Nonlinear Ultra–Long Wave and Its Stability[1]

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ABSTRACT

A stability of a nonlinear ultra–long wave and its solution are discussed in this paper by employing Burger model which is subject to heat resource. It is of interest noted that the wave solution can be described by an equation of KDV or MKDV and that conditions for the existence of the solution are related to characteristic divergences. In addition, a wave velocity expression for nonlinear ultra–long waves and some diagnostic correlations among wave parameters have been obtained.

Key words: Ultra–long wave, Stability, Burger model

I. INTRODUCTION

It is proved by weather forecast practices that the progression of a weather system, which affects local weather events, is usually influenced or controlled by a much longer–wave weather system. Indeed, a long–wave movement is responsible for a local weather event in the process of a med–term weather system, but its movement does be closely related with an ultra–long wave in which the scale of time and space is much longer than it. The ultra–long wave, therefore, can be regarded as a "background" of the long wave movement. Although great progress has been made in understanding the behavior of ultra–long waves since 1960s thanks to the unceasing studies and analyses made by meteorologists such as Burger (1958), Zhu (1964), Deland (1965), Dickinson (1968), Eliassen (1969), Clark (1978), Frederikson (1979), Zhang (1979), Call et al (1979) and Egger et al (1983), the nature of them is still not fully understood.

The purpose of this research is to study the influence of heat sources on nonlinear ultra–long waves and to wish that the result will bring some benefits to the study.

II. BASIC MODEL

In P coordinate, the Burger model depicting an ultra–long wave can be devised as follows:

$$-fv = -\frac{\partial \varphi}{\partial x} ,$$

$$\beta v + f\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) = 0 , \tag{1}$$

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$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial \omega}{\partial p}=0 ,$$

$$\left(\frac{\partial}{\partial t}+u\frac{\partial}{\partial x}+v\frac{\partial}{\partial y}\right)\frac{\partial \varphi}{\partial p}+s\omega=Q ,$$

where, s is a parameter (approximately constant) of atmospheric static stability; Q is a heat source; $\beta=\frac{\partial f}{\partial y}$ is Rossby parameter (approximately constant); the other symbols have the same meanings as they usually do.

For the sake of convenience, it is necessary to assume that Eq.(1) should be a closed equation and that the term "heat source" should be directly proportional to the vertical velocity, that is, $Q=\alpha\omega$. It is evident that there is an adiabatic atmosphere (a diabatic atmosphere) when $\alpha=0(\neq 0)$.

In regard to an ultra-long wave, there exist downdrafts in the east of a trough, while updrafts in the west (Zhang, 1983). The term $\alpha>0$ (<0), therefore, expresses a cold (heat) source in the west of the trough, and a heat (cold) source in the east.

It is supposed that there are several formal solutions for Eq.(1), which are

$$u=U(\theta),\quad v=V(\theta),\quad \omega=\Omega(\theta),\quad \varphi=\Phi(\theta) , \tag{2}$$

here, the formula $\theta=kx+ly+np-\sigma t$ is a phase angle, letters k,l,n, are wave number, and the symbol σ is a frequency.

By substituting Eq.(2) into Eq.(1) and eliminating the variables of U,V,Φ the variable Ω can then be determined by a equation:

$$\Omega''=F(\Omega)=\frac{k\beta(s+\alpha)\Omega}{n^3 f^2(\Omega+\frac{k}{n}c_x)} , \tag{3}$$

where the symbol "$''$" means to make a differential to θ; $c_x=\frac{\sigma}{k}$ is a wave velocity on x-axis; $\Omega\neq -\frac{k}{n}c_x$.

Eq.(3) is a fundamental one in this article to describe a disturbance of nonlinear ultra-long waves influenced by heat source. The following sections will discuss in details the nature of the equation.

III. STABILITY OF ULTRA-LONG WAVE DISTURBANCES INFLUENCED BY HEAT SOURCE

In general speaking, it is difficult to obtain a precise analytic solution. For this reason, a method of nonlinear expansion put forward by Liu (1982) is adopted to discuss the approximate solution in the vicinity of a balance point.

