Nondispersive Periodic Solution of a Barotropic Semi-Geostrophic Model[1]

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ABSTRACT

The existence and solution of the non-dispersive periodic solution are achieved concerning nonlinear barotropic Rossby waves of a barotropic semi-geostrophic model, demonstrating the likelihood of the Taylor evolution, together with the related dimensionless a-criterion. Finally, the wave velocity expression is proposed with some diagnostic relations among the wave parameters.

Key words: Barotropic semi-geostrophic model, a-criterion, Periodic solution

I. INTRODUCTION

On a sequence of extratropical daily weather maps it is a common observation that some patterns of the trough / ridge system remain almost unchanged and it keeps regular motion in a few days. The systems can be approximately viewed as the nondispersive periodic solution (NDPS) in this time interval, i. e., the solution of time-independent change in waveform, or the so-called nonlinear characteristic wave (Zeng, 1979). As such, the study of NDPS of atmospheric nonlinear waves is of much importance both in theoretical and practical aspects.

Zeng is the first to get the particular solution of nondispersive slow waves of the primitive equations in terms of a singular perturbation technique. Afterwards, Liu and Liu (1987) documented the approach to the particular solution of real nonlinear properties with the aid of semi-geostrophic equations, which provides a new effective way to address issues of nonlinear nature.

This paper is devoted to further investigation of the NDPS existence and solution of the barotropic semi-geostrophic model, demonstrating the possibility of series expansion in a strict sense, together with the related dimensionless a-criterion. The wave velocity expression for nonlinear features is acquired, along with some diagnostic relations among the wave parameters, providing us with additional appreciation of the problem.

II. CONDITION FOR NDPS EXISTENCE

The basic equations of the barotropic semi-geostrophic model have the form (Liu, 1987):

$$\left(\frac{\partial}{\partial t}+u\frac{\partial}{\partial x}+v\frac{\partial}{\partial y}\right)\zeta^{(0)}+\beta v^{(0)}+f_0 D=0,$$
$$\left(\frac{\partial}{\partial t}+u\frac{\partial}{\partial x}+v\frac{\partial}{\partial y}\right)\varphi+(C_0^2+\varphi)D=0, \tag{1}$$

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$$f_0\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right)=\triangle\Phi,$$

where
$$f_0 u^{(0)}=-\frac{\partial\varphi}{\partial y},\quad f_0 v^{(0)}=\frac{\partial\varphi}{\partial x} \tag{2}$$

is the condition of geostrophic balance; $C_0^2=gH$ and $\varphi=gh$, with H and h being the depth of the undisturbed fluid and the deviation from H of the disturbed depth, respectively; $\xi^{(0)}$ is the geostrophic vorticity; D the divergence and $\triangle$ the two-dimensional Laplace operator, the others being usual in the meteorological context.

To seek the NDPS of Eq.(1), we assume

$$u=U(\theta),\ v=V(\theta),\ \varphi=\Phi(\theta),\ \theta=kx+ly-\sigma t, \tag{3}$$

in which $k(l)$ is the wavenumber in the $x(y)$ direction, and σ the intrinsic frequency.

Substitution of (3) into (1), then with similar treatment given in Liu and Liu(1987), yields

$$\Phi''+\frac{[\beta(1+\frac{\Phi}{C_0^2})^2+\mu^2C_x]\Phi'}{[-k_h^2C_x(1+\frac{\Phi}{C_0^2})]}=0, \tag{4}$$

where the prime denotes the differentiation with respect to θ; $k_h^2=k^2+l^2$; $C_x=\sigma/k$ is the phase velocity in the x direction; $\mu=f_0/C_0$ and μ^{-1} the Rossby deformation radius for the barotropic atmosphere. Integrating (4) with respect to θ with the integration constant being zero, we have

$$\Phi''-\frac{\beta}{C_xk_h^2}(\Phi+\frac{\Phi^2}{2C_0^2})-\frac{\mu^2C_0^2}{k_h^2}\ln\left|1+\frac{\Phi}{C_0^2}\right|=0, \tag{5}$$

which, multiplied by Φ' and then integrated, gives

$$\frac{\Phi'^2}{2}=E+\frac{\beta}{C_xk_h^2}\left(\frac{\Phi^2}{2}+\frac{\Phi^3}{6C_0^2}\right)+\frac{\mu^2C_0^2}{k_h^2}\left[(\Phi+C_0^2)\ln\left|1+\frac{\Phi}{C_0^2}\right|-\Phi\right]=F(\Phi) \tag{6}$$

where the integration constant E is the enstrophy (and it is easy to verify that E is the Hamilton quantity of the system represented by (5); $\Phi'^2/2$ can be viewed as the kinetic energy; $F(\Phi)$ is the enstrophy influence function so that $E-F(\Phi)$ can be viewed as the potential energy. The enstrophy (the Hamilton quantity) of the system by (6) is conservative.

