Note on the Symmetric Stability of Quasi–Homogeneous and Incompressible Rotating Ocean ①

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ABSTRACT

The general property of zonally symmetric stability of quasi–homogeneous and incompressible rotating ocean can be determined by a nondimensional parameter R_r which is similar to the Richardson number in Howard's paper. The results indicate that the rotating effect leads to stabilize the basic flows and the horizontal shear effect leads to destabilize the basic flows. In addition, the most unstable growth rate is obtained and the semicircle and semiellipse theorem about the distributions of the unstable phase velocity are given in this paper.

Key words: Symmetric stability, Horizontal shear, Semicircle and semiellipse theorem

I. INTRODUCTION

Symmetric model was often used to gain the physical insight into the symmetric circulation. In this model, the acceleration of fluid in azimuthal direction can be ignored due to the quasi–axisymmetric distribution of external heating and friction. So the variables in fluid field are independent of azimuthal coordinate. Applying cylindrical coordinate, the stability problem of symmetric motion (swirling flows) has been studied extensively by Howard and Gupta (1962) by using the normal mode method, and by Eliassen and Kleinschmidt (1957), Fjortoft (1950) by using the parcel method.

The purpose of this note is to study the zonally symmetric stability property of quasi–homogeneous and incompressible rotating ocean under the local plane approximation. What we want to know is the effects of rotating and horizontal shear on the stability of basic flows. Although the motion equations here are similar to those in cylindrical coordinate, but the physical interpretation of the stability condition is somewhat different, because the rotation of fluid in ocean is no longer the solid–body rotating.

The governing equations, the stability problem, the most unstable growth rate, as well as the semicircle and semiellipse will be given in Sections II to IV, respectively. Conclusions are presented in the last section briefly.

II. GOVERNING EQUATIONS

Considering a uniformal homogeneous ocean on the earth, assuming that the radius of the earth is so large in relation to the horizontal scale of motion that the cartesian coordinate system can be accurately used and restricting our attention to a special kind of motion in which all the variables are independent of zonal direction, the perturbations (u',v',w') superimposed on the basic flows satisfy the following equations:

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$$Du' - v'\overline{M}_y = 0, \tag{2.1}$$
$$Dv' + fu' = -p'_y, \tag{2.2}$$
$$Dw' + v'\overline{w}_y = -p'_z, \tag{2.3}$$
$$v'_y + w'_z = 0, \tag{2.4}$$

where f is the Coriolis parameter and $\overline{M}_y$ is the absolute vorticity of the basic flow,

$$\overline{M}_y = f - \overline{u}_y,$$
$$D = \partial_t + \overline{w}_y(y)\partial_z,$$
$$f = f_0 + \beta y = 2\Omega\sin\varphi_0 + \beta y,$$

$\overline{u}$, $\overline{w}$ are the components of the basic flow in y and z direction respectively and are supposed only depending on y. Ω is the angular velocity of the earth and φ_0 is the latitude of the centre of the ocean locates.

By introducing a stream function ψ which satisfies

$$v' = \psi_z, \qquad w' = -\psi_y,$$

then equations (2.1)–(2.4) can be rewritten in terms of ψ

$$D^2(\partial_{yy} + \partial_{zz})\psi - \overline{w}_{yy}D\psi_z + f\overline{M}_y\psi_{zz} = 0. \tag{2.5}$$

Let $\psi(y, z) = \varphi(y)\exp[ik(z - ct)]$ be the solution to (2.5), we have

$$(\overline{w} - c)^2(\varphi'' - k^2\varphi) + (\overline{w} - c)\overline{w}_{yy}\varphi + f\overline{M}_y\varphi = 0. \tag{2.6}$$

Denoting $\varphi(y) = (\overline{w} - c)F(y)$, (2.6) can be transformed into

$$[(\overline{w} - c)^2F']' - k^2(\overline{w} - c)^2F + f\overline{M}_yF = 0. \tag{2.7}$$

If $f\overline{M}_y$ in (2.7) is replaced by $g\rho'/\rho_0$, then it becomes the equation (2.2) in Howard's paper (1961).