Demanding $\Omega'=P$, then Eq.(3) can be simplified into a first-order simultaneous equation:

$$\Omega'=P,\qquad P'=F(\Omega) . \tag{4}$$

In Eq.(4), there exists only one balance point $A(0,0)$. After making a Taylor series expansion to the nonlinear function $F(\Omega)$ near point A, a equation

$$F(\Omega)=\frac{\beta(s+\alpha)}{n^2 f^2 c_x}(\Omega-\frac{n}{kc_x}\Omega^2+\frac{n^2}{k^2c_x^2}\Omega^3+\cdots) \tag{5}$$

is obtained. If only the first term on the right side is kept, Eq.(5) then becomes:

$$\Omega'=P, \qquad P'=\frac{\beta(s+\alpha)}{n^2 f^2 c_x}\Omega\ . \tag{6}$$

The above equation is refered to a linear system, which describes the disturbances of linear ultra-long waves influenced by heat source in the vicinity of point A; If keeping the first two terms on the right side, it then becomes:

$$\Omega'=P, \quad P'=\frac{\beta(s+\alpha)}{n f^2 c_x}\left(\frac{\Omega}{n}-\frac{\Omega^2}{kc_x}\right). \tag{7}$$

It is called a square system and if keeping the first three terms it will become:

$$\Omega'=P, \quad P'=\frac{\beta(s+\alpha)}{n f^2 c_x}\left(\frac{\Omega}{n}-\frac{\Omega^2}{kc_x}+\frac{n\Omega^3}{k^2c_x^2}\right). \tag{8}$$

Which is termed a cubic system. In this section, only the disturbance stability is discussed, and the analytic expression will remain to be solved in the next section.

Through analysing the linear system (6), the characteristic root of a characteristic equation can be determined, that is,

$$\lambda^2=\frac{\beta(s+\alpha)}{n^2 f^2 c_x}\ . \tag{9}$$

In view with the term $f>0$ in the Northern Hemisphere, a criterion can be gained from Eq.(9):

$$\frac{s+\alpha}{c_x}<(>)0\ , \tag{10}$$

which means point A is a center (saddle point) or a stable (an unstable) point.

Through examing square system (7) and cubic system (8) by the use of Poincare–Bendxson's theory and symmetrical principles (Qin, 1959), it is noted that the nature of the linear system is quantitatively identical to that of the square and cubic systems near point A.

An instable criterion for the disturbance of a nonlinear ultra–long wave influenced by heat sources in the vicinity of balance point A is then obtained:

$$\frac{s+\alpha}{c_x} > 0 \ . \tag{11}$$

It is clearly seen from the criterion that the disturbance instability is subject to the parameter of atmospheric static instability, phase velocity c_x of disturbance and parameter α of heat source. Regarding the large size of an ultra-long wave disturbance, the parameter of atmospheric static stability is generally larger than zero.

Regardless of the heat source ($\alpha = 0$), an eastward-advancing ultra-long wave ($c_x > 0$) will be instable, while a westward-withdrawing one stable. In other words, there exists only a stably-bounded ultra-long wave withdrawing westward when the heat source is disregarded. Otherwise, it is different unless there is a cold source in the west of a trough and a heat source ($\alpha > 0$) in the east. While, if there is a heat source in the west, a cold source in the east ($\alpha < 0$), and their intensities reach to a certain extent ($|\alpha| > s$), the heat source will make the eastward-advancing ultra-long wave stable and the westward-withdrawing one instable. The results are contrary to those in the absence of heat sources, which imply an important effect of heat sources on the stability of an nonlinear ultra-long wave disturbance.

Plotting $c_x = c_p \frac{n}{k}$, $c_p = \frac{\sigma}{n}$ as a wave velocity in vertical direction, it can be seen from climatic mean charts that axes of a ridge and a trough of extra-long waves tilt backward with the increase of height (Zhang, 1983). In this case, the term n / k is larger than zero. The nonlinear ultra-long wave always transfers energy upward when disregarding heat sources ($c_p < 0$). This feature of vertically transfering energy upto the stratosphere is of great significance to air motions in the stratosphere. On the contrary, a nonlinear ultra-long wave tends to transfer energy downward ($c_p > 0$) if taking into account the heat source along with the formula $-\alpha > s$ (i.e., there is a heat source in the west of a trough, a cold source in the east, and thier intensities reach to a certain extent), having an influence on the air motion in the middle and lower troposphere. It is necessary to point out that a series approximation would not affect any stability criteria of Eq.(3) (He, 1993).

IV. SOLUTION OF NONLINEAR ULTRA-LONG WAVE DISTURBANCES INFLUENCED BY HEAT SOURCES

In this section, emphasis will be given to the solution of nonlinear ultra-long wave disturbances and the correlation between wave velocity and parameter.

