Preliminary Study of Reconstruction of a Dynamic System Using an One–Dimensional Time Series

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ABSTRACT

This paper concerns the reconstruction of a dynamic system based on phase space continuation of monthly mean temperature 1D time series and the assumption that the equation for the time–varying evolution of phase–space state variables contains linear and nonlinear quadratic terms, followed by the fitting of the dataset subjected to continuation so as to get, by the least square method, the coefficients of the terms, of which those of greater variance contribution are retained for use. Results show that the obtained low–order system may be used to describe nonlinear properties of the short range climate variation shown by monthly mean temperature series.

Key words: Monthly mean temperature, Time series, Phase space continuation, Dynamic system

I. INTRODUCTION

From the investigations made in recent years of weather–climate time series (Peng et al., 1989; Yan et al., 1990, 1991, 1992) abundance of information is found to be about dynamic systems of concern and can be used in the following aspects: 1) a fractal dimensionality of a phase–space attractor for a dynamic system can be determined by computing the incident function of the data treated by phase–space continuation; 2) a predictable time scale of the system can be found out by calculating Kolmogorov entropy based on Lyapunov exponent of the data for describing the attractor's structure. Obviously, these pieces of information serve as an indispensable basis for us to gain insight into the system. This is only one facet of the problem, however. More important and interesting is how to reconstruct the system in terms of 1D time sequences. Here we may conceive the problem to be a sort opposite in nature to a differential equation.

It is known that the usual differential equation is built on description of a real physical process or system's state and solved on initial and boundary conditions of its own. If the solution were existent, unique and dependent on the rhs terms and the related conditions of the equation, the problem of the definite solution would be well–posed. The commonest "opposite" problem may be that, if the known parameters and coefficients of a differential equation appear as unknown quantities, can the original parameters and coefficients be found out by means of other conditions and coefficients be found out by means of other conditions or information? Just starting from the motivation, an attempt is made to rebuild a dynamic system through phase space continuation of 1D time series.

II. BASIC ASSUMPTIONS

Following the clue, we set the dynamic model of a system to have the form

$$\frac{dX_i}{dt} = f_j(X_1, X_2, \cdots, X_n; \lambda_i) \ , \qquad (1)$$

where $i = j = 1, 2, \cdots n$ and λ_i = coefficient. The solution of the "opposite" problem means that the nonlinear function $f_j (j = 1, 2, \cdots n)$ is derived with its specific form unknown but the conditions for the particular solution of (1) known. As a preliminary study, (1) is set to have linear and quadratic nonlinear terms, with the evaluation of j depending on the system's fractal dimensionality d. Generally, $j > D$, meaning that only the minimum integer dimensionality of the phase space is taken that supports an attractor of the system.

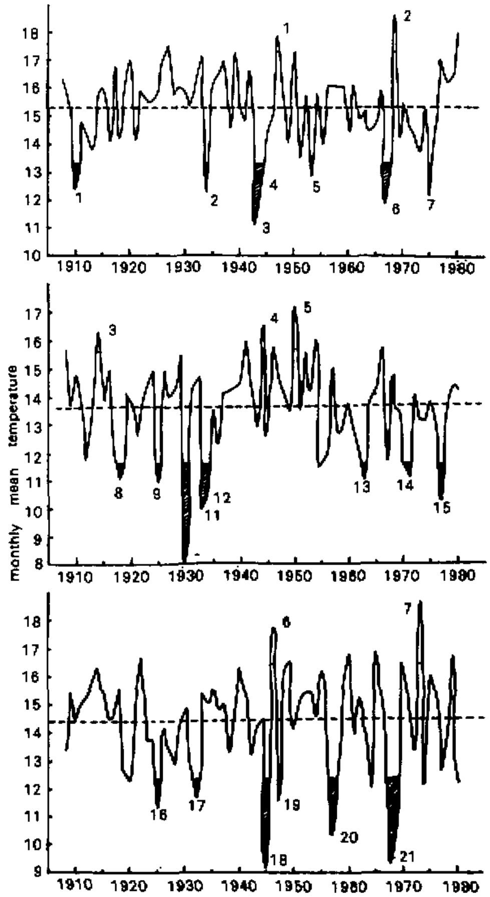

Fig. 1. Variations of 1908–1980 monthly mean temperature at Guangzhou, South China, with A for December, B for January and C for February. The climatology and observation are given by broken and solid line, respectively, with the subscript of the former 1–21 standing for 21 cold winters (thereof 3, 18, 20, 21 representing 4 severe winters) and the superscript 1–9 for 9 warm winters.