Now the demonstration will be done of the conditions of the periodic solution in (6).

Fox $C_x<0$, the enstrophy influence function has its image as shown in Fig.1.

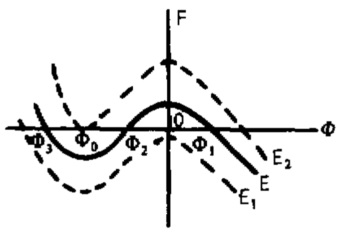

Fig.1. Representation of $F(\Phi)$ for $C_x<0$.

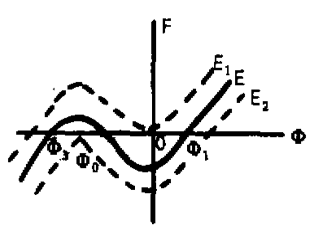

Fig.2. Representation of $F(\Phi)$ for $C_x>0$.

For E satisfying the condition

$$E_1 < E < E_2, \tag{7}$$

where $E_1 = 0$ and $E_2 = \dfrac{-2f_0^2 C_0^2(1+3a)}{3k_h^2 a}$ with $a = \dfrac{\mu^2 C_x}{\beta}$ as the dimensionless phase velocity. At this point $F(\Phi)$ is not negative in $[\Phi_2, \Phi_1]$ and $F(\Phi)$ has three real roots $(\Phi_1 > 0 > \Phi_2 > \Phi_3)$. Analysis of the phase pattern indicates a periodic solution in $[\Phi_2, \Phi_1]$. Evidently, (7) holds only for $E_2 > 0$, i. e., $0 > a > -\frac{1}{3}$. Hence the wave solution has the form $0 > C_x > \dfrac{-\beta}{3\mu^2} = C_{x0}$, suggesting that for the west-travelling Rossby waves, it is marked by the upper limit $|C_{x0}|$ for the zonal phase velocity, which is $C_{x0} \approx 20$ m/s with $\varphi = 45°$N. Therefore, the barotropic atmosphere acts as a low-pass filter for the westward waves, allowing only low-frequency slow waves to be generated and exist, suppressing the opposite.

If E is increased to the point where the solid curve is translated upward to join the broken line, i. e., $E = E_2$, then $\Phi_3 = \Phi_2 = \Phi_0 = -2C_0^2$, implying that the nonlinear effects of the system are amplified to the limit. In that case, the periodic solution is likely to be a solitary wave solution, i.e., $E = E_2$ is the necessary condition for the system to give the solitary solution. On the other hand, in $[\Phi_0, \Phi_1]$, although $\Phi = \Phi_0 / 2$ represents a singular point (non-equilibrium point) of (5), $\Phi \to \dfrac{\Phi_0}{2}$, $F(\Phi) \to = E - \dfrac{E_2}{2}$ is a bounded quantity based on the definition of $F(\Phi)$. Thus, no singular point exists in $F(\Phi)$, suggesting that only weak singularity at $\Phi = \dfrac{\Phi_0}{2}$ of (5). If, however, the definition $\Phi' = \pm[2(E - \dfrac{E_2}{2})]^{\frac{1}{2}}$ is added to (5) at $\Phi = \dfrac{\Phi_0}{2}$, then the differential system can be regarded as continuous. For this reason, there exists a steady bounded wave solution at $\dfrac{\Phi_0}{2}$ under certain conditions. For the present model, $E = E_2$ serves as the sufficient and necessary condition for the solitary wave to appear.

If E is kept growing so that $E > E_2$, then $F(\Phi)$ has merely a real root Φ_1 during which case no periodic solution is possible. If E is reduced to the point where the solid curve is translated downward to join the broken line, meaning $E = E_1$, then $\Phi_1 = \Phi_2 = 0$. In that case, the nonlinear periodic solution of the system would be degraded to the linear analogue.