Eq.(6) can be rewritten:

$$\Omega'' = \frac{\beta(s+\alpha)\Omega}{n^2 f^2 c_x} \ . \tag{12}$$

Assuming $\frac{\beta(s+\alpha)}{n^2 f^2 c_x} = -1$, then

$$c_x = -\frac{\beta(s+\alpha)}{n^2 f^2} \ . \tag{13}$$

In this case, the wave solution has become a stably-bounded cosine function. And Eq.(13)

is regarded as a phase velocity equation. When $\alpha = 0$, it will be degenerated into

$$c_x = -\frac{\beta s}{n^2 f^3} . \tag{14}$$

Eq.(14) is considered as a wave velocity expression disregarding heat sources. By comparing expressions (13) and (14), it is noted that the westward-shifting wave velocity speeds up (slows down) when there is a cold (heat) source in the west of the trough 〖$\alpha > (<) 0$〗, a heat (cold) source in the east, and relatively no heat source ($\alpha = 0$). By assuming $\frac{\beta(s+\alpha)}{n^2 f^3 c_x} = 1$, then the solution of expression (12) becomes an instable solution----"e" exponential function.

An equation can be deduced from square system (7):

$$\Omega''' + \frac{2\beta(s+\alpha)}{nkf^3 c_x^2}\Omega\Omega' - \frac{\beta(s+\alpha)}{n^2 f^3 c_x}\Omega' = 0 . \tag{15}$$

It is a famous KDV equation. In addition, another equation can be derived through intergration of square system (7):

$$(\Omega')^2 = \frac{-2\beta(s+\alpha)}{3nf^3 kc_x^2} g(\Omega) . \tag{16}$$

Here, the term $g(\Omega) = \Omega^3 - \frac{3kc_x}{2n}\Omega^2 - \frac{3Anf^3 kc_x^2}{2\beta(s+\alpha)}$ is a cubic multinomial of Ω, A is an integral constant. In expression (16) there exists a bounded periodic wave solution, which requires $g(\Omega) = 0$ and three different real roots. It can be demonstrated that the integral constant A must satisfy:

$$0 < A < \frac{-\beta(s+\alpha)c_x k^2}{3n^4 f^3} . \tag{17}$$

From expression (16), $A = (\Omega')^2 |_{\Omega=0}$ can be obtained. Furthermore, an expression $\Omega' = \frac{D}{n}$ can also be established from the third formula of Eq.(1), where, $D = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}$ is a horizontal divergence. As a consequence, $A = \frac{Do^2}{n^2}$, where, $Do = \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)\Big|_{\Omega=0}$ is a "characteristic divergence". So, expression (17) can be rewritten as follows:

$$0 < |Do| < \left[\frac{-\beta(s+\alpha)c_x k^2}{3n^2 f^3}\right]^{\frac{1}{2}} . \tag{18}$$

It shows that the condition for a bounded wave solution gives an upper limit to the characteristic divergence. Once the limit is exceeded, the instability of a nonlinear ultra-long wave will take place.

When the condition for the existence of a stable wave solution is satisfied, a stably-bounded wave solution is

$$\Omega = \Omega_2 + (\Omega_1 - \Omega_2)c_n^2 \sqrt{\frac{\beta(s+\alpha)(\Omega_3 - \Omega_1)}{6n f^3 k c_x^2}}\theta , \tag{19}$$

where, $c_n(\ )$ is Jacobi elliptic function. The term Ω_i $(i = 1,2,3)$ expresses

$$\begin{aligned} \Omega_1 &= \Omega_0 \left\{\cos[\frac{1}{3}\cos^{-1}(12h_0 - 1)] - \frac{1}{2}\right\} , \\ \Omega_2 &= -\Omega_0 \left\{\cos[\frac{1}{3}\cos^{-1}(12h_0 - 1)] + \frac{\pi}{3}] + \frac{1}{2}\right\} , \\ \Omega_3 &= -\Omega_0 \left\{\cos[\frac{1}{3}\cos^{-1}(12h_0 - 1) - \frac{\pi}{3}] + \frac{1}{2}\right\} . \end{aligned} \tag{20}$$

Here, $\Omega_0 = \frac{-c_x k}{.n} = -c_p$, $h_0 = \frac{-n^4 f^3 A}{2\beta(s+\alpha)c_x k^2}$. It clearly shows that expression (17) is equivalent to $0 < h_0 < \frac{1}{6}$.

