A Nonlinear Time–lag Differential Equation Model for Predicting Monthly Precipitation①

Peng Yongqing (彭永清), *Yan Shaojin* (严绍谨) and *Wang Tongmei* (王同美)
Nanjing Institute of Meteorology, Nanjing 210044
Received September 20, 1994; revised January 23, 1995

ABSTRACT

This paper investigates the nonlinear prediction of monthly rainfall time series which consists of phase space continuation of one–dimensional sequence, followed by least–square determination of the coefficients for the terms of the time–lag differential equation model and then fitting of the prognostic expression is made to 1951–1980 monthly rainfall datasets from Changsha station. Results show that the model is likely to describe the nonlinearity of the annual cycle of precipitation on a monthly basis and to provide a basis for flood prevention and drought combating for the wet season.

Key words: Monthly rainfall, Phase space continuation, Time–lag differential equation

I. INTRODUCTION

Monthly rainfall prediction (MRP) is undoubtedly one of the major items included in a long–range weather forecast for practically all weather services of the world (Wang, 1990).the reason lies in that the space / time patterns of the rainfall are indispensable for investigating large–scale drought / flood regularity and hence the fundamental data. Accordingly, as far as a particular season is concerned, MRP's at a variety of regions to be plotted on a map will be a scientific basis for such disaster forecasting.

In a customary sense, by MRP we mean to give its departure from the climatology, of which the causes can be sought in the persistent anomaly of synoptic–scale low– / high–pressure systems whose evolutions on a long–term basis are under the control of so–called atmospheric centers of action. Consequently, the MRP can be worked out by long–range forecasting of circulation persistent abnormality. However, in view of the fact that the long–term farecasting is still at infant stage, remedial measures in current use include the extension of medium–range numerical weather prediction, statistical–empirical method, MOS and expert systems wherewith long–term rainfall forecasting is made to meet the operational needs. Yet these techniques are now in such a dilemma that they are hardly be allowed to improve. Speaking of MRP improvement, a problem deserves attention that the development of nonlinear sciences, particularly the nonlinear prediction theory of time series has really offered us with new approaches to such research.

With regard to nonlinear prediction with time sequence, the authors effecti(1994) proposed a scenario of 1D series continuation and then establishment of a dynamic system that will be described in terms of phase space variables and first–order differential equations consisting of quadratic polynomials based on these variables. Such a scheme works quite

①This work is sponsored by the National Natural Science Foundation of China.

effectively for monthly mean temperature forecasting but great caution should be exercised when the technique is adopted for MRP because monthly rainfall has a departure big enough to be at almost the same order as the precipitation itself and besides, it is characterized by vast difference in its yearly variation and great interannual variability of monthly rainfall. For this reason, developed is a scheme that consists of constructing a phase space in the light of a monthly rainfall sequence (as with the case illustrated in Peng et al., 1994) wherewith is established a prognostic expression that is a model composed of specified nonlinear time–lag differential equations.

II. TIME–LAG DIFFERENTIAL EQUATION MODEL

Write the phase space state variable as $X_j(t)$ and we have the differential equation in the form

$$\frac{dX_j(t)}{dt} = F(X_i(t), X_i(t-\tau)) , \qquad i=j=1, 2, \cdots n \tag{1}$$

which suggests that change in the state at time t depends not merely on the state but on that at a fixed time in the past as well.

Set $X(t)$ to be a monthly rainfall series. And after its drift, we assume $E(t) = x(t)$, $I(t) = x(t-\tau)$ and $G(t) = x(t-2\tau)$ which make up a phase space. Again, the time–dependent variation in a dynamic system described by $E(t)$, $I(t)$ and $G(t)$ is let to satisfy the following differential equations (Wu, 1993).

$$\begin{cases} \dfrac{dE(t)}{dt} = -E(t) + a_1 f(b_1 G(t-\tau)) - a_2 f(b_2 I(t-\tau)) , \\ \dfrac{dI(t)}{dt} = -I(t) + a_3 f(b_3 E(t-\tau)) - a_4 f(b_4 G(t-\tau)) , \\ \dfrac{dG(t)}{dt} = -G(t) + a_5 f(b_5 E(t-\tau)) - a_6 f(b_6 I(t-\tau)) , \end{cases} \tag{2}$$