If E keeps decreasing further, resultion in $E < E_1$, then $F(\Phi)$ has only one real root. Likewise, no periodic solution is available at this time.

For $C_x > 0$, the enstrophy influence function has its image as shown in Fig.2. Similar to the discussion with $C_x < 0$, for E satisfying

$$E_2 < E < E_1, \tag{8}$$

a periodic solution exists in $[\Phi_3, \Phi_2]$. And particularly for $E = E_1$, the solution becomes the form of a solitary wave; for $E = E_2$, the solution is degraded into a linear solution. For $E > E_1$ or $E > E_2$, the wave is splitted, leading to the breakdown of the NDPS. Unlike the case of $C_x < 0$, $E_1 > E_2$ is always true when $C_x > 0$ meaning no constraint of the upper limit to the zonal phase velocity for the east-travelling waves, as opposed to the west-moving counterparts. In other words, the barotropic atmosphere acts only as a low -pass filter in one direction.

Generally, nonlinear advective effect steepens the waveform in contrast to linear dispersion. Only if the role of both the types is in equilibrium, is the time-independent nonlinear NPDS reached. The enstrophy is essentially a physical quantity showing the magnitude of the linear effect in relation to the nonlinearity.

III. ENSTROPHY INFLUENCE FUNCTION $F(\Phi)$ APPROXIMATION AND NDPS

With the conditions satisfying the nonlinear NDPS given above, we shall now deal with the analytical expression of the solution.

From the integration of (6), we have

$$\theta = \pm \int [2F(\Phi)]^{-1/2} d\Phi. \tag{9}$$

In general, the analytical results can never be achieved by integrating because of the complexity of $F(\Phi)$. Here we shall seek approximate integration analytical expression with the aid of the function approximation of $F(\Phi)$ (He, 1993). Take $C_x<0$ for example. Considering the $F(\Phi)$ curve of Fig.1, $F(\Phi)$ can be approximated by a cubic curve, ensuring that the trinomial curve has the same zero-point evaluation and the same amplitude at the equilibrium as $F(\Phi)$. In doing so, integration of (9) yields

$$\Phi = \Phi_2 + (\Phi_1 - \Phi_2) Cn^2 \sqrt{\frac{E(\Phi_1 - \Phi_3)}{2\Phi_1 \Phi_2 \Phi_3}}\,\theta, \tag{10}$$

where $E<E<E_2$, $C^2n(\ \ )$ is the Jacobian elliptic cosine function and $\Phi_i (i=1,2,3)$ is the three roots (with $\Phi_1>0>\Phi_2>\Phi_3$) of the transcendental equation

$$\frac{-\beta}{C_x k_h^2}\left(\frac{\Phi^2}{2}+\frac{\Phi^3}{6C_0^2}\right)-\frac{\mu^2 C_0^2}{k_h^2}\left[\left(\Phi+C_0^2\right)\ln\left|1+\frac{\Phi}{C_0^2}\right|-\Phi\right]=E. \tag{11}$$

Since a NDPS exists in the system, the elliptic cosine wave solution can be the approximate solution to the periodic analog which has zonal wavelength of the form

$$L=\frac{2}{k}\sqrt{\frac{2\Phi_1\Phi_2\Phi_3}{E(\Phi_1-\Phi_3)}}\,K(m), \tag{12}$$

where $K(m)$ denotes the complete elliptic integration of first kind with the modulus m squared

$$m^2=\frac{(\Phi_1-\Phi_2)}{(\Phi_1-\Phi_3)}, \tag{13}$$

in which the amplitude $\hat{\Phi}=\Phi_1-\Phi_2$.

For $E=E_2$, we have $\Phi_3=\Phi_2=\Phi_0=-2C_0^2$ and $m^2=1$. Following the discussion in the preceding section, the NDPS takes the form of a solitary wave solution

$$\Phi=-2C_0^2+(\Phi_1+2C_0^2)\mathrm{sec}^2 h\sqrt{\frac{E_2(2C_0^2+\Phi_1)}{8C_0^4\Phi_1}}\,\theta, \tag{14}$$

where Φ_1 is the root of (11) when $E=E_2$.

