Oscillatory Rossby Solitary Waves in the Atmosphere

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ABSTRACT

The linear Rossby wave frequency expression is expanded at higher accuracy based on the scale difference characteristics of atmospheric long waves in the x and y directions. That the nature of the waves represented by the expansion is identical to that of the original ones is demonstrated both in phase velocity $\vec{C}$ and wave energy dispersion speed $\vec{C}_g$, followed by the derivation of the nonlinear expression describing atmospheric long wave behaviors with the associated approximate analytic solution obtained. Then, for the first time atmospheric' oscillatory Rossby solitary wave' with its dispersion relation is obtained by numerical calculation with the aid of physical parameters of the real atmosphere. The solitary wave is found to be very close to such longwave systems as blocking highs and cut-off depressions in the actual atmosphere.

Key words: Longwave approximation, Oscillatory Rossby solitary wave

I. INTRODUCTION

Observed atmospheric motion sometimes exhibits intense nonlinear features, the examples being the blocking high and meso-scale convective system. For this reason, research into the problem of atmospheric nonlinear waves is of value to further understanding of some aspects of air motion. On the strength of the settlement of some nonlinear issues in the mathematic context, atmospheric science as an applied branch of learning has made considerable progress in the study of nonlinear waves. Using different techniques and starting from the governing equation for atmospheric motion, many authors have achieved the results of Rossby solitary waves analogous in pattern to a blocking high (Redekopp, 1977; Redekopp et al., 1978; Weidman et al., 1980; Liu and Liu, 1982; Wu, 1985; Chen, 1992; Lu, 1987), as shown in Fig. 1. These findings acquired by the theoretical models available are, however, different, to some degree, from the observed solitary wave system, as illustrated in Fig. 2. In comparison, the actual solitary system is closer to the system of oscillatory solitary waves (Kawahara, 1972; Dai et al., 1990). As such, study of the oscillatory wave is likely to contribute to the understanding of some aspects of nonlinear motion in the real atmosphere.

Based on the scale difference features in the x and y directions and through the use of longwave approximation and consideration of the influence of basic flow, the linear Rossby wave dispersion relation is approximately evolved at even higher accuracy, demonstrating the identical nature of waves involved in the expansion and the original ones both in phase velocity $\vec{C}$ and wave energy dispersion velocity $\vec{C}_g$. Then, from the combination of linear dispersion with nonlinear advection, we gent the generalized KdV equation and its approximate analytic solution. Through numerical calculation for the solution based on the physical parameters from the real atmosphere obtained, for the first time, is atmospheric oscillatory

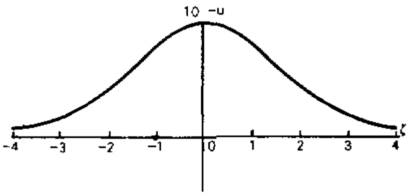

Fig. 1. Image of solitary wave with $u = -U\mathrm{sech}^2\frac{\zeta}{2}$.

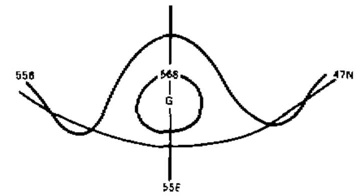

Fig. 2. The outermost contour of the 500 hPa European blocking high at 1200 GMT, 7 April 1991.

solitary wave that is closer to the observed blocking solitary wave than the common Rossby counterpart, with the propagation features similar to those of the blocking system.

II. NONLINEAR EQUATION FOR ATMOSPHERIC LONG WAVES

Set $\bar{u}$ to be the basic flow of atmospheric motion. For the sake of analysis, $\bar{u}$ is assumed constant so that linear Rossby wave has its dispersion relation of the form

$$\omega = k\bar{u} - \frac{\beta k}{k^2 + m^2}, \tag{1}$$

where $\beta = \frac{\partial f}{\partial y}$; f= the Coriolis parameter; ω= the wave frequency; $k(m)$= the wavenumber in the $x(y)$ direction with their respective form as follows:

$$k = \frac{2\pi}{L_x} = \frac{1}{L}, \qquad m = \frac{2\pi}{L_y} = \frac{1}{D}, \tag{2}$$

in which $L(D)$ denotes the horizontal characteristic scale of long wave motion in the $x(y)$ direction.