In this study, monthly mean temperature is employed as the climate variable, with phase space continuation of 1908–1980 monthly mean temperature 1D series to produce a set of particular solutions of (1), followed by determination of the coefficients of the terms via solving the contradictory expression, whereby a dynamic system is reconstructed for short term local evolution of low–latitude climate.

Fig. 1. illustrates the monthly mean temperature of 1908–1980 winter months of Guangzhou. One can see that apart from annual change there exists interannual variation, showing that 21 cold winters (out of which 4 cases are severe, i.e., 4°C lower than the climatology) and 9 warm winters, as judged by a temperature anomaly criterion in which 2°C lower (higher) than the mean in any of the three successive months is defined as a cold (warm) winter. Following the same criterion the city record shows 8 (10) warm (cold) springs, 2 (2) hot (cool) summers, and 2 (2) hot (cool) falls. Evidently, even in a tropical marine climate as in Guangzhou, short–range climatic variation displays noticeable aperiodicity as the reflection of nonlinear complex.

Feng et al. (1989) reported the fractal dimensionality ($d_g = 2.3$) of the chaotic attractor of the Guangzhou mean temperature series. Consequently, the dynamic system for the short–term change in climate there ought to be described by at least three dimensions. Accordingly, something is done with (1), with X_1, X_2 and X_3 denoting the phase space state variables whose time–varying evolutions satisfy the dynamic system of the form

$$\begin{cases} \frac{dX_1}{dt} = a_1X_1 + a_2X_2 + a_3X_3 + a_4X_1^2 + a_5X_2^2 + a_6X_3^2 + a_7X_1X_2 + a_8X_2X_3 + a_9X_3X_1 \\ \frac{dX_2}{dt} = b_1X_1 + b_2X_2 + b_3X_3 + b_4X_1^2 + b_5X_2^2 + b_6X_3^2 + b_7X_1X_2 + b_8X_2X_3 + b_9X_3X_1 \\ \frac{dX_3}{dt} = c_1X_1 + c_2X_2 + c_3X_3 + c_4X_1^2 + c_5X_2^2 + c_6X_3^2 + c_7X_1X_2 + c_8X_2X_3 + c_9X_3X_1 \ . \end{cases} \tag{2}$$

It is obvious therefrom that (2) contains both linear terms and quadratic nonlinear terms. To get a dynamic system able to describe the time–dependent evolution of state variables X_1 X_2 and X_3, we have to set the coefficients in (2) to be unknown quantities, with the variables, their products and all the left hand side terms found directly from the time sequences.

Now, how can we get the information on the state variables? A phase space continuation is done, by a drift technique, of 1D into 3 D series at time lag τ, so that

$$\begin{cases} X_1: \ x_0(t_1) \ , \ x_0(t_2) \ , \ x_0(t_3) \ , \cdots\cdots , x_0(t_N) \\ X_2: \ x_0(t_1+\tau) \ , \ x_0(t_2+\tau) \ , \ x_0(t_3+\tau) \ , \cdots\cdots , x_0(t_N+\tau) \\ X_3: \ x_0(t_1+2\tau) \ , \ x_0(t_2+2\tau) \ , \ x_0(t_3+2\tau) \ , \cdots\cdots , x_0(t_N+2\tau) \end{cases} \tag{3}$$

The first row indicates discrete values of X_1 in its temporal development, and so does the second (third) of $X_2(X_3)$. Thus, these values as particular solutions are put into (2) to find unknown coefficients.