A nonlinear ultra-long wave, therefore, can be expressed in an elliptic cosine wave. Its wavelength on X-axis is

$$L = \frac{2}{k}\sqrt{\frac{6n f^3 k c_x^2}{\beta(s+\alpha)(\Omega_3 - \Omega_1)}}\ \mathrm{K(m)} \tag{21}$$

where, $K(m)$ is a complete elliptic integral of the first kind, module $m^2 = \frac{\Omega_1 - \Omega_2}{\Omega_1 - \Omega_3}$ and wave amplitude $\hat{\Omega} = \Omega_1 - \Omega_2$. It is evident that the heat source has an influence on the wavelength and amplitude of ultra-long waves. When the value of $\Omega_3 - \Omega_1$ is given, the squared wavelength is reversely proportional to heating intensity.

If the condition for the existence of a stable wave solution (17) or (18) is not satisfied, an instable wave solution can be derived from expression (16):

$$\Omega = \Omega_2 + (\Omega_1 - \Omega_2) nc^2 \sqrt{\frac{\beta(s+\alpha)(\Omega_1 - \Omega_3)}{6n f^3 k c x^2}}\ \theta \tag{22}$$

where, function $nc = (cn)^{-1}$, module $m_2 = \frac{\Omega_2 - \Omega_3}{\Omega_1 - \Omega_3}$. It can be seen that the "e" exponential function of a nonlinear instable wave soluion is different from that of a linear instable wave solution. By using an approach similar to Lu (1987), it is calculated that in view with the linear and nonlinear instabilities the periods that need to double their initial amplitudes are 18 hrs and 60 hrs, respectively. This exhibits that the nonlinear instability tends to make a disturbance to develop slowly, which has a good agreement with the time scale of an increasingly instable disturbance in the real atmosphere (Clark, 1978).

Next, a relation between wave velocity expression of nonlinear ultra-long wave and wave parameter is studied.

From expression (20), it is easy to obtain another one:

$$\Omega_1\Omega_2\Omega_3 = 3k_0\Omega_0^3;\ \Omega_1\Omega_2 + \Omega_1\Omega_3 + \Omega_2\Omega_3 = 0;\ \Omega_1 + \Omega_2 + \Omega_3 = \frac{-3\Omega_0}{2} \ . \tag{23}$$

Supposing $\Omega_1 = r\hat{\Omega}$, then $\Omega_2 = (r-1)\hat{\Omega}$, and $\Omega_3 = \delta\hat{\Omega}$, it can be derived from expression (23) that $\delta = r\frac{1-r}{2r-1}$. According to $\Omega_1 > 0 > \Omega_2 > \Omega_3$, then

$$r > 0 > r-1 > r\frac{1-r}{2r-1} \ . \tag{24}$$

By solving expression (24), $\frac{1}{3} < r < \frac{1}{2}$. Additionally, a formula is obtained from expression (23):

$$\Omega_0 = -q(r)\hat{\Omega} \ , \tag{25}$$

where, $q(r) = \frac{2(3r^2-3r+1)}{3(1-2r)}$. The first expression of wave velocity for nonlinear ultra-long waves, therefore, can be written:

$$c_x = \frac{-nq(r)}{k}\ \hat{\Omega} \ . \tag{26}$$

This expression depicts a relation between wave velocity and amplitude when the value of $q(r)$ is given.

By substituting expression (25) into formula $g(\Omega_1) = 0$ and bring them into order, it then becomes:

$$c_x = \frac{-\beta(s+\alpha)\hat{\Omega}^2\tilde{q}(r)}{n^2 f^3 A} \ , \tag{27}$$

Where, $\tilde{q}(r) = \frac{r^2}{3}\left(\frac{3}{2} - \frac{r}{q(r)}\right)$ It describes a relation between wave velocity and β, which is known as the second expression of wave velocity.

It is also easy to set out another expression----the third expression of wave velocity----by substituting $L = \frac{2\pi}{k}$ into expression (21),

$$c_x = \frac{-\beta(s+\alpha)}{n^2 f^3}\left(\frac{\pi}{2\mathrm{K(m)}}\right)^2 q_1(r) \ . \tag{28}$$

Here, $q_1(r) = \frac{2r-3r^2}{3r^2-3r+1}$. This expression is closely similar to that (13) of linear ultra-long waves. It, however, contains some factors relating to amplitude, which reflects a feature of nonlinear waves.

