Arnol'd's Second Nonlinear Stability Theorem for General Multilayer Quasi–geostrophic Model

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ABSTRACT

Arnol'd's second nonlinear stability criterion for motions governed by a general multilayer quasi–geostrophic model is established. The model allows arbitrary density jumps and layer thickness, and at the top and the bottom of the fluid, the boundary condition is either free or rigid. The criterion is obtained by the establishment of the upper bounds of disturbance energy and potential enstrophy in terms of the initial disturbance field.

Key words: Nonlinear stability, Multi–layer Q–G model

I. INTRODUCTION

Mu, Zeng, Shepherd and Liu (1992; 1993) studied the nonlinear stability of multilayer quasi–geostrophic flow with equal density jumps, and with both rigid boundary conditions at the top and the bottom of the fluid. By using the matrix transformation theory properly, which originally appeared in Zeng (1979; 1989), they established upper bounds of disturbance energy and potential enstrophy in terms of the initial disturbance field, and obtained Arnol'd's second stability theorem. In this paper, it is shown that, without much difficulty, their method can also be utilized to treat a quite general multilayer quasi–geostrophic model, with arbitrary density jumps and layer thickness, and with either a free or rigid (including the possibility of topography) boundary condition at the top and the bottom of the fluid. Arnol'd's second theorem, which is a nonlinear stability criterion, is obtained (Criterion 3.1). The upper bounds of energy and potential enstrophy of finite–amplitude disturbance to steady basic state are established. These bounds are expressed in terms of the initial disturbance fields, and hold uniformly in time, and tend to zero uniformly as the initial disturbance decreases to zero, which is an improvement over the ones established by Mu et al. (1992; 1993) provided that the initial disturbance energy is not zero.

It is also worthwhile to point out that this paper also presents a sufficient condition for the nonlinear stability (Criterion 3.3), which is equivalent to Criterion 3.1, but is easier applicable.

Ripa (1992) also investigated the nonlinear stability of motions governed by the model

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we consider in this paper. The main results of Ripa's (1992) was corrected in Ripa (1993)①. But in that paper Ripa only considered the equivalent barotropic model in detail and only *mentioned briefly that the multilayer case could be treated without much difficulty. No* detailed result had been given there.

II. THE MODEL

Consider a stratified fluid of N superimposed layers of constant density $\rho_1 < \cdots < \rho_N$, with arbitrary density jumps and mean layer depth d_i. The flow is governed by the multilayer quasi-geostrophic potential vorticity equations (Pedlosky, 1979; Ripa, 1992).

$$\frac{\partial P_i}{\partial t} + \partial(\Phi_i, P_i) = 0, \quad i = 1, \cdots, N, \tag{2.1}$$

where $\Phi_i(x,y,t)$ is the stream function in layer i and

$$P_i = \nabla^2 \Phi_i - d_i^{-1} \sum_{j=1}^{N} T_{ij} \Phi_j + f_i(x,y), \quad i = 1, \cdots, N \tag{2.2}$$

is the potential vorticity in the ith layer. And $\partial(f,g) = f_x g_y - f_y g_x$ is the two-dimensional Jacobian, x and y are the eastward and northward coordinates, respectively, t is the time, ∇^2 the two-dimensional Laplacian operator, and

$$f_1 = f_0 + \beta y - \frac{f_0 \tau_0(x,y)}{d_1},$$
$$f_2 = f_3 = \cdots f_{N-1} = f_0 + \beta y,$$
$$f_N = f_0 + \beta y + \frac{f_0 \tau_N(x,y)}{d_N},$$

where f_0 is a representative value of the Coriolis parameter, and $\tau_0(x,y)$ and $\tau_N(x,y)$ are the topography at the top and the bottom of the fluid respectively. If the top (or bottom) boundary is free, $\tau_0(x,y) = 0$ (or $\tau_N(x,y) = 0$).

$T = (T_{ij})$is an $N \times N$ symmetric tridiagonal matrix,
$T_{ii} = f_0^2((g_{i-1})^{-1} + (g_i)^{-1}), \quad i = 1, \cdots, N,$
$T_{i+1,i} = T_{i,i+1} = -f_0^2(g_i)^{-1} < 0, \quad i = 1, \cdots, N-1,$
$T_{ij} = 0, \quad |i-j| > 1,$

where g_i is the buoyancy jump across the interface between the ith and the $(i+1)$th layer, and if the top (or bottom) boundary is rigid, then $(g_0)^{-1} = 0$ (or $(g_N)^{-1} = 0$); and when the top (or bottom) boundary is free, $(g_0)^{-1} > 0$ (or $(g_N)^{-1} > 0$).