III. THE STABILITY PROBLEM

Let $G = (\overline{w} - c)^{1/2}F$ and write (2.7) in terms of G, the result is

$$[(\overline{w} - c)G']' - k^2(\overline{w} - c)G - \frac{\overline{w}_{yy}G}{2} + (f\overline{M}_y - \frac{\overline{w}_y^2}{4})G/(\overline{w} - c) = 0. \tag{3.1}$$

Multiplying equation (3.1) by G^* and integrating over $(0, L)$ leads to

$$\int_0^L dy[(\overline{w} - c)(|G'|^2 + k^2|G|^2) + \frac{\overline{w}_{yy}|G|^2}{2} - (\overline{w} - c^*)(f\overline{M}_y - \frac{\overline{w}_y^2}{4})|G/(\overline{w} - c)|^2] = 0, \tag{3.2}$$

where the boundary conditions $\varphi(y)|_{y=0} = \varphi(y)|_{y=L} = 0$ have been used.
The imaginary part of (3.2) is

$$c_i\int_0^L dy[(|G'|^2 + k^2|G|^2) + (f\overline{M}_y - \frac{\overline{w}_y^2}{4})|G/(\overline{w} - c)|^2] = 0. \tag{3.3}$$

Suppose that $\overline{w}_y$ is not equal to zero everywhere in field, then to make $c_i = 0$ it must have

$$R_s = \min[f\overline{M}_y/\overline{w}_y^2] > \frac{1}{4}, \tag{3.4}$$

where R_s is a nondimensional parameter which is similar to Richardson number in Howard's paper (1961). The stability property of the basic flows can be determined by the value of R_s if the value of R_s is great than 0.25, then the basic flows are stable.

As the first special case, we consider that the horizontal shear of vertical velocity equals zero, then the stability condition becomes the inertial stability condition, i.e. if the absolute vorticity is positive everywhere in the fluid field, then the basic flows are stable.

As the second special case, we consider that the zonal basic flow has no horizontal shear. To make the basic flows stable, we must have

$$f_0 > \frac{|\bar{w}_y|_{max}}{2}. \tag{3.5}$$

This condition can be easily satisfied in the ocean.

As to the general case that both the zonal flow and vertical flow have horizontal shear, if we consider that the horizontal shear of zonal flow would be both positive and negative and denoting the maximum values of $|\bar{u}_y|$ and $\bar{w}_y^2$ by α and δ, respectively, then (3.4) can be put into the following form

$$R_s = f_0(f_0 - \alpha)\delta > \frac{1}{4} \tag{3.6}$$

or

$$f_0 > f_c, \tag{3.7}$$

where

$$f_c = \frac{(\alpha + \sqrt{\alpha^2 + \delta})}{2}. \tag{3.8}$$

This condition is a limitation on the location of ocean for the fixed flows. The neutral stability curves are illustrated in Fig. 1 and Fig. 2.

IV. THE ESTIMATION OF THE MOST UNSTABLE GROWTH RATE AND THE SEMICIRCLE AND SEMIELLIPSE THEOREM

If the basic flows are unstable, then from (3.3) we have

$$k^2 c_i^2 \leqslant k^2 |\bar{w} - c|^2 \leqslant (\frac{1}{4} - R_s)\delta \left[\int dy k^2 |G'|^2 \Big/ \int dy(|G'|^2 + k^2|G|^2)\right]\text{min}. \tag{4.1}$$

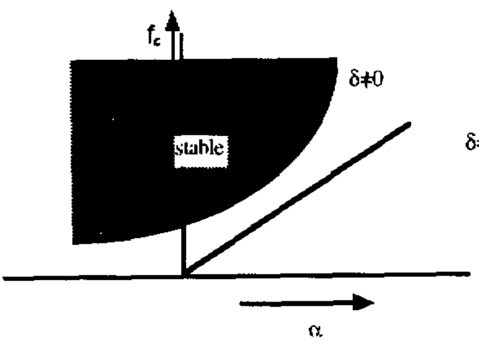

Fig. 1. Neutral stability curve for symmetric ocean in terms of f_c–α.