For $E=E_1$ we have $\Phi_1=\Phi_2=0$ and $m^2=0$. As shown in the last section, the nonlinear NDPS is degraded into a linear cosine wave periodic solution.

IV. CONDITIONS OF TAYLOR SERIES EXPANSION AND a-CRITERION

The NDPS analytical expression is found in the previous section. Since $\Phi_i(i=1,2,3)$ represents three roots of (11), it is generally hard to get the analytical root-seeking formula. To analytically explore the NDAP, i.e., the expression of velocity of nonlinear waves and diagnostic relationships amidst the wave parameters, it is a common practice to change the transcendental equation into an algebraic one with the aid of Taylor series expansion. But the following aspects deserve our attention: i) the conditions for Taylor expansion should not damage those for the periodic solution of the equation at issue, and ii) the conditions for the algebraic equation solution to exist obtained by the expansion must be reconcilable with those for the periodic solution, and, besides, the problem of discontinuous points should not be ignored. In a word, the Taylor series serves, in truth, as a tool to relax some constraints of the system. As such, the domain wherein the approximate solution of the algebraic equation is available has to be within the scope defined by the constraints of the problem.

In the following the demonstration of the likelihood of the expansion for the barotropic semi-geostrophic model is performed with the related dimensionless a-criterion. Detailed discussion is made with $C_x<0$ as example and similar examination is done of the case of $C_x>0$.

The transcendental function $\ln\left|1+\dfrac{\Phi}{C_0^2}\right|$ of (11) is treated by the Taylor evolution, and with the cubic term retained we have

$$g(\Phi)\equiv\frac{\mu^2(1-a)}{6K_h^2C_0^2a}\Phi^3+\frac{\mu^2(1+a)}{2K_h^2a}\Phi^2+E=0. \tag{15}$$

To obtain (15), $|\Phi|<C_0^2$ (i.e., $h<H$) must be available, and meanwhile $C_0^2<\Phi_1$ is also necessary, meaning that $F(C_0^2)<F(\Phi_1)=0$, with

$$F(C_0^2)=E+\frac{2C_0^2f_0^2}{3K_h^2a}+\frac{C_0^2f_0^2}{K_h^2}(2\ln2-1)<0, \tag{16}$$

and

$$E<E_3=\frac{C_0^2f_0^2}{K_h^2a}\left[-\frac{2}{3}+a(2\ln2-1)\right]. \tag{17}$$

We shall prove that the condition (17) is contained in the condition (7). It is easy to find $E_3-E_2=\dfrac{C_0^2f_0^2}{K_h^2}(3-2\ln2)>0$, that is $E_2<E_3$. Consequently, when (7) is satisfied, so is (17) for the expansion.

Next, we come to the second problem — The periodic solution arising after the evolution should not argue against the periodic solution before the operation for the system.

In the same way as indicated before, we can find (15) to have the condition of a periodic solution (see Fig.3) of form

$$E_1<E<E_4, \tag{18}$$

where $E_4=\dfrac{-2C_0^2f_0^2(1+a)^3}{3K_h^2a(1-a)^2}$. Besides $\Phi^*=-2C_0^2\dfrac{(1+a)}{(1-a)}=\Phi_0\eta$ in which $\eta=(1+a)/$

$(1-a)$ is the translation factor. Since $E_4 > E_1 = 0$, we have $-1 < a < 0$ and $\eta < 1$, leading to $0 > \Phi^* > \Phi_0$. In the system treated with the series expansion, therefore, the equilibrium origin remains unchanged but the other point is moved from Φ_0 in the direction opposite to that of the phase velocity with the amplitude depending on η. In view of the fact that the untreated equation has its periodic solution satisfying $a > -\frac{1}{3}$, $\eta = \frac{1}{2}$ for the maximum amplitude of $a = -\frac{1}{3}$, meaning $\Phi^* = \frac{\Phi_0}{2}$ so that $\frac{\Phi_0}{2} > \Phi^* > \Phi_0$ is available in general. In other words, in the original system the break point $\frac{\Phi_0}{2}$ is still kept between the equilibria $(\Phi^*, 0)$. On the other hand, no breakage is found in the series-expanded (15), which somewhat differs from the untreated system. As stated earlier, the original system shows little singularity of the breakage and can be regarded as continuous system with added definition. As such, the treatment of the expansion has not very significant effects on the problem of breakage.