The horizontal domain D under consideration is a bounded, multiply (or simply) connected domain on the beta-plane, with a smooth boundary ∂D consisting of $J+1$ simple closed curves ∂D_j, $j = 0, \cdots, J$. The boundary conditions are the usual ones of no normal flow and conservation of circulation in each layer, namely

①Ripa, P. 1993. Arnol'd's second stability theorem for the equivalent barotropic model. *J. Fluid Mech.*, to appear.

$$\left.\frac{\partial \Phi_i}{\partial s}\right|_{\partial D}=0,\quad \frac{d}{dt}\int_{\partial D_j}\nabla\Phi_i \vec{n} ds=0, \tag{2.3}$$

where s is the arc length along the boundary ∂D, and $\vec{n}$ the outward unit normal. Now suppose that $(\Phi_i, P_i)=(\Psi_i, Q_i)$ is a steady solution to the system (2.1)–(2.3), and we further assume that there exist continuously differentiable functions Ψ_i (.) such that

$$\Psi_i(x,y)=\Psi_i(Q_i(x,y)),\qquad \forall(x,y)\in D;\quad i=1,\cdots,N. \tag{2.4}$$

And the disturbance (ψ_i, q_i) to this basic steady state is defined by

$$\Phi_i=\Phi_i+\psi_i,\quad p_i=Q_i+q_i, \tag{2.5}$$

with

$$q_i=\nabla^2\psi_i-d_i^{-1}\sum_{j=1}^{N}T_{ij}\psi_j,\quad i=1,\cdots,N. \tag{2.6}$$

III. NONLINEAR STABILITY CRITERIA

Corresponding to the hypothesis of Arnol'd's second theorem, we now assume that the functional relation (2.4) are monotonic with a negative slope, and there exist positive constants C_{1i} and C_{2i} such that

$$0<C_{1i}\leqslant-\frac{d\Psi_i}{dQ_i}\leqslant C_{2i}<\infty,\quad i=1,\cdots,N, \tag{3.1}$$

where $d\Psi_i/dQi=\nabla\Psi_i/\nabla Q_i$. The purpose now is to establish upper bounds for the disturbance energy

$$E(t)=\frac{1}{2}\int_D\Big[\sum_{i=1}^{N}d_i|\nabla\psi_i|^2+f_0^2((g_0)^{-1}\psi_1^2+(g_N)^{-1}\psi_N^2$$
$$+\sum_{i=1}^{N-1}(g_i)^{-1}(\psi_{i+1}-\psi_i)^2)\Big]dxdy \tag{3.2}$$

and the disturbance enstrophy

$$Z(t)=\frac{1}{2}\int_D\sum_{i=1}^{N}d_iq_i^2\,dxdy \tag{3.3}$$

in terms of the initial disturbance fields.

To this end, first define the functions $G_i(\eta)=\int_0^{\eta}\Psi_i(\tau)d\tau$

and

$$A(t)=\int_D\sum_{i=1}^{N}d_i\Big[G_i(Q_i+q_i)-G_i(Q_i)-G'_i(Q_i)q_i\Big]dxdy. \tag{3.4}$$

Using conservations of total energy, total potential enstrophy and total circulation in each layer, it is easy to show that

$$\frac{d}{dt}(E(t)+A(t))=0, \quad t\geqslant 0. \tag{3.5}$$

That is, $E+A$ is the energy–Casimir functional.

Following Mu et al. (1992; 1993), we decompose the disturbance (ψ_i, q_i) into two parts

$$\psi_i=\psi_i'+\psi_i^*, \quad q_i=q_i'+q_i^*, \quad i=1,\cdots,N, \tag{3.6}$$

where $q^*=\int_D q_{io}\,dxdy \Big/ \int_D dxdy$, $q_{io}=q_i(x,y,0)$, and ψ_i' is the solution of the problem

$$\nabla^2\psi_i'-d_i^{-1}\sum_{j=1}^{N}T_{ij}\psi_i'=q_i', \quad \left.\frac{\partial\psi_i'}{\partial s}\right|_{\partial D}=0, \tag{3.7}$$

$$\int_{\partial D_j}\nabla\psi_i'\cdot\vec{n}ds=0, \quad j=0,\cdots,J; \quad i=1,\cdots,N.$$

The existence and "uniqueness" of such a solution can be established as in Mu et al. (1993).

