# On the Nonlinear Stability of Three–Dimensional Quasigeostrophic Motions in Spherical Geometry [1]

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ABSTRACT

Nonlinear stability criteria for the motions geoverned by three–dimensional quasigeostrophic model in spherical geometry are obtained by using Arnol'd's variational principle and a priori estimate method. The results gained in this paper are parallel to Arnol'd's second theorem and better than the known results. Especially, under the approximation of vertically integrated nondivergency, criteria corresponding to Arnol'd's second theorem are first established by a detailed analysis.

**Key words:** Nonlinear stability, Quasigeostrophic motion, Spherical geometry

## I. INTRODUCTION

In recent years, there has been considerable interest in elucidating the nonlinear stability properties of fluid motions. Since Arnol'd (1965, 1966) established two theorems for the nonlinear stability of plane flow by employing the variational principle and a priori estimate method, many authors have worked in this field (for example, Holm et al., 1985; McIntyre and Shepherd, 1987; Mu and Zeng, 1991; Mu and Wang, 1992; Shepherd, 1988; Zeng, 1979, 1989).

For the three–dimensional quasigeostrophic motion, which is of importance in geophysical fluid dynamics, nonlinear stability criteria were established in spherical geometry (Zeng, 1989, Mu and Zeng, 1991) and on beta–planes (McIntyre and Shepherd, 1987; Mu and Wang, 1992). Generally speaking, it is much more difficult to establish Arnol'd's second theorem than to the first one. Mu and Zeng (1991) derived a criterion corresponding to Arnol'd's second theorem for three–dimensional quasigeostrophic motion in spherical coordinates. This paper continues this line of investigation and derives new nonlinear stability criteria analogous to Arnol'd's second theorem. The results, which is better than the known results, are given by an accurate priori estimate and the improvement is shown to be of considerable importance. Moreover we first obtain criteria parallel to Arnol'd's second theorem under the approximation of vertically integrated nondivergency by a detailed analysis.

This paper is organized as follows. Section 2 is concerned with the three–dimensional quasigeostrophic model in spherical coordinates. In Section 3, the criteria parallel to Arnol'd's second theorem under the approximation of vertically integrated nondivergency are established. In Section 4, we present a better criterion for the three–dimensional quasigeostrophic model. Appendix A and Appendix B are devoted to solving two systems of inequalities, which are of crucial importance to the establishment of the criteria.

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II. THE MODEL

The governing equation of three–dimensional quasigeostrophic model in the spherical coodinates is the conservation of potential vorticity (see Zeng, 1989)

$$\frac{\partial q}{\partial t}+\frac{1}{a^2}J(\psi,q)=0, \tag{2.1}$$

where $\psi$ is the stream function and

$$q=\nabla^2\psi+\frac{\partial}{\partial\zeta}\left(\frac{f_0^2}{c^2}\zeta^2\frac{\partial\psi}{\partial\zeta}\right)+2\omega\cos\theta \tag{2.2}$$

is the potential vorticity. Where $0\leqslant\zeta\leqslant 1$, $c^2\equiv\alpha R\tilde{T}$, $\alpha=R(\gamma_\alpha-\tilde{\gamma})/g$, $\gamma_\alpha=g/c_p$, $\tilde{T}$ and $\tilde{\gamma}$ are the mean temperature and its vertical gradient respectively. $J(f,g)=\frac{1}{\sin\theta}\left(\frac{\partial f}{\partial\theta}\frac{\partial g}{\partial\lambda}-\frac{\partial f}{\partial\lambda}\frac{\partial g}{\partial\theta}\right)$ is the Jacobian in spherical coodinates, $\lambda$ the longitude, $\theta$ the colatitude; $f_0$ the mean Coriolis parameter, $\omega$ the Earth's angular velocity, $\nabla^2$ the Laplacian on the spherical surface with radius $a$.

The initial condition is

$$\psi|_{t=0}=\psi_0, \tag{2.3}$$

and the boundary condition is

$$\frac{\partial b}{\partial t}+\frac{1}{a^2}J(\psi_s,b)=0, \tag{2.4}$$

where $b\equiv(\partial\psi/\partial\zeta)_s+\kappa\alpha_s\psi_s$, the subscript $s$ denotes the functions defined at the bottom boundary $\zeta=1$, and the orographic influences are omitted for simplicity. $\kappa=0$ or $1$, $\kappa=0$ corresponds to the approximation of vertically integrated nondivergency.