III. RECONSTRUCTION OF A DYNAMIC SYSTEM

Put (2) into difference form and it is evident from $\frac{dX_1}{dt}$ of (2) that for X_1 we have

$$\frac{X_1(n+1)-X_1(n-1)}{2\Delta t} = a_1X_1(n) + a_2X_2(n) + a_3X_3(n) + a_4X_1^2(n) + a_5X_2^2(n) + a_6X_3^2(n) + a_7X_1(n)X_2(n) + a_8X_2(n)X_3(n) + a_9X_3(n)X_1(n) \ . \tag{4}$$

Put the values of (3) from continuation of the series into (4) and express in matrix form. Then the lhs of (4) can be given as

$$D=\begin{bmatrix} d_1 \\ d_2 \\ \vdots \\ d_M \end{bmatrix}=\begin{bmatrix} \dfrac{X_1(3)-X_1(1)}{2\Delta t} \\ \dfrac{X_1(4)-X_1(2)}{2\Delta t} \\ \vdots \\ \dfrac{X_1(m-2)-X_1(m-4)}{2\Delta t} \end{bmatrix} , \tag{5}$$

in which $X_1(1)=x_0(t), X_1(2)=x_0(t_2)$, ⋯ and Δt = the sampling interval of time. The known time series length $m=876$, of which at least 2 will be lost during drifting, leading to $m_1=m-2$ and $M=m_1-1$. As a result, the lhs of (4) can be found and made known, so can the related variables and their products on the rhs through the use of the sequences subjected to continuation. For convenience we denote Q as

$$\begin{aligned} Q &= (Q_1,Q_2,Q_3,Q_4,Q_5,Q_6,Q_7,Q_8,Q_9) \\ &= \left(X_1(n),X_2(n),X_3(n),X_1^2(n),X_2^2(n),X_3^2(n),X_1(n)X_2(n),X_2(n)X_3(n),X_3(n)X_2(n)\right) . \end{aligned} \tag{6}$$

Corresponding to (5), (3) is substituted into (6), yielding a matrix of k columns and M rows, where $k=1, 2, \cdots, 9$. Thus,

$$Q=\begin{bmatrix} Q_{11} & Q_{12} & \cdots & Q_{1k} \\ Q_{21} & Q_{22} & \cdots & Q_{2k} \\ \vdots & \vdots & \cdots & \\ Q_{M1} & Q_{M2} & \cdots & Q_{Mk} \end{bmatrix} . \tag{7}$$

At this point, the coefficients of the terms of (4) are unknown quantities to be obtained. We have the coefficient in matrix form

$$A=\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_9 \end{bmatrix} . \tag{8}$$

Then, a system of contradictory equations is formulated from (5), (7) and (8) and given in matrix form as

$$D=QA , \tag{9}$$

which is then solved by the least square method, i.e., to take the minimum of the error squared sum R that has the form

$$R=(D-QA)^T(D-QA) . \tag{10}$$

Therefore, the regular equation relative to (9) takes the form

$$Q^T Q A = Q^T D \ . \qquad (11)$$

Therein we denote $S = Q^T Q$, where superscript T represents the transposed matrix. From (11) we have

$$A = S^{-1} Q^T D \ , \qquad (12)$$

where S^{-1} refers to the inverse form of the matrix S. If S were nonsingular, A would be found directly, and if not, according to the procedure shown in Chou (1986).

Similarly, the coefficient matrices $B = (b_1, b_2, \cdots, b_k)$ and $C = (c_1, c_2, \cdots c_k)$ are determined via $\frac{dX_2}{dt}$ and $\frac{dX_3}{dt}$ (2), separately, following the procedure shown in (4) through (12).

IV. CALCULATION RESULTS AND DYNAMIC MODEL

Table 1 summarizes all the calculations, indicating that some of the coefficients are very small while others are quite large, spanning two orders of magnitude. It is a common practice in constructing a dynamic model to only make use of the terms with higher order. However, the problem is whether the importance of the terms with their coefficients at different orders is linked to their contribution to the system's evolution. In other words, it is unclear that the terms of greater coefficients contribute more. The answer may not be affirmative. For this reason, we examine it in terms of the relative error of each of the terms SR_i that is in the form

$$SR_i = \frac{1}{m} \sum_{n=1}^{m} \left[Q_i^2 / \sum_{i=1}^{9} Q_i^2 \right]_n \ , \qquad (13)$$

where Q_i denotes the i-th term on the rhs of (4) with $i = 1, 2, \cdots, 9$, the results given in the second column of the table.