Let λ be the least positive eigenvalue of the problem

$$\nabla^2\varphi+\lambda\varphi=0 \quad \text{in} \quad D; \quad \left.\frac{\partial\varphi}{\partial s}\right|_{\partial D}=0, \quad \int_{\partial D_j}\nabla\varphi\cdot\vec{n}ds=0, \quad j=0,\cdots,J. \tag{3.8}$$

Define the matrix

$$K=\mathrm{diag}(d_1^{-1/2},\cdots,d_N^{-1/2}), \tag{3.9}$$

and the disturbance energy associated with (ψ', q') is

$$E'(t)=\frac{1}{2}\int\Big(\sum_{i=1}^{n}d_i\left|\nabla\psi_i'\right|^2+f_0^2((g_0)^{-1}(\psi_1')^2+(g_N)^{-1}(\psi_N')^2+\sum_{i=1}^{N-1}(g_i)^{-1}$$
$$(\psi_{i+1}'-\psi_i')^2)\Big)dxdy. \tag{3.10}$$

Analogues to Mu et al. (1992; 1993), we can prove that

$$E'(t)\leqslant\frac{1}{2}\int_D(\vec{q}')^TK^{-1}(\lambda E+KTK)K^{-1}\vec{q}', \tag{3.11}$$

where $\vec{q}'=\mathrm{col}(q_1',\cdots q_N')$ and $\vec{\psi}'=\mathrm{col}(\psi_1',\cdots,\psi_N')$ are column vectors.

Denote

$$E^*=\frac{1}{2}\int_D\Big(\sum_{i=1}^{N}d_i\left|\nabla\psi_i^*\right|^2+f_0^2((g_0)^{-1}(\psi_i^*)^2+(g_N)^{-1}(\psi_N^*)^2+\sum_{j=1}^{N-1}(g_i)^{-1}$$
$$(\psi_{j+1}^*-\psi_j^*)^2\Big)dxdy,$$

$$Z^*=\frac{1}{2}\int_D\sum_{i=1}^{N}d_i(q_i^*)^2dxdy,$$

$$Z'(t)=\frac{1}{2}\int_D\sum_{i=1}^{N}d_i(q_i')^2)dxdy,$$

then we have

$$E(t) \leqslant ((E'(t))^{1/2} + (E^*)^{1/2})^2 \tag{3.12}$$

$$Z(t) = Z'(t) + Z^* . \tag{3.13}$$

Since E^* and Z^* are independent of time (see Mu et al. (1992, 1993), by (3.11)–(3.13), we need only to estimate $Z'(t)$ so that the upper bounds of $Z(t)$ and $E(t)$ follow immediately. By Assumption (3.1) and (3.5), we have

$$\frac{1}{2}\int_D \sum_{i=1}^{N} C_{1i} d_i (q_i)^2 dxdy \leqslant -A(t) = E(t) - E(0) - A(0). \tag{3.14}$$

In the following, a slightly different method is adopted to estimate $Z'(t)$ and $E'(t)$, which yields a better result:

Denote

$$H = E^* - E(0) - A(0) - \frac{1}{2}\int_D \sum_{i=1}^{N} C_{1i} d_i (q_i^*)^2 dxdy,$$

then by (3.12) and (3.14), we have

$$\frac{1}{2}\int_D \sum_{i=1}^{N} C_{1i} d_i (q_i')^2 dxdy \leqslant E'(t) + 2(E^* E'(t))^{1/2} + H. \tag{3.15}$$

Let $C = \mathrm{diag}(C_{11}, \cdots, C_{1N})$, by (3.11) and (3.15) we have

$$\frac{1}{2}\int_D (\vec{q}')^T K^{-1}(C - (\lambda E + KTK)^{-1})K^{-1}\vec{q}' dxdy \leqslant 2(E^*)^{1/2}$$
$$\left(\frac{1}{2}\int_D (\vec{q}')K^{-1}(\lambda E + KTK)^{-1}K^{-1}\vec{q}' dxdy\right)^{1/2} + H.$$

Obviously, λ_1, the smallest eigenvalue of matrix KTK, is positive or zero (cf. Liu and Mu (1992), Lemma 3.1). We also assume the matrix $C - (\lambda E + KTK)^{-1}$ is positive definite and its smallest eigenvalue is k_1, then

$$k_1 Z'(t) - 2(E^* / (\lambda + \lambda_1))^{1/2} (Z'(t))^{1/2} - H \leqslant 0, \tag{3.16}$$

since $Z'(t)$ is real, this quadratic inequality with respect to $(Z'(t))^{1/2}$ yields

$$Z'(t) \leqslant ((E^*)^{1/2} + (E^* + \bar{\lambda} k_1 H)^{1/2})^2 / (\bar{\lambda} k_1^2), \tag{3.17}$$

where $\bar{\lambda} = \lambda + \lambda_1$. Note also that (3.17) guarantees that $E^* + \bar{\lambda} k_1 H \geqslant 0$ holds.

