

## Some Splitting Methods for Equations of Geophysical Fluid Dynamics<sup>①</sup>

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### ABSTRACT

In this paper, equations of atmospheric and oceanic dynamics are reduced to a kind of evolutionary equation in operator form, based on which a conclusion that the separability of motion stages is relative is made and an issue that the tractional splitting methods established on the physical separability of the fast stage and the slow stage neglect the interaction between the two stages to some extent is shown. Also, three splitting patterns are summed up from the splitting methods in common use so that a comparison between them is carried out. The comparison shows that only the improved splitting pattern (ISP) can be in second order and keep the interaction well. Finally, the applications of some splitting methods on numerical simulations of typhoon tracks made clear that ISP owns the best effect and can save more than 80% CPU time.

**Key words:** Evolution equation, Splitting method, Fast and slow stages.

### 1. INTRODUCTION

Computational Geophysical Fluid Dynamics is a new and developing branch of cross discipline. One of its major research objects is the large scale atmospheric and oceanic motions. Usually, the equation of atmospheric and oceanic dynamics used for numerical simulations or for numerical predictions belongs to the nonstationary equation set of geophysical fluid dynamics. For example, the barotropic shallow water equation set describing the atmospheric motion approximately is:

$$\begin{cases} \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = -\frac{\partial \varphi}{\partial x} + f v, \\ \frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} = -\frac{\partial \varphi}{\partial y} - f u, \\ \frac{\partial \varphi}{\partial t} + \frac{\partial u \varphi}{\partial x} + \frac{\partial v \varphi}{\partial y} = 0. \end{cases} \quad (1)$$

The equation set includes both the slow-developing Rossby waves and the fast-changing inertia-gravitational waves, and has a remarkable feature that the time for numerical integration of it is very long and in general hundreds of thousands of integration steps and more are needed. Thereby, the stability and the time-saving effect of the computing scheme to the equation become two key issues for successful numerical computations, because they directly affect the cost and the time effectiveness of predictions. In order to resolve the stability issue, a species of numerical methods that are called the perfect energy conservative difference

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methods have been developed successfully in Institute of Atmospheric Physics, Chinese Academy of Sciences, which are introduced in detail in some relevant references<sup>①</sup>. To resolve another issue, many economical schemes and methods have been established in recent years and the splitting method is one of the most effective economical methods.

On splitting method, the Russian mathematician Yaneko et al. (1967) made a construction in early time. He constructed the split-time-interval method that was generalized and applied to the numerical computations of atmospheric and oceanic problems, and later constructed by Marchuk (1982), the former president of Russian Academy of Sciences. One of the earliest works in this field in China refers to Zeng et al. (1980). Following the appearance and development of parallel computers, parallel algorithm becomes a popular subject, which includes an important method i.e. split-operator method. The splitting method we consider here is also a split-operator method. It is mainly suitable for series computers, but it is not difficult to generalize it to parallel computers and more CPU time is expected to be saved.

## II. PHYSICAL FUNDAMENTALS OF SPLITTING METHOD

The splitting method we study is different from the general split-operator methods, although it is a split-operator method, it is implemented according to separability of physical states. Therefore, not only this algorithm would be a mathematical method but also have clear physical significant. The motion described by Eq.(1) includes the adjustment stage and the development stage. These two stages can be split into time scale, which is put forward early by Ye et al. (1988) and Zeng et al. (1980). For large-scale atmospheric motions, the characteristic time of the adjustment stage is  $10^4$ 's and that of the development stage is  $10^5$ 's (Ye et al., 1988). Due to the difference between two kinds of characteristic time, Eq.(1) can be separated into two equation sets. One is mainly to describe the adjustment stage and a short time interval is selected for the time integrations. Another is mainly to describe the development stage and a longer time interval is chosen for the time integrations. By this way, all the integrations for splitting method may decrease in cost more than that for non-splitting method.