From the governing equations (2.1)–(2.2) and the boundary condition (2.4), the conservation of total energy $E$, generalized enstrophy $F$ and generalized boundary energy $B$ are as follows

$$E\equiv\frac{1}{2}\left[\int_M\left(|\vec{v}|^2+\left|\frac{f_0}{c}\zeta\frac{\partial\psi}{\partial\zeta}\right|^2\right)dM+\int_{s^2}\kappa\frac{f_0^2}{c_s^2}\alpha_s|\psi_s|^2ds\right], \tag{2.5}$$

$$F\equiv\int_M F(q)dM, \tag{2.6}$$

$$B\equiv\int_{s^2}G(b)ds, \tag{2.7}$$

where, $\int_M(\cdot)dM=\int_0^1\int_{s^2}(\cdot)dsd\zeta$, $M=s^2\times[0,\ 1]$, $s^2$ stands for the surface of the sphere with radius $a$, $F$ and $G$ are two arbitrary first–order continuously differentiable functions of their arguments, and

$$\vec{v}=\left(\frac{1}{a}\frac{\partial\psi}{\partial\theta},\ -\frac{1}{a\sin\theta}\frac{\partial\psi}{\partial\lambda}\right). \tag{2.8}$$

Now suppose $(\bar{\psi},\ \bar{q})$ is a basic steady state to systems (2.1)–(2.4); it follows

$$J(\bar{\Psi},\bar{q})=0, \tag{2.9}$$

$$J(\bar{\Psi}_s,\bar{b})=0, \tag{2.10}$$

where $\bar{q}=\nabla^2\bar{\Psi}+\frac{\partial}{\partial\zeta}\left(\frac{f_0^2}{c^2}\zeta^2\frac{\partial\bar{\Psi}}{\partial\zeta}\right)$, $\bar{b}=\left(\frac{\partial\bar{\Psi}}{\partial\zeta}\right)_s+\kappa\alpha_s\bar{\Psi}_s(\lambda,\theta)$.

We further assume that there exist continuously differentiable functions $P(\cdot)$ and $Q(\cdot)$ such that

$$\overline{\psi}(0,\lambda,\zeta) = Q(\overline{q}(0,\lambda,\zeta)), \tag{2.11}$$

$$\overline{\psi}_s(0,\lambda) = P(\overline{b}(0,\lambda)). \tag{2.12}$$

Corresponding to the hypothesis of Arnol'd's second theorem, we assume that there exist positive constants $C_i\,(i = 1,\cdots,4)$, such that

$$0 < C_1 \leqslant -\frac{dQ}{d\overline{q}} \leqslant C_2 < \infty, \tag{2.13}$$

$$0 < C_3 \leqslant \frac{dP}{d\overline{b}} \leqslant C_4 < \infty, \tag{2.14}$$

where $\frac{dQ}{d\overline{q}} = \frac{\nabla\overline{\psi}}{\nabla\overline{q}}, \frac{dP}{d\overline{b}} = \frac{\nabla\overline{\psi}_s}{\nabla\overline{b}}$.

## III NONLINEAR STABILITY CRITERIA IN THE CASE OF $\kappa = 0$

From (2.17), (2.17)' and (2.18) of Mu and Zeng (1991), omitting the perturbation of parameters, we have

$$H(t) = H_3(t) + H_4(t), \tag{3.1}$$

where

$$H(t) = \frac{1}{2}\int_M \left[ |\vec{v}|^2 - |\overline{\vec{v}}|^2 + \left|\frac{f_0}{c}\zeta\frac{\partial\psi}{\partial\zeta}\right|^2 - \left|\frac{f_0}{c}\zeta\frac{\partial\overline{\psi}}{\partial\zeta}\right|^2 \right] dM$$

$$+ \int_M [F(q) - F(\overline{q})] dM + \frac{f_0^2}{c_s^2}\int_{s^2} [G(\overline{b}) - G(b)] ds, \tag{3.2}$$

$$H_3(t) = \frac{1}{2}\int_M \left[ |\nabla(\psi - \overline{\psi})|^2 + \left|\frac{f_0}{c}\zeta\frac{\partial}{\partial\zeta}(\psi - \overline{\psi})\right|^2 \right] dM, \tag{3.3}$$

$$H_4(t) = \int_M [F(q) - F(\overline{q}) - Q(\overline{q})(q - \overline{q})] dM$$

