t_n^{(1)} < \cdots < t_n^{(m)} < \cdots < t_n^{(M-1)} < t_n^{(M)} = t_{n+1} \ , \tag{5}$$

where

$$t_n^{(m)} = t_n + m\tau_0 = t_n + \frac{m}{M}\tau_s \tag{5'}$$

$$(m = 0, 1, 2, \cdots, M-1, M) \ .$$

The key theorem to introduce the new method is shown in the following, whose proof is omitted due to the limited space.

Theorem: If L_1 and L_2 are spatial difference operators of 2nd order precision in space, compatible with A_1 and A_2, respectively and B is a harmonious dissipative operator① , then the program of the improved splitting method based on the completely square-conservative difference scheme in an explicit way (Wang et al.1990 and Ji et al. 1991) can be established as the following

$$\tilde{F}^{n,0} = F^n \ , \tag{6}$$

$$\frac{\tilde{F}^{n,m+1} - \tilde{F}^{n,m}}{\tau_0} + L^* \tilde{F}^{n,m} + \varepsilon_{n,m}^* \tau_0 B\tilde{F}^{n,m} = 0 \ (m = 0,1,2,\cdots,M-1) \ , \tag{7}$$

$$F^{n+1} = \tilde{F}^{n,M} - \frac{\tau_s}{2}(L_2 \tilde{F}^{n,M} - L_2 F^n) \ , \tag{8}$$

which is of 2nd order precision both in the time direction and in the space direction, where

① Here, the concept of harmonious dissipative operator will be introduced simply. An operator B is called a (K-th order) harmonious dissipative operator if it can be expanded in Taylor form

$$BF = -2\left[\frac{1}{2!}\frac{\partial^2 F}{\partial t^2} + \frac{\tau}{3!}\frac{\partial^3 F}{\partial t^3} + \cdots + \frac{\tau^{K-1}}{(K+1)!}\frac{\partial^{K+1} F}{\partial t^{K+1}}\right] + O(\tau^K) \ ,$$

where K is an integer and it is greater than or equal to 1. It is easy to prove that a harmonious dissipative operator is generally positively definite and it can heighten the precision of an explicitly and completely square-conservative difference scheme in the time direction. Due to the limited space, the detailed discussion on it will not be given here, readers may refer to Wang et al. (1993).

$t\in[t_n, t_{n+1}]$, $(t=t_n+\tau,\ 0\leqslant\tau\leqslant\tau_s)$,$(G)^n=G|_{t=t_n}$, $(G)_{n,m}=G|_{t=t_n^{(m)}}$, $\varepsilon^*_{n,m}$ is calculated according to the methods of Wang et al. (1990) and Ji et al. (1991), and L^* is defined as

$$L^*\tilde{F}^{n,m}=L_1\tilde{F}^{n,m}+L_2F^n \ . \tag{9}$$

Obviously, the term L_2F^n in Schemes (6)–(9), denoting the slow stage, is constant when m changes, i.e., it does not need to be calculated at each small step, which plays the role of splitting calculation in the new method. Meanwhile, in this splitting calculation the slow stage is not absolutely separated from the fast stage, because it takes part in the integrations of the fast stage on each small step, which really shows the feature that the evolution of the fast stage includes the effect of the slow stage. Now, the problem is that the time-direction precision of the solution to Scheme (7) is not heightened yet since the precision that Scheme (7) approaches to Eq.(1) in the time direction is still one-order. A way to solve the problem is to calibrate the solution of Scheme (7) at each large step according to the truncated error of Scheme (7) to Eq.(1), which is implemented in Scheme (8), so the precision becomes two-order at each large step and keeps one-order on each small step between t_n and t_{n+1}. By this way, not only can good results be acquired but also can a lot of CPU time be lessened. This is an economical way. On the other hand, the evolution of the slow stage, denoted by Scheme (8), also includes the effect of the fast stage, because the result from the evolution of the fast stage is used in Scheme (8). Therefore, it is of mathematical and physical significance because good computational effect can be acquired and much CPU time can be saved.