Now let us compare the magnitudes of E_2 and E_4 by use of inequality. One can see that for $-1/3 < a < 2-\sqrt{5}$ $(2-\sqrt{5} < a < 0)$ we have $E_4 > E_2$ $(E_4 < E_2)$. In the former case $E_1 < E < E_2 < E_4$ indicates that original equation has the periodic solution and the series-expanded form corresponds with the condition of periodic solution, too, meaning that no doubt the condition (7) satisfies the condition (18); and $E_2 < E < E_4$ shows that (18) satisfies (7) in a natural manner, suggesting the periodic solution available in the system both with and without the expansion except that for the availability in the treated equation the evaluation of E is limited (In reality for a bigger scope $E_4 < E < E_2$, the untreated equation still has the periodic solution). But in this scope of the evaluation the untreated equation has the periodic solution as opposed to the series-expanded system.

From the foregoing analysis we can get the possible condition for the Taylor expansion, i.e., the dimensionless a-criterion

$$
\begin{aligned}
&-1/3 < a < 2-\sqrt{5}, \qquad E_1 < E < E_2, \\
&2-\sqrt{5} < a < 0, \qquad E_1 < E < E_4.
\end{aligned} \tag{19}
$$

For a system where (19) is not satisfied the condition of the periodic solution argues against each other before and after the series evolution. In this case the expansion is unapplicable. For $\varphi = 45°N$, $-\frac{1}{3} < a < 2-\sqrt{5}$ $(2-\sqrt{5} < a < 0)$ is equivalent to $-20 < C_x < -12$ $(-12 < C_x < 0)$.

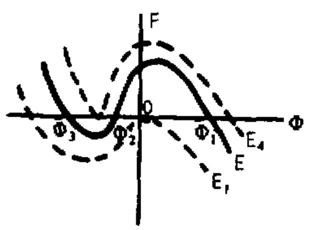

Fig. 3. $F(\Phi)$ image based on the series expansion for $C_x < 0$.

V. EXPRESSION VELOCITY OF NONLINEAR WAVE

If the a-criterion given above is satisfied, the three roots of the cubic algebraic equation $g(\Phi)$ are used in place of those of (11). The root-seeking formula is given below.

Transformation is mode of (15) as follows

$$Q = \frac{1}{2} - \frac{\Phi}{\Phi^*}, \tag{20}$$

from which we find

$$4Q^3 - 3Q = 2\varepsilon - 1, \tag{21}$$

where $\varepsilon (= E/E_4)$ is the number of dimensionless enstrophy.

Inspecting (21), we get

$$Q \in [-1,1] <=> 2\varepsilon - 1 \in [-1,1] <=> \varepsilon \in [0,1].$$

Assuming $Q = \cos\alpha$ and $\alpha \in [0,\pi]$, and using the triplication formula, we obtain

$$\cos 3\alpha = 4\cos^3\alpha - 3\cos\alpha = 2\varepsilon - 1. \tag{22}$$

Hence, for $0 < \varepsilon < 1$, i.e., $0 < E < E_4$, (22) has three real roots, which is consistent with (18) derived in the last section ($E_1 = 0$). From (22) and using (20), we have

$$\begin{aligned} \Phi_1 &= -\Phi^* \left\{ \cos\left[\frac{1}{3}\arccos^{-1}(2\varepsilon - 1)\right] - \frac{1}{2} \right\}, \\ \Phi_2 &= -\Phi^* \left\{ -\cos\left[\frac{1}{3}\arccos^{-1}(2\varepsilon - 1) + \frac{\pi}{3}\right] - \frac{1}{2} \right\}, \\ \Phi_3 &= -\Phi^* \left\{ -\cos\left[\frac{1}{3}\arccos^{-1}(2\varepsilon - 1) - \frac{\pi}{3}\right] - \frac{1}{2} \right\}. \end{aligned} \tag{23}$$

Obviously $\Phi_1 > 0 > \Phi_2 > \Phi_3$. For $\varepsilon = 0$, $\Phi_1 = \Phi_2 = 0$, $\Phi_3 = \frac{3\Phi^*}{2}$ and $m^2 = 0$, and for $\varepsilon = 1$, $\Phi_1 = -\frac{1}{2}\Phi^*$, $\Phi_2 = \Phi_3 = \Phi^*$ and $m^2 = 1$.