where (x) represents a nonlinear function, taken to be in the form of $tgh(x)$. It was a nut for meteorologists to prepare a monthly rainfall forecast in a certain technique available for no other reasons than the big departure and interannual variability of monthly precipitation, as stated earlier, so that it is practically impossible to have an appropriate function for the nonlinear properties. To alleviate the situation, a hyperbolic tangential function $tgh(x)$ is introduced that is shown in Fig. 1. It is apparent that for $x \in (0, 0.3)$ the function is in a linear relation to x; with $x \in (0.3, 3.0)$ it is related to x in a nonlinear manner; when $x \in (3.0, +\infty)$ it takes unity as the asymptote. Almost the same is true of the relations with the domain $(-\infty, 0)$ except that values of x and $tgh(x)$ are negative. It follows that, if we choose appropriate coefficients b_i ($i = 1, 2, \cdots, 6$) in such a way that the product of b_i and variables E, I and G is over the range of 0.3 to 3.0, then the difficulty will be avoided in association with great departure and interannual variability of monthly precipitation.

Next, (2) is a system of equations with active memory because the second and third terms on the right–hand side involve nonlinear effect at time $(t-\tau)$ prior to t such that (2) is applicable to monthly rainfall forecasting.

Then, in attempt to reveal the flood / drought evolutions in terms of monthly rainfall departures, the positive, negative, zero anomalies are denoted by F, D and R, respectively, to investigate the Changsha 1951–1980 datasets, deriving the results only with four series at a decade interval presented below:

1951: DDFFDRFFDFFF; 1961: DFFFDDDFFDFD;
1970: FFFFDDFDFDFF; 1980: FFFFDDDFDDDD,

which are represented by solid lines of Fig.2.

From the above sequences one can see that both the positive and negative anomalies are of persistence. To our knowledge, drought and flood are attributed utterly to the continuous abnormality of rainfall departure. Therefore, the persistence should be considered in describing the evolution of such systems as precipitation, or, in other words, the time–lag influence ought to be involved in establishing the related prognostic equation, which serves as another important reason that this type of model is employed to construct a monthly rainfall forecast.

III. DETERMINATION OF THE FROGNOSTIC EXPRESSION AND ITS COEFFICIENTS

The second–order Runge–Kutta formulation is used for solving (2) , of which the terms on rhs are denoted as

$$\begin{aligned} E1 &= -E(t) + a_1 f(b_1 G(t-\tau)) - a_2 f(b_2 I(t-\tau)) \;, \\ I1 &= -I(t) + a_3 f(b_3 E(t-\tau)) - a_4 f(b_4 G(t-\tau)) \;, \\ G1 &= -G(t) + a_5 f(b_5 E(t-\tau)) - a_6 f(b_6 I(t-\tau)) \;. \end{aligned} \tag{3}$$

Eq.(3), with a timestep length added to, will have the form

$$\begin{aligned} E2 &= -(E(t) + hE1) + a_1 f(b_1 G(t-\tau+h) - a_2 f(b_2 I(t-\tau+h)) \;, \\ I2 &= -(I(t) + hI1) + a_3 f(b_3 E(t-\tau+h) - a_4 f(b_4 G(t-\tau+h)) \;, \\ G2 &= -(G(t) + hG1) + a_5 f(b_5 E(t-\tau+h) - a_6 f(b_6 I(t-\tau+h)) \;. \end{aligned} \tag{4}$$

Following Runge–Kutta, we get, from the difference form of (2), the following prognostic expression

$$\begin{aligned} E(t+h) &= E(t) + h(E1 + E2)/2 \;, \\ I(t+h) &= I(t) + h(I1 + I2)/2 \;, \\ G(t+h) &= G(t) + h(G1 + G2)/2 \;. \end{aligned} \tag{5}$$

For clarity, we set $\tau = mh$, $t = nh$, $E(nh) = E_n$, $G(nh) = G_n$, and $I(nh) = I_n$ in such a manner that (3) and (4) are substituted into (5), yielding

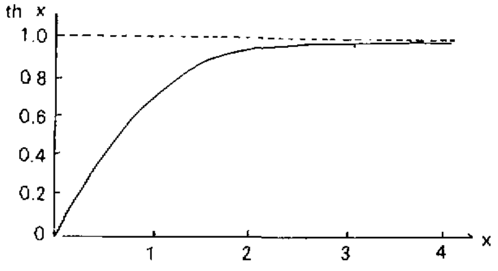

Fig. 1. Properties of hyperbolic tangential function $tgh(x)$.