In addition, the improved splitting method is of universal significance, because the second-order-precision splitting methods of the implicit scheme and of the frog leap scheme can be constructed in the similar way as follows:

$$\begin{cases}\tilde{F}^{n,0}=F^n\\ \dfrac{\tilde{F}^{n,m-1}-\tilde{F}^{n,m}}{\tau_0}=-L^*\dfrac{\tilde{F}^{n,m}+\tilde{F}^{n,m+1}}{2}\quad(m=0,1,2,M-2,M-1)\\ F^{n+1}=\tilde{F}^{n,M}-\tau_s\left(L_2\dfrac{F^n+F^{n+1}}{2}-L_2F^n\right)\ ,\end{cases} \tag{10}$$

$$\begin{cases}\tilde{F}^{n,0}=F^n,\ \tilde{F}^{n,-1}=F^{n-1,M-1}\\ \dfrac{\tilde{F}^{n,m+1}-\tilde{F}^{n,m-1}}{2\tau_0}+L^*\tilde{F}^{n,m}=0\qquad(m=0,1,2,\cdots M-2,M-1)\\ F^{n+1}=\tilde{F}^{n,M}-\dfrac{\tau_s}{2}(L_2\tilde{F}^{n,M}-L_2F^n)\\ F^{n,M-1}=\tilde{F}^{n,M-1}-\dfrac{\tau_s}{2}(L_2\tilde{F}^{n,M-1}-L_2\tilde{F}^{n,-1})\ .\end{cases} \tag{11}$$

Similar to Schemes (6)–(9), Scheme (10) and Scheme (11) own the major advantages which are their second-order precision and the maintenance of the important physical feature. They also have better time benefits than the traditional splitting methods of these two schemes have. However, they are less economical than Schemes (6)–(9), because an implicit scheme is difficult to be solved and a frog leap scheme has a worse computational stability and its time step is limited greatly. Thus, in practical computations, Schemes (6)–(9) should be used firstly.

III. APPLICATION IN NUMERICAL SIMULATIONS FOR CURRENTS OF THE SOUTH CHINA SEA

In this section, the improved splitting method based on the explicitly and completely square-conservative difference scheme is used to simulate the monthly mean currents and surface-elevations of the South China Sea and good results are obtained. The simulated results show that not only can the computational effect of the improved splitting method based on the explicitly and completely square-conservative difference scheme be the same as that of the traditional splitting method based on the implicitly and completely square-conservative difference scheme but also can good time benefits be gained from the new method that costs only 30% CPU time of the traditional method.

The equation set of oceanic motion can be written in operator form

$$\frac{\partial \vec{G}}{\partial t} + A\vec{G} = \vec{P} , \tag{12}$$

where $\vec{P}$ is the physical term including the lateral eddy viscosity, the bottom friction and the sea-surface wind stress, the left term is the barotropic shallow water equation set in the IAP operator form. The operator A is anti-symmetrical under the rigid boundary condition. The energy-conservatism can be kept in Eq.(12) if the right term is set to be zero. Also, the operator A can be split into the adjustment (fast) stage A_1 and the development (slow) stage A_2. The physical term can be added to the slow stage to form a combined slow stage term

$$A'_2\vec{G} = A_2\vec{G} - \vec{P} , \tag{13}$$

because it changes very slowly. From Eq.(13), Eq.(12) can be rewritten as

$$\frac{\partial \vec{G}}{\partial t} + A_1\vec{G} + A'_2\vec{G} = 0 . \tag{12$'$}$$

Then, the improved splitting method can be directly used on Eq.(12)$'$. By this way, the monthly mean currents and surface-elevations of the South China Sea in January, April, July and October are simulated and the results are satisfactory. Here, only the results in January are shown due to the limited space.

Figs.1-4 show the results of the monthly mean currents and surface-elevations in January, from the one-month integrations, by using the traditional method and the improved method respectively. From the figures, it is found that the effects of the two methods are almost the same. On one hand, it is clearly seen that there are an intensive wind-driven coastal current flowing southwestwards in the northern and western areas of the South China Sea (Zeng et al.,1989 and Xu et al.,1980) and a counter-wind northward current in the eastern area, of which a large scale cyclonic circulation is composed. The circulation includes two mesoscale closed cyclonic circulations, one located at (16.5°N, 116.5°E) in the northern area of the South China Sea and the other located at (6.5°N, 109°E) in the southern area (Zeng et al.,1989; Xu et al., 1980 and Wang, 1985). Furthermore, a northeastward, narrow and bending, counter-wind current out of the down-wind coastal current in the northern area of the South China Sea, similar to the observed South China Sea Warm Current (Zeng et al., 1989; Xu et al., 1980; Dale, 1956; Kwan, 1978 and Guo et al., 1985), exists obviously. On the other hand, driven by southwestward wind, the whole South China Sea is in the state that the surface-elevations are high in the southern area and low in the northern area. This conclusion is the same as that of Zeng et al. (1989).