From (23) or directly from Wida theorem, we get

$$\Phi_1\Phi_2\Phi_3 = -\frac{\varepsilon}{2}\Phi^{*3}, \quad \Phi_1 + \Phi_2 + \Phi_3 = \frac{3}{2}\Phi^*, \quad \Phi_1\Phi_2 + \Phi_1\Phi_3 + \Phi_2\Phi_3 = 0, \tag{24}$$

with which the NDPS (10) can be rewritten as

$$\Phi = \Phi_2 + \hat{\Phi} C n^2 \sqrt{\frac{(\Phi_3 - \Phi_1)\mu^2(1-a)}{12K_h^2 C_0^2 a}}\,\theta, \tag{25}$$

where $\hat{\Phi}(= \Phi_1 - \Phi_2)$ is the amplitude with $\Phi_i (i = 1,2,3)$, as shown in (23). However, Eq. (14) for the solitary wave solution can be put as

$$\Phi = \Phi^* \left[1 - \frac{3}{2}\sec^2 h \sqrt{\frac{-\mu^2(1+a)}{4K_h^2 a}}\right]\theta. \tag{26}$$

Clearly the solitary wave amplitude has the form

$$\hat{\Phi} = -\frac{3}{2}\Phi^* \quad \text{or} \quad \Phi^* = -\frac{2}{3}\hat{\Phi}. \tag{27}$$

Assume $\Phi_1 = \gamma\hat{\Phi}$, where γ has the physical implication of the aspect ratio of elliptic cosine wave and we have $\Phi_2 = (\gamma - 1)\hat{\Phi}$. Letting $\Phi_3 = \delta\hat{\Phi}$ and with the aid of (24), we get $\delta = \frac{\gamma(1-\gamma)}{2\gamma - 1}$. From $\Phi_1 > 0 > \Phi_2 > \Phi_3$ we find

$$\gamma > 0 > \gamma - 1 > \frac{\gamma(1-\gamma)}{2\gamma - 1}. \tag{28}$$

Solving (28) yields $\frac{1}{3} < \gamma < \frac{1}{2}$. And $\gamma \to \frac{1}{3}$ and $m^2 = 1(\gamma \to \frac{1}{2}$ and $m^2 = 0)$ are associated with solitary (linear) waves.

From $g(\Phi_1) = g(\gamma\hat{\Phi}) = 0$ and $g(\Phi_2) = g[(\gamma - 1)\hat{\Phi}] = 0$, we get

$$g(\gamma\hat{\Phi}) - g[(\gamma - 1)\hat{\Phi}] = 0, \tag{29}$$

Solution of (29) gives

$$\Phi^* = -\frac{2}{3}\hat{\Phi}P(\gamma), \tag{30}$$

in which $P(\gamma) = \frac{(3\gamma^2 - 3\gamma + 1)}{(1 - 2\gamma)}$ and $1 < P(\gamma) < \infty$. And the smaller the P(γ), the stronger the nonlinearity. In view of $P(1/3) = 1$, (30) contains (27). Besides, for the same $\hat{\Phi}$, the solitary wave moves more slowly. From the definition of Φ^*, (30) can also be put into the form

$$C_x = -\frac{\beta}{\mu^2}\left[1 - \frac{2}{(1 + \frac{3C_0^2}{P(\gamma)\hat{\Phi}})}\right], \tag{31}$$

which is the expression of velocity of nonlinear waves.
From (31) and with $-1/3 < a < 0$, one can also get

$$3C_0^2 \geqslant \frac{3C_0^2}{P(\gamma)} > \hat{\Phi} > \frac{3C_0^2}{2P(\gamma)} \geqslant 0, \tag{32}$$

which indicates that the waves described are of finite amplitude with its maximum of $3C_0^2$. Also, from (31) we can show that the velocity is the smaller, the bigger the amplitude, a fact in close concord with weather observation.

On the other hand, it is hard for (31) to describe the wave motion with $\gamma \to \frac{1}{2}$, $P(\gamma) \to \infty$ and $\hat{\Phi} \to 0$. We give another expression for the velocity as follows.