$$\begin{aligned}
E_{n+1} &= \left(\frac{1-h+h^2}{2}\right)E_n + \frac{((1-h)f(b_1 G_{n-m}) + f(b_1 G_{n-m+1}))a_1 h}{2} \\
&\quad - \frac{((1-h)f(b_2 I_{n-m}) + f(b_2 I_{n-m+1}))a_2 h}{2} \\
I_{n+1} &= \left(\frac{1-h+h^2}{2}\right)I_n + \frac{((1-h)f(b_3 E_{n-m}) + f(b_3 E_{n-m+1}))a_3 h}{2} \\
&\quad - \frac{((1-h)f(b_4 G_{n-m}) + f(b_4 G_{n-m+1}))a_4 h}{2} \\
G_{n+1} &= \left(\frac{1-h+h^2}{2}\right)G_n + \frac{((1-h)f(b_5 E_{n-m}) + f(b_5 E_{n-m+1}))a_5 h}{2} \\
&\quad - \frac{((1-h)f(b_6 I_{n-m}) + f(b_6 I_{n-m+1}))a_6 h}{2} .
\end{aligned} \tag{6}$$

Eq.(6) represents the final form of (2) subjected to manipulation. At this point, what should be done is nothing but determine coefficients a_i and $b_i (i = 1, 2, \cdots, 6)$ for Eq.(6).

Coefficients a_i and b_i of the recognition equation (6) can be acquired by means of the nonlinearity least square method. However, considering the properties of monthly rainfall and the hyperbolic function used, we, by setting b_i to be fixed, identify a_i in virtue of the least square technique.

Based on the Changsha 1951–1980 monthly rainfall record, and by assuming $b_1 = 0.015, b_2 = 0.025, b_3 = 0.035, b_4 = 0.015, b_5 = 0.025$ and $b_6 = 0.035$ with $\tau = 1, h = 0.5$ and $m = 2$, we get from (6) using the least square method the following

$$a_1 = -7.10; \quad a_2 = 14.10; \quad a_3 = 280.55;$$
$$a_4 = 8.43; \quad a_5 = 0.48; \quad a_6 = 10.33.$$

For details the reader is referred to Nie (1984).

IV. CALCULATIONS AND DISCUSSION

To have an idea about the applicability of (6), we investigate the Changsha record again to make prognostic experiments with the original sequences for two different cases. In the first case, the real magnitudes at the $n-$, $(n-1)-$ and $(n-2)-th$ steps are used as initial values to predict (or fit) the $(n+1)-th$ value through (6), i. e., to give the rainfall for the next month, as shown in Fig. 2. And in the second, when the magnitude at the $(n+1)-th$ step, which results from those at the $(n-1)-$, $n-$ and $(n+1)-th$ steps, is found, it will be employed for succeeding values at the $(n+2)-$, $(n+3)-th$, $\cdots$ until the December rainfall prediction of the year is finished, with the results illustrated in Fig. 3.

As we know, stochastic are the weather systems responsible for the Changsha precipitation, including westerly troughs, South–China quasi–stationary front, SW vortices and the related cyclones, typhoons and their inverted troughs, etc. Nevertheless, the annual variation is fixed. From solid lines of Fig. 2, one can see that the rainfall has a monopeak in the May–June wet season in most of the years, only with a limited number of years (e. g., 1953, 1972, 1975 and 1980) exhibiting a b_i–peak in the May–June and autumnal rainy seasons. Further, the wet–season rainfall intensity on a monthly basis varies from year to year in the study record both for the mono–and b_i–peak categories. For instance, the monthly precipitation in the 1954, 1956, 1958 and 1969 rainy season reaches as high as 350 mm or more, corresponding to the amounts of six events of torrential rain within one month, thus leading to