For extratropic latitudes Rossby long wave is marked by the basic feature of the x-perturbation scale L greater than the y-perturbation scale D, viz.,

$$\frac{D}{L} = \frac{k}{m} < 1, \tag{3}$$

$$(\frac{D}{L})^2 = (\frac{k}{m})^2 < 1. \tag{4}$$

Transformation of (1) yields

$$\omega = k\bar{u} - \frac{k\beta}{m^2}\left[\frac{1}{1 + (\frac{k}{m})^2}\right]. \tag{5}$$

From (4) and in virtue of mathematics $1 / 1 + (\frac{k}{m})^2$ of (5) can be series-expanded. And for $(\frac{k}{m})^2 \leqslant 0.2$, taking the first three terms from the representation is enough to ensure considerable accuracy (with relative error $< 0.9\%$). Since extratropical Rossby wave has a small value of $(\frac{k}{m})^2$, only the first three terms are considered in this present work. In such a case (5) is represented approximately as

$$\omega = k\bar{u} - \frac{k\beta}{m^2}\left[1 - (\frac{k}{m})^2 + (\frac{k}{m})^4\right]. \tag{6}$$

Let u be the wave quantity of linear Rossby waves and we have

$$u = Ae^{(ikx - i\omega t)}, \tag{7}$$

so that

$$\frac{\partial u}{\partial t} = -i\omega u; \qquad \frac{\partial u}{\partial x} = iku. \tag{8}$$

In terms of (8) and (6) the governing equation in relation to (6) takes the form

$$\frac{\partial u}{\partial t} + \bar{u}\frac{\partial u}{\partial x} - \frac{\beta}{m^2}\frac{\partial u}{\partial x} - \frac{\beta}{m^4}\frac{\partial^3 u}{\partial x^3} - \frac{\beta}{m^6}\frac{\partial^5 u}{\partial x^5} = 0, \tag{9}$$

which is the wave equation for linear Rossby wave propagation equivalent essentially to the vorticity analog and we can have the dispersion expression of linear Rossby waves either from (9) or the vorticity equation.

If u of (9) is viewed as the x–component of wind velocity, then (9) is in reality the momentum equation for the direction and also the equation of motion in that orientation that occurs under the same physical conditions as Rossby waves. The reader is referred to Wu (1985) for the detailed proof (which indicates their full similarity when basic flow is available).

Rossby wave is known to be dispersive in nature with the phase velocity $\vec{C}$ differing from wave energy dispersion speed $\vec{C}_g$ and energy dispersed outward during wave propagation. Hence, to ensure the identity of the wave described by the expansion to the original wave produced in evolving the frequency equation with the dispersion wave of $\vec{C} \neq \vec{C}_g$, it is necessary to keep the representation of the equation identical in phase velocity $\vec{C}$ and group velocity $\vec{C}_g$ to the original waves. Otherwise, the expansion will distort the properties of the original wave. Then, a problem arises. Is (9) resulting from (1) reasonable? It is an issue that needs proof.

From the foregoing manipulation one can see that the phase velocity ($C = \omega / k$) characterized by (6) is roughly equal to that by (1). Then, what happens to the group velocity ($C_g = \frac{d\omega}{dk}$)? At first differentiation is made of (1), and after some manipulations we find

$$C_{gx} = \frac{d\omega}{dk} = \bar{u} - \frac{\beta}{m^2} \cdot \frac{1 - (\frac{k}{m})^2}{[1 + (\frac{k}{m})^2]^2}; \tag{10}$$

$$C_{gy} = \frac{d\omega}{dm} = \frac{2\beta k}{m^2} \cdot \frac{1}{[1 + (\frac{k}{m})^2]^2}. \tag{11}$$

Then, a series–expansion is performed of $\frac{1}{1 + (\frac{k}{m})^2}$ and taking the first three terms, we get

$$\frac{1}{1 + (\frac{k}{m})^2} = 1 - \frac{k^2}{m^2} + \frac{k^4}{m^4}. \tag{12}$$