## III. RELATIVE SEPARABILITY OF FAST STAGE AND SLOW STAGE

Equations of atmospheric and oceanic dynamics can be reduced to the following evolution equation in operator form:

$$\frac{\partial F}{\partial t} + \mathcal{L}F = 0, \quad (2)$$

where the operator  $\mathcal{L}$  can be split into the operator  $\mathcal{L}_1$  denoting the fast stage and the operator  $\mathcal{L}_2$  denoting the slow stage:  $\mathcal{L} = \mathcal{L}_1 + \mathcal{L}_2$ . Suppose that  $F$  is a function of the time variable  $t$ , then Eq.(2) becomes

$$\frac{dF}{dt} + \mathcal{L}_1(F) + \mathcal{L}_2(F) = 0. \quad (3)$$

Expand  $F(t)$  in Taylor series at the time to

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① Ji Zhongzhen, Wang Bin and Zeng Qingcun (1993), Perfectly Energy-Conservative Difference Methods and Their Applications (submitted to *Some Progress in New Techniques for Numerical Prediction*, Meteorological Press, Beijing, 1994 (in Chinese)).

$$F(t) = F^o + \Delta t \left( \frac{dF}{dt} \right)^o + \frac{\Delta t^2}{2} \left( \frac{d^2 F}{dt^2} \right)^o + O(\Delta t^3), \quad (4)$$

where  $\Delta t = t - t^o$ , and the following expressions are obtained:

$$\frac{dF}{dt} = -\mathcal{L}_1(F) - \mathcal{L}_2(F), \quad (5)$$

$$\begin{aligned} \frac{d^2 F}{dt^2} &= \frac{d\mathcal{L}_1(F)}{dF} \mathcal{L}_1(F) + \frac{d\mathcal{L}_1(F)}{dF} \mathcal{L}_2(F) + \frac{d\mathcal{L}_2(F)}{dF} \mathcal{L}_1(F) + \frac{d\mathcal{L}_2(F)}{dF} \mathcal{L}_2(F) \\ &\equiv R_1 + r_1 + r_2 + R_2. \end{aligned} \quad (6)$$

It is derived from substituting (5) and (6) into (4) that

$$F(t) = F^o + \Delta t \left( \mathcal{L}_1(F) + \mathcal{L}_2(F) \right)^o + \frac{\Delta t^2}{2} \left( R_1^o + R_2^o + r_1^o + r_2^o \right) + O(\Delta t^3). \quad (7)$$

Exp.(7) shows that the two stages can be separated only when the small terms with second order or higher are truncated, and they can not be split completely if the second order term including the interaction of the two stages is left.

#### IV. ESTABLISHMENT OF AND COMPARISON BETWEEN THREE SPLITTING PATTERNS<sup>①</sup>

From Eq.(3), three splitting patterns can be constructed

(1) CSP (Conservative Splitting Pattern)

$$\begin{cases} \frac{dP}{dt} + \mathcal{L}_1(P) = 0, & P^o = F^o \\ \frac{dQ}{dt} + \mathcal{L}_2(Q) = 0, & Q^o = P^1 \end{cases} \quad t \in [t_o, t_1] \quad (8)$$

to which the general solution is

$$F^* = P + Q - Q^o = Q + P - P^1, \quad (9)$$

clearly,  $(F^*)^o = F^o$ ,  $(F^*)^1 = Q^1$ .

(2) ESP (Economical Splitting Pattern)

$$\frac{dF^*}{dt} + \mathcal{L}_1(F^*) + \mathcal{L}_2(F^o) = 0, \quad (F^*)^{-1} = (F)^{-1}, \quad t \in [t_{-1}, t_1] \quad (10)$$

(3) ISP (Improved Splitting Pattern)

$$\begin{cases} \frac{dP}{dt} + \mathcal{L}_1(P) + \mathcal{L}_2(F^o) = 0, & P^o = F^o \\ \frac{dQ}{dt} + \mathcal{L}_2(F^*) - \mathcal{L}_2(F^o) = 0, & Q^o = P^1 \end{cases} \quad t \in [t_o, t_1] \quad (11)$$

$$F^* = P + Q - Q^o = Q + P - P^1.$$

<sup>①</sup>Wang Bin, Ji Zhongzhen and Zeng Qingcun (1993), A Preliminary Study on the Theory of Splitting Method (submitted to *Chinese Journal of Computational Mathematics*).