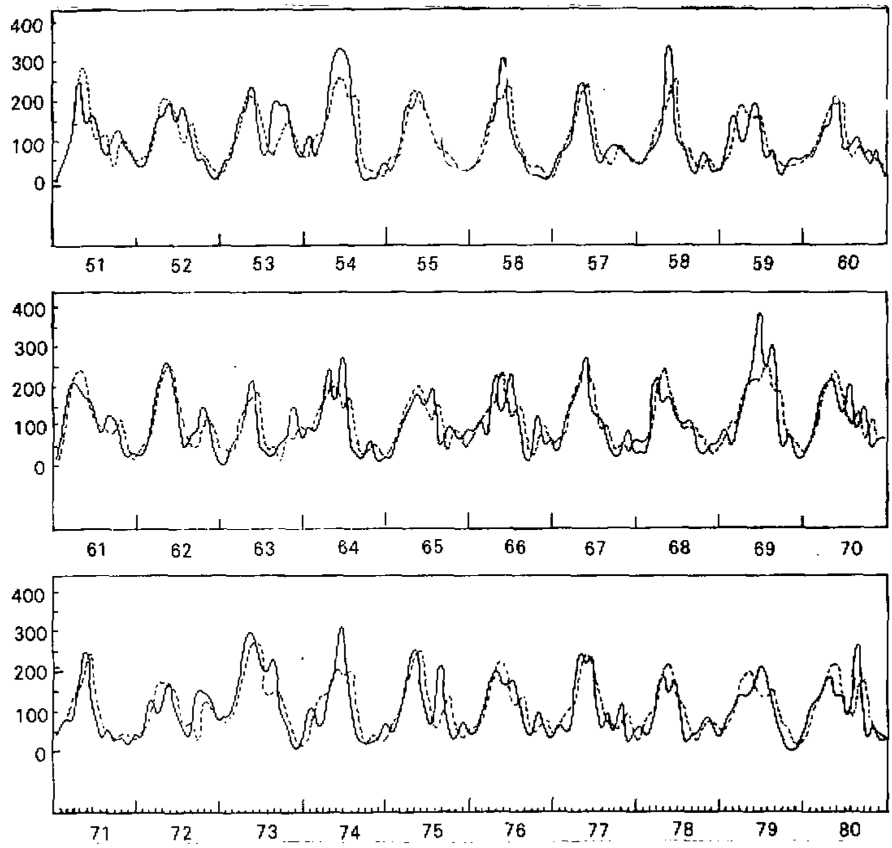

Fig. 2. Fittings of 1951–1980 monthly precipitation for Changsha.

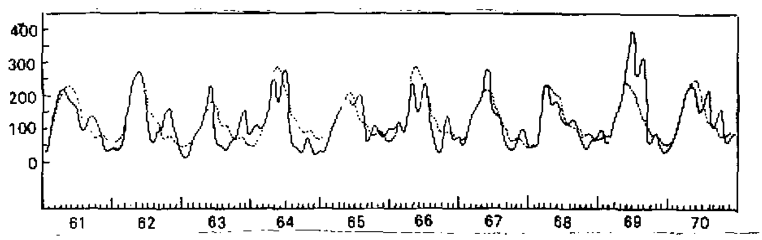

Fig. 3. Results of April–December monthly rainfall derived through Eq.(2) with the monthly values of January, February and March as initial ones on an annual basis over 1961–1971 for Changsha.

deluge on a regional basis (flood–year category), and in contrast, the drought–year category takes place in 1963, 1972, and 1978 with the rainfall reduced twofold. For this reason, it is necessary to construct a deterministic but non–linear system capable of describing the yearly course of rainfall.

From Fig. 2 we see that the time–lag differential equation model is able to illustrate the annual variation in monthly rainfall for the research area. For example, the salient features are revealed by fittings (broken lines) that show both the yearly trend of the rainfall and the intensity more satisfactorily. As such, the results are encouraging, considering the dilemma for rainfall forecasting.

Inspection of Fig. 3 indicates that the scheme is feasible in constructing a six–month rainfall forecast. The data used cover 10 years (1961–1970), differing in initial values for each of the years and for the April– December monthly precipitation, integration is performed in virtue of (6) and a_i. Results show that the May– June prediction is quite good for the scheme is responsible for monthly rainfall rather than the approximate categorization of "below normal, normal and above normal". The derived intensity can be up to 300 mm per month. It should be noted that the autumnal wet span, if available, is likely to be ignored for the study area. On this basis, however, the skill is likely to be improved.

With regard to the usefulness of the present model, it rests on operational divisions for assessment in practice. Our article aims at offering a means for monthly precipitation prediction in the wet season.

REFERENCES

Nie Tiejun (1984), *Methods of Calculation*, National Defence Industry Press, Beijing, 305–409 (in Chinese).

Peng Yongqing, Yan Shaojin and Zhu Yufeng (1994), Preliminary study of reconstruction of a dynamic system using an 1D time series, *Advances in Atmospheric Sciences.*, **11(3)**: 277–284.

Wang Shaowu (1990), WMO long–term prediction project, Meteor, *Sci. / Tech.*, **1**: 52–56.

Wu Xiangbao (1993), Nonlinear prediction of chaotic time sequence and its application to functions of human brain, *J. Graduate School*, Academia Sinica, **10(2)**: 189–196.