Substituting (12) into (10) yields

$$C_{gx} = \bar{u} - \frac{\beta}{m^2} \cdot \frac{1}{1+\frac{k^2}{m^2}}(1-\frac{k^2}{m^2})(1-\frac{k^2}{m^2}+\frac{k^4}{m^4})$$

$$= \bar{u} - \frac{\beta}{m^2} \cdot \frac{1}{1+\frac{k^2}{m^2}}(1-\frac{2k^2}{m^2}+\frac{2k^4}{m^4}-\frac{k^6}{m^6}).$$

Because of the smallness of $\frac{k}{m}$, $(\frac{k}{m})^6$ is exceptionally small and negligible so that only the terms of $(\frac{k}{m})$ lower than fourth power are retained (the same below). Thus, we find

$$C_{gx} = \bar{u} - \frac{\beta}{m^2} \cdot \frac{1}{1+\frac{k^2}{m^2}} \cdot (1-\frac{2k^2}{m^2}+\frac{2k^4}{m^4}). \tag{13}$$

Similarly, evolving (11) produces

$$C_{gy} = \frac{2\beta k}{m^3}(1-\frac{2k^2}{m^2}+\frac{3k^4}{m^4}). \tag{14}$$

Differentiation is done of (6), and after some manipulations we have

$$C'_{gx} = \bar{u} - \frac{\beta}{m^2}(1-\frac{k^2}{m^2}+\frac{k^4}{m^4}) - \frac{\beta}{m^2}(-\frac{2k^2}{m^2}+\frac{4k^4}{m^4}); \tag{15}$$

$$C'_{gy} = \frac{2\beta k}{m^3}(1-\frac{2k^2}{m^2}+\frac{3k^4}{m^4}). \tag{16}$$

Substitution of (12) into (15) yields

$$C'_{gx} = \bar{u} - \frac{\beta}{m^2}\left[\frac{1}{1+\frac{k^2}{m^2}} + (-\frac{2k^2}{m^2}+\frac{4k^4}{m^4})\right]$$

$$= \bar{u} - \frac{\beta}{m^2} \cdot \frac{1}{1+\frac{k^2}{m^2}} \cdot \left[1+(1+\frac{k^2}{m^2})(-\frac{2k^2}{m^2}+\frac{4k^4}{m^4})\right]$$

$$\approx \bar{u} - \frac{\beta}{m^2} \cdot \frac{1}{1+\frac{k^2}{m^2}} \cdot (1-\frac{2k^2}{m^2}+\frac{2k^4}{m^4}), \tag{17}$$

where C'_{gx} and C'_{gy} represent the group velocity, or wave energy dispersion speed, described by the expansion in the x and y direction, separately.

Comparison of (13), (14), (16) and (17) shows

$$C'_{gx} = C'_{gx}; \qquad C_{gy} = C'_{gy}. \tag{18}$$

Hence, (6) is reasonable, thereby leading to the identity in propagation velocity and energy dispersion features for the waves denoted by (9) and the original ones.

Since atmospheric motion is marked by nonlinearity, waves given by (9) are subject to nonlinear effects in their communication, which cause wave deformation in the course and appear as an advection term in the hydrodynamic context. Thus, with nonlinear advective effects considered (Wu, 1985) we find

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+\bar{u}\frac{\partial u}{\partial x}-\frac{\beta}{m^2}\frac{\partial u}{\partial x}-\frac{\beta}{m^4}\frac{\partial^3 u}{\partial x^3}-\frac{\beta}{m^6}\frac{\partial^5 u}{\partial x^5}=0, \tag{19}$$

which is the nonlinear equation for describing atmospheric long waves, i.e., Rossby waves, referred to as the generalized KdV equation. With no effects of the basic flow ($\bar{u}=0$) involved and dropping the last term, (19) is degenerated into another KdV form as shown in Wu.

