Vol.11 No.2 Advances in Atmospheric Sciences May 1994

Improving the Vorticity–Streamfunction Method to Solve Two–Dimensional Anelastic and Nonhydrostatic Model

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Received March 10, 1993; revised November 18, 1993

ABSTRACT

The potential temperature vorticity has been introduced to polish the (momentum) vorticity–streamfunction method for solving the two–dimensional and nonhydrostatic model with much accuracy but not many increments of computation. The three–step procedure introduced in the present paper can be used to solve both shallow and deep dynamic models.

Key words: Potential temperature vorticity, Anelastic model, Nonhydrostatic model

I. INTRODUCTION

The numerical simulation method has been very beneficial to our understanding the evolution of meso–and–small scale systems in the past decades, and it will play a more and more important role in future research. Although we can run three–dimensional models on very powerful scientific computers, two–dimensional models seem to be more efficient in some fields such as the cumulus dynamics, frontogenesis, symmetric instability, mountain–valley breeze, land and sea breeze, and so on. In the practice of numerical computation, the vorticity– streamfunction method is widely used because it is more efficient and accurate than the solution of primitive equations. For nonhydrostatic models, if one integrates the primitive equations, he must first obtain the disturbed pressure from the elliptic equation, and then find velocity fields from the momentum equations, finally calculate the potential temperature. Thus in one time step, one elliptic equation and three hyperbolic equations must be solved. By contrast, if one use the vorticity–streamfunction method, he just needs to solve two hyperbolic equations, i.e. the prognostic equation of vorticity and that of potential temperature. Furthermore, since the vorticity calculated from streamfunction satisfies the continuity equation automatically, the vorticity–streamfunction method is more efficient and accurate than the former (Roache, 1974). These advantages enable it to be widely used.

However, There exist some troubles in its application to integrating the deep models because the disturbed pressure in the buoyancy term cannot be ignored as in shallow models (Schlesinger, 1973). This unknown disturbed pressure makes it difficult to integrate the vorticity equation with no decrease of accuracy and not too many increases of computation. Previous three ways have ever been tried to overcome this barrier, but none of them has reached supposed destination successfully. One way was to estimate the unknown disturbed pressure in the vorticity equation from the momemtum equations (Schlesinger, 1973) with lower accuracy because of its artificial restriction to the disturbed pressure. Another way was to ignore these unknown terms in the vorticity equation, but this is not available for deep models. The third one was to find the disturbed pressure from the elliptic equation. Maybe this procedure can give results with sufficient accuracy, but it needs too much computation to solve the two elliptic equations in one time step. The problem still remains unsolved.

In order to solve this problem, the potential temperature vorticity is introduced here to polish the vorticity–streamfunction method.

II. IMPROVEMENT OF THE VORTICITY–STREAMFUNCTION METHOD

The governing equations of two–dimensional and energy–conservation anelastic models for both shallow and deep convections can be written as follows (Durran, 1989)

$$\frac{\partial u}{\partial t}=-u\frac{\partial u}{\partial x}-w\frac{\partial u}{\partial z}-c_p\theta\frac{\partial \pi'}{\partial x}\ , \tag{1}$$

$$\frac{\partial w}{\partial t}=-u\frac{\partial w}{\partial x}-w\frac{\partial w}{\partial z}-c_p\theta\frac{\partial \pi'}{\partial z}+g\frac{\theta'}{\bar\theta}\ , \tag{2}$$

$$\frac{\partial \theta'}{\partial t}=-u\frac{\partial \theta'}{\partial x}-w\frac{\partial \theta'}{\partial z}-w\frac{d\bar\theta}{dz}\ , \tag{3}$$

$$\frac{\partial u}{\partial x}+\frac{\partial w}{\partial z}+\delta\frac{1}{\bar\rho\bar\theta}\frac{d\bar\rho\bar\theta}{dz}=0\ . \tag{4}$$

where $\delta=0$ for shallow models and $\delta=1$ for deep models.