$$- \int_{s^2} \frac{f_0^2}{c_s^2} [G(b) - G(\overline{b}) - P(\overline{b})(b - \overline{b})] ds, \tag{3.4}$$

and

$$F(\xi) = \int^{\xi} Q(\tau)d\tau, \quad G(\xi) = \int^{\xi} P(\tau)d\tau.$$

It follows from (2.5)–(2.7) that

$$\frac{d}{dt}H(t) = 0. \tag{3.5}$$

Let $F(q) = q$, $G(b) = b$ in (2.6) and (2.7), we have

$$\frac{d}{dt}\int_{s^2} \frac{\partial}{\partial\zeta}\left( \zeta^2\frac{f_0^2}{c^2}\frac{\partial\psi}{\partial\zeta} \right) ds = 0, \tag{3.6}$$

$$\frac{d}{dt}\int_{s^2} \left.\frac{\partial\psi}{\partial\zeta}\right|_{\zeta=1} ds = 0. \tag{3.7}$$

Denote $\varphi = \frac{1}{4\pi a^2}\int_{s^2} (\psi_0 - \psi) ds$, and define $\tilde{\psi} \equiv \psi + \varphi$, we have

$$\int_{s^2} \tilde{\psi} ds \equiv \int_{s^2} \psi_0 ds. \tag{3.8}$$

It follows from (3.6) and (3.7) that

$$\frac{\partial}{\partial \zeta}\left(\zeta^2 \frac{f_0^2}{c^2} \frac{\partial \varphi}{\partial \zeta}\right) \equiv 0, \tag{3.9}$$

$$\frac{\partial}{\partial \zeta} \varphi|_{\zeta=1} \equiv 0. \tag{3.10}$$

It is easy to verify that

$$\tilde{q} \equiv \nabla^2 \tilde{\psi} + \frac{\partial}{\partial \zeta}\left(\frac{f_0^2}{c^2} \zeta \frac{\partial \tilde{\psi}}{\partial \zeta}\right) + 2\omega\cos\theta = q, \tag{3.11}$$

$$\tilde{b} \equiv \left(\frac{\partial \tilde{\psi}}{\partial \zeta}\right)_s = b. \tag{3.12}$$

Integrating by parts and using (3.9) and (3.10), we have

$$\int_M \frac{f_0^2}{c^2} \zeta^2 \left|\frac{\partial \tilde{\psi}}{\partial \zeta}\right|^2 d\zeta \equiv \int_M \frac{f_0^2}{c^2} \zeta^2 \left|\frac{\partial \psi}{\partial \zeta}\right|^2 d\zeta. \tag{3.13}$$

From (3.11) and (3.13), we can prove that

$$H_3(t) = \tilde{H}_3(t), \qquad H_4(t) = \tilde{H}_4(t), \tag{3.14}$$

where $\tilde{H}_3(t)$ and $\tilde{H}_4(t)$ are obtained by replacing $\psi$, $q$ with $\tilde{\psi}$, $\tilde{q}$ in $H_3(t)$ and $H_4(t)$, respectively.

Also note that

$$\tilde{q} - \bar{q} \equiv \nabla^2(\tilde{\psi} - \bar{\psi}) - \frac{\partial}{\partial \zeta}\left(\frac{f_0^2}{c^2} \zeta^2 \frac{\partial}{\partial \zeta}(\tilde{\psi} - \bar{\psi})\right). \tag{3.15}$$

Multiplying (3.15) by $(\tilde{\psi} - \bar{\psi})$ and integrating by parts, we obtain

$$\begin{aligned} \tilde{H}_3(t) = & \frac{1}{2}\int_{s^2} \frac{f_0^2}{c_s^2}(\tilde{\psi}_s - \bar{\psi}_s)(\tilde{b} - \bar{b}_s) ds \\ & - \frac{1}{2}\int_M (\tilde{q} - \bar{q})(\tilde{\psi} - \bar{\psi}) dM. \end{aligned} \tag{3.16}$$

To estimate $\tilde{H}_3(t)$, we establish

**Proposition 1** Suppose $\tilde{\psi}$, $\bar{\psi}$, $\psi$ and $\psi_0$ are defined as above, then

$$\begin{aligned} \int_M |\tilde{\psi} - \bar{\psi}|^2 dM \leqslant & \frac{a^2}{2}\int_M |\nabla(\tilde{\psi} - \bar{\psi})|^2 dM \\ & + \frac{1}{4\pi a^2}\int_0^1 d\zeta \left|\int_{s^2} (\psi_0 - \bar{\psi}) ds\right|^2. \end{aligned} \tag{3.17}$$