III. OSCILLATORY ROSSBY SOLITARY WAVES

For convenience, (19) is reduced to a dimensionless equation. Set $x=Lx_1, u=Uu_1$ and $t=Tt_1=\frac{L}{U}t_1$ which are inserted into (19), resulting in

$$\frac{U}{\beta D^2}\frac{\partial u_1}{\partial t_1}+\frac{U}{\beta D^2}u_1\frac{\partial u_1}{\partial x_1}+\frac{\bar{u}-\beta D^2}{\beta D^2}\frac{\partial u_1}{\partial x_1}-(\frac{D}{L})^2\frac{\partial^3 u_1}{\partial x_1^3}-(\frac{D}{L})^4\frac{\partial^5 u_1}{\partial x_1^5}=0, \tag{20}$$

of which both sides are multiplied by $(\frac{D}{L})^2$, leading to

$$\frac{UL^2}{\beta D^4}\frac{\partial u_1}{\partial t_1}+\frac{UL^2}{\beta D^4}u_1\frac{\partial u_1}{\partial x_1}+\frac{(\bar{u}-\beta D^2)L^2}{\beta D^4}\frac{\partial u_1}{\partial x_1}-\frac{\partial^3 u_1}{\partial x_1^3}-(\frac{D}{L})^2\frac{\partial^5 u_1}{\partial x_1^5}=0,$$

where, if $\alpha=UL^2/\beta D^4$, $\gamma=(\bar{u}-\beta D^2)L^2/\beta D^4$ and $\varepsilon=D^2/L^2$ are assumed, then this expression can be rewritten as

$$\alpha\frac{\partial u_1}{\partial t_1}+\alpha u_1\frac{\partial u_1}{\partial x_1}+\gamma\frac{\partial u_1}{\partial x_1}-\frac{\partial^3 u_1}{\partial x_1^3}-\varepsilon\frac{\partial^5 u_1}{\partial x_1^5}=0. \tag{21}$$

It is hard to get an accurate analytic solution of (21). According to the scale characteristics of long waves, however, it is known that $\varepsilon=\frac{D^2}{L^2}<1$, representing quite a small value. As such, ε can be regarded as a small parameter. It is noted here that in numerous studies of the past, ε was viewed as a parameter a lot smaller than unity but its value is in fact not nearly so (with $D/L=0.5$, for example, $\varepsilon=0.25$) such that ε is evaluated appropriately for the use in this study. With ε as a small parameter, we proceed to seek the approximate analytic solution to (21) with higher accuracy in a way similar to the treatment of Dai (1990), and when the accuracy reaches $O(\varepsilon^3)$, (21) slightly differs from (4.1) of Dai with the consequence that (4.1) cannot be used directly with no necessary change.

Now let

$$\begin{aligned}&u_1=-\alpha v(\zeta),\quad \zeta=\sqrt{\lambda}(x_1+qt_1);\\&\lambda=\frac{\alpha a}{3}+\varepsilon\lambda_1+\varepsilon^2\lambda_2;\\&q=\frac{1}{\alpha}(-\gamma+\frac{\alpha a}{3}+\varepsilon\beta_1+\varepsilon^2\beta_2);\end{aligned} \tag{22}$$

for the atmospheric solitary wave solution, we assume

$$v_{(0)} = 1. \tag{23}$$

Substitution of (22) into (21) yields

$$v''' + 3vv' - v' + \varepsilon\left[\frac{\alpha a}{3}v^{(5)} + \frac{3\lambda_1}{\alpha a}v''' - \frac{3\beta_1}{\alpha a}v'\right]$$
$$+ \varepsilon^2\left[2\lambda_1 v^{(5)} + \frac{3\lambda_2}{\alpha a}v''' - \frac{3\beta_2}{\alpha a}v'\right] = O(\varepsilon^3), \tag{24}$$

in which $v^{(5)} = \dfrac{d^5 v}{d\zeta^5}$, indicating that (24) is the special case of (4.3) of Dai for $\mu = 1$. Consequently, we can have, with the aid of the findings of Dai, the third-order approximate analytic solution to (21) in the form

$$u_1(x_1, t_1) = -a\,\mathrm{sech}^2\frac{\zeta}{2}\left[1 - \frac{5\alpha a}{4}\varepsilon\,\mathrm{tg}h^2\frac{\zeta}{2}\right.$$
$$\left.- \alpha^2 a^2 \varepsilon^2\left(\frac{35}{16}\mathrm{tg}h^2\frac{\zeta}{2} - \frac{155}{48}\mathrm{tg}h^4\frac{\zeta}{2}\right)\right] + O(\varepsilon^3), \tag{25}$$

where

$$\zeta = \sqrt{\lambda}(x_1 + qt_1); \tag{26}$$

$$\lambda = \frac{\alpha a}{3} - \frac{5}{36}\alpha^2 a^2 \varepsilon + \frac{5}{108}\alpha^3 a^3 \varepsilon^2; \tag{27}$$

$$q = \frac{1}{\alpha}\left(-\gamma + \frac{\alpha a}{3} - \frac{1}{36}\alpha^2 a^2 \varepsilon - \frac{5}{108}\alpha^3 a^3 \varepsilon^2\right); \tag{28}$$