According to the continuity equation (4), we can easily define the streamfunction as

$$\alpha u=\frac{\partial \psi}{\partial z}\ ,$$

$$\alpha w=-\frac{\partial \psi}{\partial x}\ , \tag{5}$$

and the vorticity η as

$$\eta=\frac{\partial \alpha u}{\partial z}-\frac{\partial \alpha w}{\partial x}=\nabla^2\psi\ , \tag{6}$$

where $\alpha=(1-\delta)+\delta\cdot\bar\rho\bar\theta$.

After simple differential operations, one can easily obtain the following vorticity eqution,

$$\frac{\partial \eta}{\partial t}=-u\frac{\partial \eta}{\partial x}-w\frac{\partial \eta}{\partial z}-c_p\frac{\partial \pi'}{\partial x}\frac{\partial}{\partial z}(\alpha\theta)+c_p\frac{\partial \pi'}{\partial z}\frac{\partial}{\partial x}(\alpha\theta)+f(u,w,\theta,\eta)\ , \tag{7}$$

where $f(u,w,\theta,\eta)$ is independent of the disturbed pressure for both shallow and deep models. Since the disturbed pressure can not be ignored in deep models, it is difficult to obtain η without decrease of accuracy and too many increases of computation. In order to integrate the vorticity equation accurately and efficiently, the following three steps are proposed:

Firstly, define the potential temperature vorticity as

$$\eta_\theta=\frac{\partial}{\partial z}(\theta^{-1}u)-\frac{\partial}{\partial x}(\theta^{-1}w)\ . \tag{8}$$

It must be noted that η_θ is not the same as the vorticity η.

Secondly, eliminate the disturbed pressure by using the momentum equations (1) and (2), the prognostic equation of potential temperature vorticity can be written as follows

$$\begin{aligned}\frac{\partial \eta_\theta}{\partial t}&=-u\frac{\partial \eta_\theta}{\partial x}-w\frac{\partial \eta_\theta}{\partial z}-\frac{g}{\bar\theta}\frac{\partial}{\partial x}\left(\frac{\theta'}{\theta}\right)-\frac{\partial}{\partial x}(\theta^{-1}u)\frac{\partial u}{\partial x}\\&-\frac{\partial}{\partial z}(\theta^{-1}u)\frac{\partial w}{\partial z}+\frac{\partial}{\partial x}(\theta^{-1}w)\frac{\partial u}{\partial x}+\frac{\partial}{\partial z}(\theta^{-1}w)\frac{\partial w}{\partial x}\ .\end{aligned} \tag{9}$$

Finally, we can obtain the elliptic equation from the diagnostic relation between η_θ and η. For shallow models, we have

$$\eta_\theta = \theta^{-1}\eta + u\theta_z^{-1} - w\theta_x^{-1} \quad ,$$

By using Eq.(6), the elliptic equation pertaining to ψ reads

$$\nabla^2\psi - \frac{\partial\psi}{\partial z}\frac{\partial \ln\theta}{\partial z} - \frac{\partial\psi}{\partial x}\frac{\partial \ln\theta}{\partial x} = \theta\eta_\theta \quad . \tag{11}$$

But for deep models, we have

$$\eta_\theta = \frac{1}{\theta\bar{\rho}\bar{\theta}}(\eta - u\frac{d\bar{\rho}\bar{\theta}}{dz}) + u\frac{\partial\theta^{-1}}{\partial z} - w\frac{\partial\theta^{-1}}{\partial x} \tag{12}$$

and

$$\nabla^2\psi - \frac{\partial\psi}{\partial z}\frac{\partial \ln(\bar{\rho}\bar{\theta}\theta)}{\partial z} - \frac{\partial\psi}{\partial x}\frac{\partial \ln\theta}{\partial x} = \bar{\rho}\bar{\theta}\theta\eta_\theta \quad . \tag{13}$$

It is obvious that the streamfunction can be obtained from Eqs.(13) and (15) without additional limits to the disturbed pressure and almost no increase of computation.

III. CONCLUDING REMARKS

The potential temperature vorticity was introduced to polish the vorticity-streamfunction method without any limit to the disturbed pressure. The three-step procedure in the present paper seems to be more accurate and efficient, and available for both shallow and deep models.

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