**Proof:** Denote $e_i$ $(i = 0, 1, \cdots)$ the normal orthogonal eigenfunctions of Laplacian on the spherical surface with radius $a$ and $\lambda_i = i(i+1)/a^2$ are the corresponding eigenvalues. For any square integrable function $u$, we have

$$u=\sum_{i=0}^{i}u_i e_i$$

with $u_i=\int_{s^2}ue_i ds$, and

$$\nabla^2 u=-\sum_{i=0}^{i}\lambda_i e_i u_i,$$

thus

$$\int_{s^2}|\nabla u|^2 ds=-\int_{s^2}\nabla^2 u\cdot uds=\sum_{i=0}^{i}\lambda_i u_i^2.$$

On the other hand,

$$\int_{s^2}|\nabla u|^2 ds=|u_0|^2+\sum_{i=1}^{\infty}|u_i|^2,$$

$$\lambda_1\int_{s^2}|u|^2 ds-\lambda_1|u_0|^2=\lambda_1\sum_{i=1}^{x}|u_i|^2\leqslant\int_{s^2}|\nabla u|^2 ds.$$

thus

$$\int_{s^2}|u|^2 ds=|u_0|^2+\frac{1}{\lambda_1}\int_{s^2}|\nabla u|^2 ds.$$

Since $e_0=\frac{1}{2a\sqrt{\pi}}$, we have

$$u_0=\int_{s^2}\frac{u}{2a\sqrt{\pi}}ds.$$

Let $u=\tilde{\psi}-\bar{\psi}$ in above inequality, we can derive inequality (3.17) by noting

$$\int_{s^2}(\tilde{\psi}-\bar{\psi})ds\equiv\int_{s^2}(\psi_0-\bar{\psi})ds.$$

We continue to estimate $\tilde{H}_3(t)$. By (3.16) and (3.17), we obtain

$$\begin{aligned}\tilde{H}_3(t)\leqslant&\frac{1}{4}\frac{f_0^2\varepsilon_1}{c_s^2}\|\tilde{b}-\bar{b}\|_{s^2}^2+\frac{f_0^2}{4\varepsilon_1 c_s^2}\|\tilde{\psi}_s-\bar{\psi}_s\|_{s^2}^2\\&+\frac{\varepsilon_2}{4}\|\tilde{q}-\bar{q}\|_M^2+\frac{1}{4\varepsilon_2}\|\tilde{\psi}-\bar{\psi}\|_M^2\\\leqslant&\frac{1}{4}\frac{f_0^2\varepsilon_1}{c_s^2}\|\tilde{b}-\bar{b}\|_{s^2}^2+\frac{f_0^2}{4\varepsilon_1 c_s^2}\|\tilde{\psi}-\bar{\psi}\|_{s^2}^2\\&+\frac{\varepsilon_2}{4}\|\tilde{q}-\bar{q}\|_M^2+\frac{a^2}{4\varepsilon_2}\frac{1}{2}\|\nabla(\tilde{\psi}-\bar{\psi})\|_M^2\\&+\frac{1}{16\varepsilon_2\pi a^2}\int_0^1 d\zeta\left|\int_{s^2}(\psi_0-\bar{\psi})ds\right|^2,\end{aligned}\tag{3.18}$$

where $\varepsilon_1$ and $\varepsilon_2$ are positive constants to be determined later.

By the definition of $\tilde{H}_3(t)$ and (3.18), we have

$$\left(1-\frac{a^2}{4\varepsilon_2}\right)\frac{1}{2}\|\nabla(\tilde{\psi}-\bar{\psi})\|_M^2+\frac{1}{2}\left\|\frac{f_0}{c}\zeta\frac{\partial}{\partial\zeta}(\tilde{\psi}-\bar{\psi})\right\|_M^2$$

$$\leqslant \frac{1}{4}\frac{f_0^2\varepsilon_1}{c_s^2}\|\tilde{b}-\bar{b}\|_{s^2}^2+\frac{\varepsilon_2}{4}\|\tilde{q}-\bar{q}\|_M^2+\frac{f_0^2}{4\varepsilon_1 c_s^2}\|\tilde{\psi}_s-\bar{\psi}_s\|_{s^2}^2$$
$$+\frac{1}{4\varepsilon_2}\frac{1}{4\pi a^2}\int_0^1 d\zeta\left|\int_{s^2}(\psi_0-\bar{\psi})ds\right|^2, \tag{3.19}$$