Take $L = \frac{2\pi}{k}$ and also use $L = \frac{2}{k}\sqrt{\frac{12K_h^2 C_0^2 a}{(\Phi_3 - \Phi_1)\mu^2(1-a)}}\,K(m)$ derived by (25) in conjunction with $\Phi_1 - \Phi_3 = (\gamma - \delta)\hat{\Phi}$ and (30) employed and we arrive at

$$C_x = -\frac{\beta}{\mu^2 + K_h^2\left[\frac{2K(m)}{\pi}\right]^2 q(\gamma)}, \tag{33}$$

where $q(\gamma) = \frac{(3\gamma^2 - 3\gamma + 1)}{\gamma(2\gamma - 3)}$. With $\frac{1}{3} < \gamma < \frac{1}{2}$, we get $\frac{1}{2}\sqrt{3} \leqslant q(\gamma) \leqslant 1$.

With $E = E_1$, $\hat{\Phi} \to 0$, $\gamma \to \frac{1}{2}$, $m^2 \to 0$, $K(m) \to \frac{1}{2}\pi$ and $q(\frac{1}{2}) \to 1$, we find

$$C_x = -\frac{\beta}{\mu^2 + K_h^2}, \tag{34}$$

which is none other than the expression for the velocity of linear Rossby wave. Inspection of (33) and (34) based on $q(\gamma) \leqslant 1$ and $K(m) \leqslant \frac{1}{2}\pi$ shows that the linear wave travels fastest.

Further, the next expression can be found with the aid of $m^2 = \frac{q(\gamma)}{P(\gamma)}$ and (12) and is in the form

$$L = \frac{2}{k} K\left(\sqrt{\frac{q(\gamma)}{p(\gamma)}}\right) \left[\frac{2\gamma(\gamma-1)\delta}{E(\gamma-\delta)}\right]^{\frac{1}{2}} \hat{\Phi}, \tag{35}$$

which indicates that the wavelength L is proportional to the amplitude $\hat{\Phi}$. Evidently, with other conditions unchanged, the wave of larger amplitude is related to that of longer wavelength. This is revealed in weather maps where the ridge / trough of bigger amplitude are associated with longer wavelength.

For intuitive purposes, the numerical correspondences of the aspect ratio γ, dimensionless enstrophy number ε, modulus m^2, $p(\gamma)$ and $q(\gamma)$ are in part tabulated in the following.

Table 1. Numerical Correspondences of γ, ε, m^2, p, q and Waveform

γ	1/2	0.46	$1-(\sqrt{3}/3)$	0.38	1/3
ε	0	$(2-\sqrt{2})/4$	1/2	$(2+\sqrt{2})/4$	1
$p(\gamma)$	∞	3.19	$\sqrt{3}$	1.22	1
$q(\gamma)$	1	0.89	$\sqrt{3}/2$	0.9	1
m^2	0	0.28	1/2	0.74	1
waveform	cosine	elliptic cosine			solitary

Through ε, Φ_i can be found from (23), with which γ and m^2 are determined so that ε and m^2 are in well-defined one-to-one correspondence, and with magnitude of ε the condition of a periodic solution can be specified, leading to a good correspondence between ε and γ. However, Φ_i can be obtained much more easily directly from ε than from γ. As a result, it is appropriate to use ε in lieu of γ in characterizing the waveform aspect ratio. In a word, it is advisable to employ ε to describe nonlinear waves.

The conclusions presented agree mostly with observational facts, which indicates that nonlinear nondispersive periodic solution wave is likely to be in the atmosphere or it is perhaps the approximate form of real nonlinear wave therein. Wu (1985) reported there is motion of semi-geostrophic similarity in nature in the air. The paper shows that it is quite possible to better describe nonlinear low-frequency Rossby slow wave in terms of a semi-geostrophic model.

VI. CONCLUDING REMARKS

The discussion presented provides us with much understanding of the problem of nonlinear wave studied in a barotropic semi-geostrophic model. Yet the analysis is preliminary. The issue of the velocity expression is settled only partially in finding the approximate solution by means of function approximation, a problem that remains to be investigated. The issue of nonlinear wave in baroclinic semi-geostrophic model is not dealt with in this article because of space limitation and will be attached in a separate paper.

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