In view of the fact that (21) is nondimensional and thus u_1 the dimensionless wind with the actual wind under the control of the characteristic quantity U, u_1 is therefore assigned to be unity for the amplitude, i.e., $a = 1$. With $x_1 = \frac{1}{L}x$, $u_1 = \frac{1}{U}u$ and $t_1 = \frac{U}{L}t$ inserted in, (25) and (26) are reduced into the dimensional quantity of the form

$$u(x,t) = -U\mathrm{sech}^2\frac{\zeta}{2}\left[1 - \frac{5\alpha}{4}\varepsilon\,\mathrm{tg}h^2\frac{\zeta}{2} - \alpha^2\varepsilon^2\left(\frac{35}{16}\mathrm{tg}h^2\frac{\zeta}{2} - \frac{155}{48}\mathrm{tg}h^4\frac{\zeta}{2}\right)\right], \tag{29}$$

in which

$$\zeta = \frac{\sqrt{\lambda}}{L}(x + Uqt); \tag{30}$$

$$\lambda = \frac{\alpha}{3} - \frac{5}{36}\alpha^2\varepsilon + \frac{5}{108}\alpha^3\varepsilon^2; \tag{31}$$

$$q = -\frac{\gamma}{\alpha} + \frac{1}{3} - \frac{\alpha}{36}\varepsilon - \frac{5}{108}\alpha^2\varepsilon^2; \tag{32}$$

$$\alpha = UL^2 / \beta D^4; \tag{33}$$

$$\gamma = (\bar{u} - \beta D^2)L^2 / \beta D^4. \tag{34}$$

Thus, (29) is none other than the approximate analytic solution to (19), which, evidently, characterizes a kind of solitary waves, whose form is different from the standard type in the form $\mathrm{sec}h^2\frac{\zeta}{2}$.

It follows from (29) and (30) that the velocity of the wave has the form

$$C = -Uq. \tag{35}$$

Substituting (32)–(34) into (35) yields

$$c = \bar{u} - \frac{U}{3} - \beta D^2 + \frac{\alpha U}{36}\varepsilon + \frac{\alpha^2 U}{108}\varepsilon^2, \tag{36}$$

and the width of the wave is given by

$$d = 2L / \sqrt{\lambda}. \tag{37}$$

Putting (31) and (33) into (37) results in

$$d = 2L \Big/ \sqrt{\frac{UL^2}{3\beta D^4} - \frac{5}{36}\alpha^2\varepsilon + \frac{5}{108}\alpha^3\varepsilon^2}. \tag{38}$$

If, following (36) and (38), we refer to the terms containing no ε as the basic phase velocity and basic wave width, then those with ε^1 and ε^2 are the modified versions of these quantities that imply the wave widening in dimension and quickening in travel.

Ignoring the terms with ε of (29), (36) and (38) gives

$$u(x,t) = -U\mathrm{sech}^2\frac{\zeta}{2}; \tag{39}$$

$$C = \bar{u} - \frac{U}{3} - \beta D^2; \tag{40}$$

$$d = 2\sqrt{\frac{3\beta}{U}D^2}. \tag{41}$$

Then, removing the effects of basic flow ($\bar{u} = 0$) and substituting $m^2 = 1/D^2$ into (40), we have

$$C = -\left(\frac{\beta}{m^2} + \frac{1}{3}U\right). \tag{42}$$

(39) and (42) indicate the results obtained by Wu who took the first two terms in his expanded expression of the phase velocity.