Regarding a disturbance with small amplitude, that is, $\hat{\Omega}\to 0, \Omega_1, \Omega_2 \to 0, m^2 \to 0$, K(m) $\to\frac{\pi}{2}, \Omega_3 \to \frac{-3\Omega_0}{2}, r\to\frac{1}{2}, q_1(\frac{1}{2})=1$. Then, expression (28) is simplified as

$$c_x = \frac{-\beta(s+\alpha)}{n^2 f} \ . \tag{29}$$

It is noted that this expression is consistent with expression (13), which explains that the linear ultra-long wave is a special case of the nonlinear extra-long wave ($\hat{\Omega}\to 0,\ r\to\frac{1}{2}$).

It can be concluded from expression (19) that the shorter is the wavelength, the slower is the wave velocity; the bigger the amplitude, the slower the wave velocity; and the higher the latitude, the slower the wave velocity. The wave velocity contains not only an amplitude, but also a wave number as well, which mirrors a feature of wave fluctuations with a limited amplitude and responses quite well to real weather events. Practices of weather forecasting show (Zhang, 1983) that as is the case with some ultra-long wave systems such as blocking high and cut-off low the moving velocities of stable nonlinear ultra-long waves often speed up substantially due to the rapid diminution of their wavelength and amplitude as they subside and breakdown, and vice versa as they are in the initial and development stages.

Finally, the cubic system----expression (8)----will be examined, from which a formula is derived:

$$\tilde{\Omega}''' = \frac{2\beta(s+\alpha)}{3nf c_x^3}\tilde{\Omega}' + \frac{3\beta(s+\alpha)}{k^2 f c_x^3}\tilde{\Omega}^2\tilde{\Omega}' \ , \tag{30}$$

where, $\tilde{\Omega}=\Omega+\frac{\Omega_0}{3}$. Formula (30) is a famous MKDV equation. A nonlinear extra-long wave, therefore, can be described not only by KDV equation taking into account the square system, but also by a more precise MKDV equation in consideration of the cubic term. In fact, the latter is much better to describe ultra-long waves in middle and high latitudes. It is not difficult to get a special solution by solving Eq.(30):

$$\Omega = \left\{\frac{2}{\sqrt{3}}\operatorname{csch}\left(\sqrt{\frac{2\beta(S+\alpha)}{3n^2 f c_x}}\,\theta\right)-\frac{1}{3}\right\}\Omega_0 \ . \tag{31}$$

Discussions about the condition for the existence of wave solution and for wave velocity expression are the same with those about square system. It is hence unnecessary to go into details here.

V. CONCLUSION

An ultra-long wave is an important wave form in the atmosphere and plays a critical role in medium-range weather forecasting. Through making a study on stability and solution of nonlinear ultra–long waves, preliminary results have been achieved:

(1). In the vicinity of a balance point, a cold source in the west (east) of a trough and a heat source in the east (west) of a trough in the static stable atmosphere will have a stable (an instable) influence on westward–withdrawing ultra–long waves and an instable (stable) influence on eastward–advancing ultra–long waves. Regardless of heat sources, there only exists a stable westward–withdrawing ultra–long wave.

(2). A nonlinear ultra–long wave can be approximately depicted by a well known KDV or MKDV equation. Its bounded periodic solution is an elliptic cosine wave and would be degenerated into a linear cosine wave in extreme cases.

(3). The existence of a nonlinear ultra–long wave has restrained to a certain extent the characteristic divergence. Once a horizontal divergence exceeds the limitation, an instable nonlinear ultra–long wave will occur. The instable wave solution is different from that of "e" exponential function, the former is able to slow down the development of a disturbance.

(4). The wave velocity expression contains wave amplitudes, heat source parameer, and other factors, which reflects that the nonlinear feature of ultra–long waves and the heat source have an influence on wave disturbances. The analysis shows that a nonlinear ultra–long wave movement will speed up as its wavelenth gets shorter and amplitude gets smaller, and vice versa. These phenomena response quantitatively to observational facts.

As mentioned above, a nonlinear ultra–long wave has been discussed by the use of a simple model. Consequently, a good result that is impossible to be obtained if employing a linearity has been achieved. It is however still quite a preliminary because many physical processes have not been taken into account and heat source parameters are rather limited. All of these problems remain to be further studied hereafter. It is my hope anyway that this result will benefit the study on nonlinear ultra–long waves.

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