On the other hand,

$$\int_{s^2}|\tilde{\psi}-\bar{\psi}|^2 ds=\int_M\frac{\partial}{\partial\zeta}[\zeta(\tilde{\psi}-\bar{\psi})]^2 dM$$
$$=\int_M(\tilde{\psi}-\bar{\psi})^2 dM+\int_M 2\zeta(\tilde{\psi}-\bar{\psi})\frac{\partial}{\partial\zeta}(\tilde{\psi}-\bar{\psi})dM$$
$$\leqslant\int_M(\tilde{\psi}-\bar{\psi})^2 dM+\int_M\frac{\zeta(\tilde{\psi}-\bar{\psi})^2}{\varepsilon_3}dM+\varepsilon_3\int_M\left|\zeta\frac{\partial}{\partial\zeta}(\tilde{\psi}-\bar{\psi})\right|^2 dM$$
$$\leqslant\left(1+\frac{1}{\varepsilon_3}\right)\frac{a^2}{2}\|\nabla(\tilde{\psi}-\bar{\psi})\|_M^2+\left(1+\frac{1}{\varepsilon_3}\right)\frac{1}{4\pi a^2}\int_0^1 d\zeta\left|\int_{s^2}(\psi_0-\bar{\psi})ds\right|^2$$
$$+\varepsilon_3\max\left(\frac{c^2}{f_0^2}\right)\int_M\frac{f_0^2}{c^2}\left|\zeta\frac{\partial}{\partial\zeta}(\tilde{\psi}-\bar{\psi})\right|^2 dM. \tag{3.20}$$

In deriving (3.20) inequality (3.17) has been employed. From inequalities (3.19) and (3.20), we have

$$\left[1-\frac{a^2}{4\varepsilon_2}-a^2\left(1+\frac{1}{\varepsilon_3}\right)\frac{f_0^2}{4\varepsilon_1 c_s^2}\right]\frac{1}{2}\|\nabla(\tilde{\psi}-\bar{\psi})\|_M^2$$
$$+\left[1-2\varepsilon_3\left(\max\frac{c^2}{f_0^2}\right)\frac{f_0^2}{4\varepsilon_1 c_s^2}\right]\frac{1}{2}\left\|\frac{f_0}{c}\zeta\frac{\partial}{\partial\zeta}(\tilde{\psi}-\bar{\psi})\right\|_M^2$$
$$\leqslant\frac{1}{4}\frac{f_0^2\varepsilon_1}{c_s^2}\|\tilde{b}-\bar{b}\|_{s^2}^2+\frac{\varepsilon_2}{4}\|\tilde{q}-\bar{q}\|_M^2+\frac{1}{16\pi\varepsilon_2 a^2}\int_0^1 d\zeta\left|\int_{s^2}(\psi_0-\bar{\psi})ds\right|^2$$
$$+\frac{f_0^2}{4\varepsilon_1 c_s^2}\left(1+\frac{1}{\varepsilon_3}\right)\frac{1}{4\pi a^2}\int_0^1 d\zeta\left|\int_{s^2}(\psi_0-\bar{\psi})ds\right|^2. \tag{3.21}$$

Denote

$$M=\min\left\{1-\frac{a^2}{4\varepsilon_2}-a^2\left(1+\frac{1}{\varepsilon_3}\right)\frac{f_0^2}{4\varepsilon_1 c_s^2},\ 1-2\varepsilon_3\left(\max\frac{c^2}{f_0^2}\right)\frac{f_0^2}{4\varepsilon_1 c_s^2}\right\},$$

and let $M>0$, we obtain,

$$\frac{1}{2}\|\nabla(\tilde{\psi}-\bar{\psi})\|_M^2+\frac{1}{2}\left\|\frac{f_0}{c}\zeta\frac{\partial}{\partial\zeta}(\tilde{\psi}-\bar{\psi})\right\|_M^2$$
$$\leqslant\frac{1}{4M}\frac{f_0^2\varepsilon_1}{c_s^2}\|\tilde{b}-\bar{b}\|_{s^2}^2+\frac{\varepsilon_2}{4M}\|\tilde{q}-\bar{q}\|_M^2$$
$$+\left[\frac{f_0^2}{4M\varepsilon_1 c_s^2}\left(1+\frac{1}{\varepsilon_3}\right)+\frac{1}{4M\varepsilon_2}\right]\frac{1}{4\pi a^2}\int_0^1 d\zeta\left|\int_{s^2}(\psi_0-\bar{\psi})ds\right|^2. \tag{3.22}$$