To facilitate the analysis of effects of the amplitude and width of the solitary wave upon the velocity, we insert (41) into (40) for eliminating D with the result

$$C = \bar{u} - \frac{U}{3} - \sqrt{\frac{\beta}{12}} \cdot \sqrt{U} \cdot d, \tag{43}$$

which shows that the velocity C, amplitude U and width d are inter-dependent, that is, the bigger the amplitude and the wider the width, the slower the velocity as in the case of a blocking high and cut-off depression, a result that is analogous to that of Chen (1992). Then, from (43) one can see that when U and d are intensified to a certain point, we find

$$\frac{U}{3} + \sqrt{\frac{\beta}{12}} \cdot \sqrt{U} \cdot d > \bar{u}, \tag{44}$$

in which case $C < 0$ suggests the westward retreat of the wave, an observed fact in close coincidence with the westward shift of the long-wave system of considerable strength and longer wavelength.

As indicated earlier, there is some difference between the solitary waves given by (29) and those documented previously by other researchers ($\mathrm{sech}^2\frac{\zeta}{2}$). And what is the discrepancy? Based on the behaviors of atmospheric long waves, the parametric values are given as follows:

$$U = 10\mathrm{m\ s^{-1}}, \quad \beta = 1.6 \times 10^{-11}\mathrm{m^{-1}s^{-1}} (\varphi = 45^\circ \mathrm{N});$$

$$L = 1.5 \times 10^6 \mathrm{m}; \quad \varepsilon = (\frac{D}{L})^2 = 0.2.$$

Numerical computation is performed for (29) with the result showing that a system of oscillatory solitary long waves is most likely to exist in the atmosphere (see Fig. 3), which we call oscillatory Rossby solitary wave′ that is significantly different from the usual Rossby solitary wave reported by other investigators.

Does the displacement of such waves bear resemblance to the propagation of the observed analogs in the atmosphere, e.g., the blocking high and cut-off depression?

A numerical calculation is made for (36) by use of the parameters just mentioned, resulting in

$$C = \bar{u} - 10.05. \tag{45}$$

In general, the 500-hPa basic westerly flow $\bar{u}$ moves at ~ 10 m / s. In that case, $C \approx 0$ indicates that the solitary wave system is in a quasi-stationary state. In the presence of strong (weaker) basic flow the wave progresses slowly eastward (westward) along it, so do the blocking high and cut-off depression. In view of the fact that the solitary wave system is very close in configuration and shift to the blocking system, it can be inferred that an oscillatory Rossby solitary wave exists in the atmosphere, the examples being the isolated system, as of the blocking high and cut-off depression.

IV. CONCLUDING REMARKS

By virtue of the scale difference characteristics of atmospheric long waves both in x and y directions and by high-accuracy approximately evolving the frequency relation of linear Rossby waves, the identity is demonstrated of the waves given by the representation and the original ones both in phase velocity $\vec{C}$ and wave energy dispersion velocity $\vec{C}_g$, and thus the theoretical support is established for the expansion. Next, by means of the combination of the nonlinear advective term with the linear equation, we acquire the nonlinear expression, or the generalized KdV form, for describing atmospheric long wave motion, and after a set of transformations and manipulations, we get the approximate analytic solution of the nonlinear equation, viz.,

$$u(x,\ t) = -U\mathrm{sech}^2\frac{\zeta}{2}[1 - \frac{5\alpha}{4}\varepsilon \mathrm{tg}h^2\frac{\zeta}{2} - \alpha^2\varepsilon^2(\frac{35}{16}\mathrm{tg}h^2\frac{\zeta}{2} - \frac{155}{48}\mathrm{tg}h^4\frac{\zeta}{2})].$$

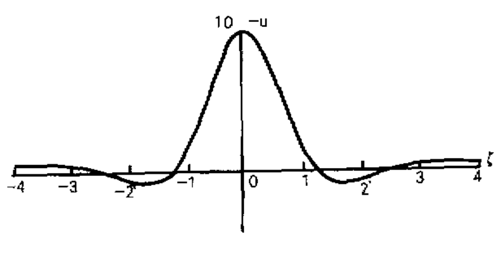

Fig. 3. Image of oscillatory Rossby solitary wave in the atmosphere.

Therewith, a numerical calculation is done using the physical parameters of the real atmosphere and an oscillatory Rossby solitary wave' is obtained, for the first time, together with its dispersion relation. Also, calculation results indicate the close resemblance of the theoretical wave to the blocking high and cut-off depression as the isolated systems in the atmosphere. This suggests that the oscillatory type does exist therein. No doubt, the problem awaits further study.

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