Table 1. Values of Coefficients (a_i, b_i and c_i) of Terms for State Variables (X_1, X_2, and X_3), Respectively, of (2), with Their Relative Error SR_i.

term on rhs	$\frac{d}{dt}X_1$		$\frac{d}{dt}X_2$		$\frac{d}{dt}X_3$	
	a_i	SR_i	b_i	SR_i	c_i	SR_i
1	−0.051187	0.7997 E−01	−0.500012	0.4957 E+00	−0.003155	0.1309 E−02
2	0.473615	0.8791 E+00	−0.000000	0.1075 E−11	−0.469822	0.8904 E+00
3	0.010207	0.5912 E−02	0.500012	0.5042 E+00	0.050215	0.8387 E−01
4	−0.018283	0.2633 E−01	0.000002	0.1167 E−10	−0.001837	0.2165 E−02
5	0.013409	0.1047 E−02	−0.000000	0.7736 E−11	−0.010252	0.6261 E−03
6	0.003843	0.2761 E−02	−0.000002	0.1648 E−11	0.011100	0.9836 E−02
7	−0.018910	0.1421 E−02	0.000001	0.1658 E−11	0.003354	0.5594 E−04
8	−0.024859	0.3106 E−02	−0.000002	0.1175 E−10	0.021260	0.2047 E−02
9	0.003212	0.3430 E−03	0.000000	0.1821 E−12	0.014709	0.9630 E−02

Then, terms of bigger coefficient and variance are singled out from the table to establish an equation for a dynamic system describing short−range climate variation which has the form

$$\begin{cases} \dfrac{dX_1}{dt} = -a_1 X_1 + a_2 X_2 \\ \dfrac{dX_2}{dt} = -b_1 X_1 + b_3 X_3 \\ \dfrac{dX_3}{dt} = -c_2 X_2 + c_3 X_3 + c_9 X_1 X_3 \ , \end{cases} \tag{14}$$

where $a_1 = 0.051187$, $a_2 = 0.473615$, $b_1 = 0.500000$, $b_3 = 0.500000$, $c_2 = 0.469822$, $c_3 = 0.050215$ and $c_9 = 0.014709$.

After setting up this system, we could not help thinking of a well-known model of chaotic dynamics—the system that has been intensively studied by Rossles, viz.,

$$\begin{cases} \dfrac{dX}{dt} = -(Y+Z) \\ \dfrac{dY}{dt} = X + aY \\ \dfrac{dz}{dt} = bX - cZ + ZX \ . \end{cases}$$

His numerical results show that when certain values are assigned to the coefficients a and b with c evaluated differently, the system will experience bifurcation, leading to chaos. Interestingly, with $X = x_2$, $Y = x_1$ and $Z = x_3$, (14) is transformed into

$$\begin{cases} \dfrac{dX}{dt} = -\tilde{a}_1 Y + \tilde{a}_2 Z \\ \dfrac{dY}{dt} = -\tilde{b}_1 X + \tilde{b}_2 Y \\ \dfrac{dZ}{dt} = -\tilde{c}_1 X + \tilde{c}_2 Z + \tilde{c}_3 ZX \ , \end{cases} \tag{15}$$

in which $\tilde{a}_1 = b_1$, $\tilde{a}_2 = b_3$, $\tilde{b}_1 = a_1$, $\tilde{b}_2 = a_2$, $\tilde{c}_1 = c_2$, $\tilde{c}_2 = c_3$ and $\tilde{c_3} = c_9$. Evidently, (15) is quite similar formally to the Rossles's system, most obvious being the existence of a nonlinear term therein in addition to the linear terms. Therefore, we assume that the reconstruction attempted in this paper can be possibly used to describe the nonlinear properties of short range climatic evolution characterized by monthly mean temperatures.

The reconstruction based on 1D time series is in fact a quantitative study in an attempt to establish a low-order model of state variables of our concern so as to construct a certain link to a large model of very high freedom or act as a "bridge". For this reason, we attemt to experiment with prediction in terms of (14) in an effert to make assessment of the system.

Following a difference scheme, (14) is put into difference form as

$$\begin{cases} X_1(n+1) = X_1(n-1) + [a_2 X_2(n) - a_1 X_1(n)]2\Delta t \\ X_2(n+1) = X_2(n-1) + [b_3 X_3(n) - b_1 X_1(n)]2\Delta t \\ X_3(n+1) = X_3(n-1) + [c_3 X_3(n) - c_2 X_2(n) + c_9 X_1(n) X_3(n)]2\Delta t \ . \end{cases} \tag{16}$$

At this stage our aim is just to investigate the capacity of this system to describe monthly mean temperature change through predictive experimentation. Without the loss of generality, prediction is made at $\Delta t = 1$ for the last phase point of the 3D phase space series from continuation, with monthly mean temperatures of the 1979–80 winter as initial values and the results of April through December 1980 are given in Fig. 2.