On the other hand, the inequality (2.27) in Mu and Zeng (1991) can be written as

$$\frac{C_1}{2}\|\tilde{q}-\bar{q}\|_M^2+\frac{C_3}{2}\frac{f_0^2}{c_s^2}\|\tilde{b}-\bar{b}\|_{s^2}^2$$

$$\leqslant\frac{1}{2}\|\nabla(\tilde{\psi}-\bar{\psi})\|_M^2+\frac{1}{2}\|\frac{f_0}{c_s}\zeta\frac{\partial}{\partial\zeta}(\tilde{\psi}-\bar{\psi})\|_M^2-H_3(0)-H_4(0). \tag{3.23}$$

Combining (3.22) with (3.23), and using (3.11), (3.12) and (3.13) yields

$$\left(\frac{C_3}{2}\frac{f_0^2}{c_s^2}-\frac{1}{4M}\frac{f_0^2\varepsilon_1}{c_s^2}\right)\|\tilde{b}-\bar{b}\|_{s^2}^2+\left(\frac{C_1}{2}-\frac{\varepsilon_2}{4M}\right)\|\tilde{q}-\bar{q}\|_M^2$$

$$\leqslant\left[\frac{f_0^2}{4M\varepsilon_1 c_s^2}\left(1+\frac{1}{\varepsilon_3}\right)+\frac{1}{4M\varepsilon_2}\right]\frac{1}{4\pi a^2}\int_0^1 d\zeta\left|\int_{s^2}(\psi_0-\bar{\psi})ds\right|^2$$

$$-H_3(0)-H_4(0). \tag{3.24}$$

Obviously if there exist $\varepsilon_1>0$, $\varepsilon_2>0$, $\varepsilon_3>0$ such that

$$\begin{cases} C_1-\dfrac{\varepsilon_2}{2M}>0 \\ C_3-\dfrac{\varepsilon_1}{2M}>0 \\ M>0, \end{cases} \tag{3.25a,b,c}$$

then, by (3.22) and (3.24), the disturbance energy, disturbance potential enstrophy and disturbance boundary energy could be upper bounded in terms of the initial disturbance field. In Appendix A, we have proved

**Lemma 1.** There exists a positive solution ($\varepsilon_1$, $\varepsilon_2$, $\varepsilon_3$) to the system of inequalities (3.25) if and only if

$$C_1\left(\frac{2}{a^2}-\frac{1}{C_3}\frac{f_0^2}{c_s^2}-\frac{1}{C_3^2}\frac{f_0^4}{c_s^4}\max\frac{c^2}{f_0^2}\right)>1. \tag{3.26}$$

By Lemma 1, we have

**Theorem 3.1** Suppose the basic flow satisfies (2.11)–(2.14) and (3.26), then it is nonlinearly stable; i. e. for any $\varepsilon>0$, there exists $\delta>0$ such that

$$\|\nabla^2(\psi-\bar{\psi})\|_M^2<\varepsilon,\qquad \|\frac{f_0}{c}\frac{\partial}{\partial\zeta}(\psi-\bar{\psi})\|^2<\varepsilon$$

$$\|q-\bar{q}\|_M^2<\varepsilon,\qquad \|b-\bar{b}\|_{s^2}^2<\varepsilon$$

provided that

$$\|\nabla^2(\psi_0-\bar{\psi})\|_M^2<\delta,\qquad \|\frac{f_0}{c}\zeta\frac{\partial}{\partial\zeta}(\psi_0-\bar{\psi})\|_M^2<\delta,$$

$$\|\psi_0-\bar{\psi}\|^2<\delta.$$

## IV. NONLINEAR STABILITY CRITERIA IN THE CASE OF $\kappa=1$

In this case, (2.11) of Mu and Zeng (1991) holds, hence there is no need to introduce $\tilde{\psi}$ and $\tilde{H}_3(t)$. From (2.17) in Mu and Zeng (1991), we have