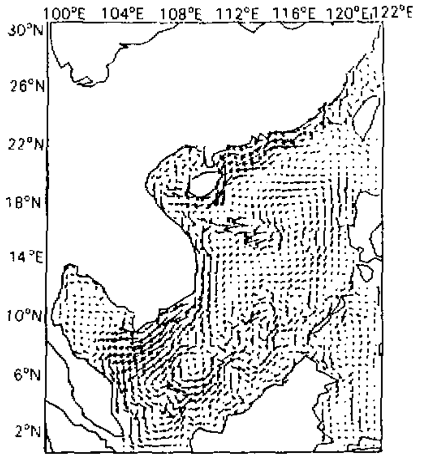

Fig.1. The monthly mean ocean currents in January (the original implicit scheme).

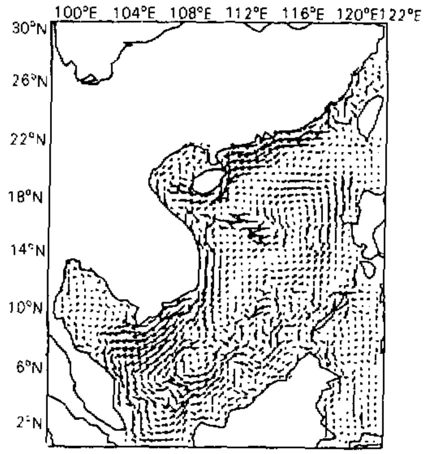

Fig.2. The monthly mean ocean currents in January (the explicitly improved splitting scheme).

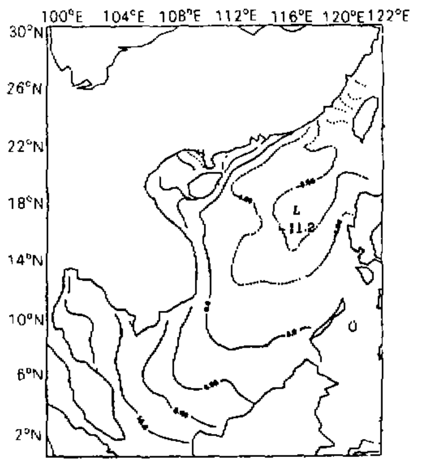

Fig.3. The monthly mean surface-elevations in January (the original implicit scheme).

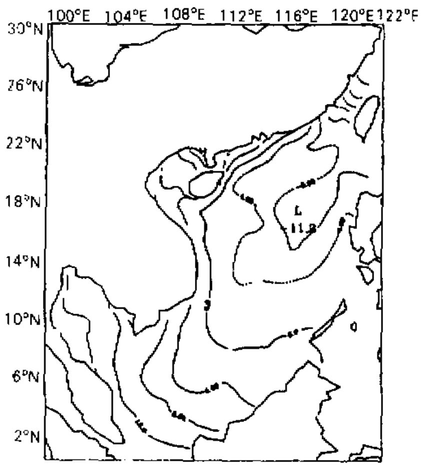

Fig.4. The monthly mean surface-elevations in January (the explicitly improved splitting scheme).

Table 1 shows the comparison of the results of the two methods, from which it is known that the time benefit of the new method are much better than that of the traditional one, because the CPU time the new method costs is only 30% of that the traditional method costs. Additionally, subtle differences of computational results between the two methods can be found from the table, which are difficult to make out from Figs.1–4.

Table 1. Comparison of the Results of the Two Methods

	The original implicit scheme	The explicitly improved splitting scheme
Integration days	30	30
CPU time(min)	9.80	2.92
$\|\overline{V}\|_{max}$ (cm / s)	25.75	25.88
Location (I, J)	(26, 23)	(26, 23)
H_{max} (cm)	18.06	18.04
Location (I, J)	(5, 47)	(5, 47)
H_{min} (cm)	-41.09	-41.05
Location (I, J)	(47, 4)	(47, 4)

Note: $\overline{V}$ is the velocity of the ocean currents, H is the elevation of the sea surface.

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