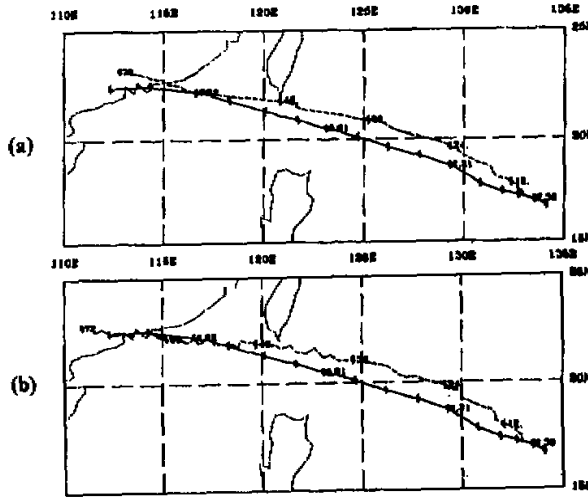


Fig. 1. 72-hour Prediction of 7908 Typhoon Track (a) from splitting scheme (b) from non-splitting scheme solid line: observed track broken line: simulated track (b) from non-splitting scheme solid line: observed track broken line: simulated track

Now, compare the three splitting patterns.

From the Taylor expansion of  $F^*$  :

$$F^* = (F^*)^o + \Delta t \left( \frac{dF^*}{dt} \right)^o + \frac{\Delta t^2}{2} \left( \frac{d^2 F^*}{dt^2} \right)^o + O(\Delta t^3), \quad (12)$$

it can be proved that ISP is in second order while CSP and ESP are in only first order, because of for CSP

$$\begin{cases} \left( \frac{dF^*}{dt} \right)^o = -\mathcal{L}_1(P)^o - \mathcal{L}_2(Q)^o = -\mathcal{L}_1(F^o) - \mathcal{L}_2(P^1), \\ \left( \frac{d^2 F^*}{dt^2} \right)^o = \left( \frac{d\mathcal{L}_1}{dP} \mathcal{L}_1(P) \right)^o + \left( \frac{d\mathcal{L}_2}{dQ} \mathcal{L}_2(Q) \right)^o = R_1^o + R_2^o, \end{cases} \quad (13)$$

for ESP

$$\begin{cases} \left( \frac{dF^*}{dt} \right)^o = -\mathcal{L}_1(F^*)^o - \mathcal{L}_2(F^o), \\ \left( \frac{d^2 F^*}{dt^2} \right)^o = \left( \frac{d\mathcal{L}_1}{dF^*} \mathcal{L}_1(F^*) \right)^o + \left( \frac{d\mathcal{L}_2}{dF^*} \mathcal{L}_2(F^o) \right)^o = R_1^o + r_1^o, \end{cases} \quad (14)$$

and for ISP

$$\begin{cases} \left(\frac{dF^*}{dt}\right)^o = -\mathcal{L}_1(P)^o - \mathcal{L}_2(F^*)^o, \\ \left(\frac{d^2 F^*}{dt^2}\right)^o = \left(\frac{d\mathcal{L}_1}{dP}\mathcal{L}_1(P) + \frac{d\mathcal{L}_1}{dP}\mathcal{L}_2(F^*) + \frac{d\mathcal{L}_2}{dF^*}\mathcal{L}_2(F^*) + \frac{d\mathcal{L}_2}{dF^*}\mathcal{L}_1(P)\right)^o \\ = R_1^o + r_1^o + R_2^o + r_2^o. \end{cases} \quad (15)$$

The splitting patterns are applied to simulate typhoon tracks so that their computing effects are examined. It is testified in examinations that ISP is the best and it saves 80% CPU time (refer to Fig. 1).

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