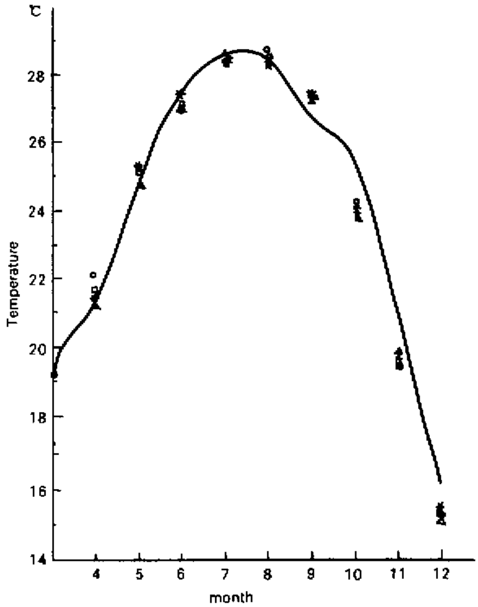

Fig. 2. Experimental results of monthly mean temperature forecasting in April–December 1980, with •, Δ, □ and ○ denoting prediction values of X_1, X_2, X_3 and their average, respectively.

It is quite obvious from this figure that the dynamic system is effective in illustrating the change tendency of Guangzhou monthly mean temperature. To demonstrate this, analysis is carried out of errors of the results as summarized in Table 2. Inspection of the system–forecasted values in comparison to the climatology and measurments indicates that their mean absolute error and mean square deviation ate identical. And as far as the mean relative error and correlation coefficient are concerned, the prediction is even closer to the measurement, with their correlation much higher than its correlation to the climatic value. Further, the fitting of historical data using (14) shows a rate greater than 80%. Together, all these reveal noticeable advantage of the system presented in forecasting short–term climatic evolution.

Table 2. Error Analysis of Experimental Predictions

type*	prediction vs climatology	prediction vs measurement
MAR	0.932566	0.932566
MRE	4.124365	0.051586
MSD	1.276853	1.278653
CC	0.372751	0.787630

* MAR = mean absolute error, MRE = mean relative error,
MSD = mean square deviation, and CC = correlation coefficient.

V. CONCLUDING REMARKS AND DISCUSSIONS

From the foregoing one can see that the reconstructed dynamic system is nonlinear that serves as a model for exploring short-range change in tropical climate with the aid of monthly mean temperature as the climate variable. Though only one nonlinear quadratic term is available in (14), nonlinearity is clearly displayed that can describe the features, i.e., the evolution is periodic but nonrepeating and marked by certain complexity.

Our target is to connect the reconstructed dynamic system built upon 1D time series to forecasting, with a preliminary result presented here. Error analysis of the findings indicates encouraging merits of the system.

Moreover, two points concerning phase space continuation analysis are of note: i) the length of the data used, m, is closer to the needs of providing the minimum limiting condition on m proposed by Ruelle (1990), $2\log_{10}m \gg D$, or $m \gg 10^{D/2}$ with D being the dimensionality of the phase space; ii) in doing continuation from 1D into multi-dimensional series, the selection of time lag τ is highly important. In view of the not too long dataset adopted and for convenience in experimenting $\tau = 1$ is assumed. The calculations by backward correlation show that the coefficient $r = 0.07191$ (0.00159) at $\tau = 1$ (3). On the whole, hence, $\tau = 1$ meets the needs of mutual independence of the state variables. Of course, $\tau = 3$ will do as well.

Finally, mention should be made of the fact that the reconstruction of the dynamic system based on the phase space continuation of 1D time series of climatic observations with monthly mean temperatures as the variable differs from the scheme of Huang et al. (1991) in which the known Lorenz system was resolved and retrieved, with the examples illustrating the retrieval from the output of the known model rather than in situ observation. Evidently, of more practical interest is the effort at getting acquainted with the reconstruction through phase space continuation of 1D time series of all kinds which come into use most frequently in our daily routine and research.

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