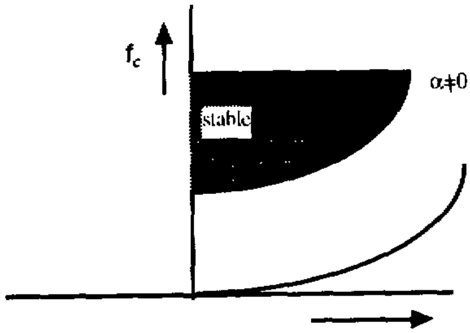

Fig. 2. Neutral stability curve for symmetric ocean in terms of f_c–δ.

Since

$$\min\left[\int dy|G'|^2 \Big/ \int dy|G|^2\right]=\frac{\pi^2}{4L^2}=k_0^2,$$

from (4.1) we have

$$k^2 c_i^2 \leqslant (\frac{1}{4} - R_s)\delta / (1 + k_0^2 / k^2). \tag{4.2}$$

When the motion scale in y direction is large enough, $k_0^2 \to 0$, (4.2) becomes

$$k^2 c_i^2 \leqslant (\frac{1}{4} - R_s)\delta.$$

Therefore, the difference between 0.25 and the parameter R_s is the measure of the instability of basic flows. Eq. (2.6) is a standard Taylor–Gordestein equation, so the distribution of unstable modes satisfies the Howard semicircle theorem

$$\left(c_r - \frac{\overline{w}_{max} + \overline{w}_{min}}{2}\right)^2 + c_i^2 \leqslant \left(\frac{\overline{w}_{max} - \overline{w}_{min}}{2}\right)^2, \tag{4.3}$$

where c_r and c_i are the real part and imaginary part of c, $\overline{w}_{max}$ and $\overline{w}_{min}$ represent the maximum and minimum values of vertical velocity, respectively.

In the light of the Makov et al's work (1984), (4.3) can be modified by considering the omitted term in Howard's paper if the value of R_s exists in the whole field

$$\left(c_r - \frac{\overline{w}_{max} + \overline{w}_{min}}{2}\right)^2 + \left(1 + \frac{2Rs}{1 - 2Rs + (1 - 4Rs - 4k^2 c_i^2 / \delta)^{1/2}}\right)c_i^2 \leqslant \left(\frac{\overline{w}_{max} - \overline{w}_{min}}{2}\right)^2. \tag{4.4}$$

This is so called semiellipse theorem, and this theorem indicates that the parameter R_s has some influence on the distribution of unstable mode. This theorem can be further refined for the specific velocity profiles. The detailed derivation is omitted here, readers may refer to the Makov's paper (Makov, 1984).

V. CONCLUSIONS AND REMARKS

Some conclusions can be drawn from the above results.

(1) From the stability conditions, we can see that the rotating effect leads to stabilize the basic flows. Since $f_0 = 2\Omega\sin\varphi_0$, the ocean in high latitude is more easily to become stable than the ocean in low latitude with the same shear.

(2) Shear effect leads to destabilize the basic flows. Ocean with large horizontal shears is more easily to become unstable.

(3) There are two kinds of stability mechanism in this system, i.e. inertial and symmetric stability. Because $\overline{w}_y^2 > 0$, if the basic flows are symmetric stable, it must be inertially stable. But this is not true in the nonlinear case in which the $\overline{w}_y^2$ is replaced by $\overline{w}\,\overline{w}_{yy}$, therefore, it is possible that even both the values of $\overline{w}\,\overline{w}_{yy}$ and the absolute vorticity are negative, the basic flows are still stable. That means in the nonlinear case, the basic flows may be inertially unstable while the basic flows are still symmetrically stable (Ren, 1993).

(4) The motion scale in horizontal direction has influence on the unstable growth rate. The ocean with large horizontal scale possesses large value of unstable growth rate.

(5) Some important factors, such as vertical stratification and dissipation are not included in this simplified model. There is no desire to regard it as a good model for the real ocean on the earth, its primary goal is to show the effects of rotating and horizontal shear on the symmetric stability property of the system. A more precise model must take the factors mentioned above into account.

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