By (3.11), we have (cf. Mu et al.(1993))

$$E'(t) \leqslant \frac{Z'(t)}{\bar{\lambda}}.$$

Using this inequality, together with (3.12), (3.13) and (3.17), we obtain the bounds of $E(t)$ and $Z(t)$ as follows:

$$E(t) \leqslant ((E^*)^{1/2}(1 + \bar{\lambda} k_1) + (E^* + \bar{\lambda} k_1 H)^{1/2}) / (\bar{\lambda} k_1))^2 \tag{3.18}$$

$$Z(t) \leqslant ((E^*)^{1/2} + (E^* + \bar{\lambda} k_1 H)^{1/2})^2 / (\bar{\lambda} k_1^2) + Z^* . \tag{3.19}$$

Since E^*, H and Z^* depend only on the initial disturbance, and tend to zero as the initial disturbance tends to zero, the bounds of the disturbance energy $E(t)$ and disturbance potential enstrophy $Z(t)$ tend to zero as the initial disturbance does. Thus, take this as the definition of nonlinear stability, we have

Criterion 3.1. Suppose the basic state (Ψ_i, Q_i) satisfies (2.4), (3.1) and the matrix $C - (\lambda E + KTK)^{-1}$ is positive definite, then (Ψ_i, Q_i) is nonlinearly stable. And the upper bounds of the finite-amplitude disturbance energy and potential enstrophy are given by (3.18) and (3.19), respectively.

Similar to the argument of Mu et al. (1992; 1993), we conclude that k_1, the smallest eigenvalue of the matrix $C - (\lambda E + KTK)^{-1}$, satisfies

$$k_1 \geqslant \tilde{k}_1 = \min C_{1i} - \frac{1}{\bar{\lambda}} .$$

So if $\bar{\lambda} \min C_{1i} > 1$, then the matrix $C - (\lambda E + KTK)^{-1}$ is positive definite and we have

Criterion 3.2. If $\bar{\lambda} \min C_{1i} > 1$, then the basic state is nonlinearly stable in the sense described in Criterion 3.1. And (3.18) and (3.19) also yield upper bounds on $E(t)$ and $Z(t)$ after k_1 is replaced by $\tilde{k}_1$.

It is worthwhile to point out that the functional H defined in this paper is always less than or equal to the functional H defined in Mu et al. (1992, 1993), the bounds (3.18) and (3.19) contain the bounds there as a special case. If the initial disturbance energy is not zero, the H defined here is smaller, so the bounds established in this paper is better than the corresponding ones there.

Sometime, leaving other matter to further studies, we are only interested in determining whether the basic state is nonlinearly stable. In this case, the following result is convenient, since we need not to find out the inverse matrix of $\lambda E + KTK$ for the verification of it.

Criterion 3.3. Suppose (2.4) and (3.1) hold. If matrix $\lambda E + KTK - C^{-1}$ is positive definite, then the basic state is nonlinearly stable.

To get this conclusion, we need only to prove that

$\lambda E + KTK - C^{-1}$ is positive definite iff $C - (\lambda E + KTK)^{-1}$ is. This is demonstrated as follows. For any matrix X, let $I(X)$ denote the number of its positive eigenvalues. By matrix theory (cf. Horn et al. 1985), $I(XY) = I(Y)$ if X is a positive definite matrix and Y is a symmetric matrix. Now since KTK is a positive or semipositive definite matrix, $\lambda E + KTK$ is positive definite and $\lambda E + KTK - C^{-1}$ is a symmetric matrix, we have

$$\begin{aligned} I(C - (\lambda E + KTK)^{-1}) &= I((\lambda E + KTK)^{-1}(\lambda E + KTK - C^{-1})C) \\ &= I((\lambda E + KTK - C^{-1})C) = I(\lambda E + KTK - C^{-1}). \end{aligned}$$

This is the desired result.

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