$$H_3(t)=\frac{1}{2}\int_M [|\nabla(\psi-\bar{\psi}|^2+\left|\frac{f_0}{c}\zeta\frac{\partial}{\partial\zeta}(\psi-\bar{\psi})\right|^2]dM+\frac{1}{2}\int_{s^2}\frac{f_0^2}{c_s^2}\alpha_s|\psi_s-\bar{\psi}_s|^2 \tag{4.1}$$

and

$$H_3(t)=\frac{1}{2}\int_{s^2}\frac{f_0^2}{c_s^2}(\psi_s-\bar{\psi}_s)(b-\bar{b})ds-\frac{1}{2}\int_M(q-\bar{q})(\psi-\bar{\psi})dM, \tag{3.16$'$}$$

$$\int_M|\psi-\bar{\psi}|^2dM\leqslant\frac{a^2}{2}\int_M|\nabla(\psi-\bar{\psi})|^2dM$$

$$+\frac{1}{4\pi a^2}\int_0^1 d\zeta\left|\int_{s^2}(\psi_0-\bar{\psi})ds\right|^2, \tag{3.17$'$}$$

$$\int_{s^2}|\psi_s-\bar{\psi}_s|^2ds\leqslant\left(1+\frac{1}{\varepsilon_3}\right)\frac{a^2}{2}\|\nabla(\psi-\bar{\psi})\|_M^2+\left(1+\frac{1}{\varepsilon_3}\right)\frac{1}{4\pi a^2}$$

$$\times\int_0^1 d\zeta\left|\int_{s^2}(\psi_0-\bar{\psi})ds\right|+\varepsilon_3\max\frac{c^2}{f_0^2}\int_M\frac{f_0^2}{c^2}\left|\zeta\frac{\partial}{\partial\zeta}(\psi-\bar{\psi})\right|^2dM. \tag{3.20$'$}$$

It follows from (3.16)′, (3.17)′ and (3.20)′, that

$$H_3(t)\leqslant\frac{1}{4}\frac{f_0^2\varepsilon_1}{c_s^2}\|b-\bar{b}\|_{s^2}^2+\left(\frac{f_0^2\chi_1}{4\varepsilon_1 c_s^2}+\frac{f_0^2\chi_2}{4\varepsilon_1 c_s^2}\right)\|\psi_s-\bar{\psi}_s\|_{s^2}^2$$

$$+\frac{\varepsilon_2}{4}\|q-\bar{q}\|_M^2+\frac{1}{4\varepsilon_2}\frac{a^2}{2}\|\nabla(\psi-\bar{\psi})\|^2$$

$$+\frac{1}{4\varepsilon_2}\frac{1}{4\pi a^2}\int_0^1 d\zeta\left|\int_{s^2}(\psi_0-\bar{\psi})ds\right|^2, \tag{4.2}$$

where $\chi_1>0$ and $\chi_2>0$ are constants to be defined later, and

$$\chi_1+\chi_2=1. \tag{4.3}$$

Combining (4.2) with (3.20)′ yields

$$H_3(t)\leqslant\frac{1}{4}\frac{f_0^2\varepsilon_1}{c_s^2}\|b-\bar{b}\|_{s^2}^2+\frac{\varepsilon_2}{4}\|q-\bar{q}\|_M^2+\frac{\chi_2 f_0^2}{4\varepsilon_1 c_s^2}\|\psi_s-\bar{\psi}_s\|^2$$

$$+\frac{f_0^2\chi_1}{4\varepsilon_1 c_s^2}a^2\left(1+\frac{1}{\varepsilon_3}\right)\frac{1}{2}\|\nabla(\psi-\bar{\psi})\|_M^2+\frac{f_0^2\chi_1}{2\varepsilon_1 c_s^2}\varepsilon_3\left(\max\frac{c^2}{f_0^2}\right)\frac{1}{2}\left\|\frac{f_0}{c}\zeta\frac{\partial}{\partial\zeta}(\psi-\bar{\psi})\right\|_M^2$$

$$+\frac{1}{4\varepsilon_2}\frac{a^2}{2}\|\nabla(\psi-\bar{\psi})\|^2+\frac{1}{4\varepsilon_1}\frac{1}{4\pi a^2}\int_0^1 d\zeta\left|\int_{s^2}(\psi_0-\bar{\psi})ds\right|^2$$

$$+\frac{f_0^2\chi_1}{4\varepsilon_1 c_s^2}\left(1+\frac{1}{\varepsilon_3}\right)\frac{1}{4\pi a^2}\int_0^1 d\zeta\left|\int_{s^2}(\psi_0-\bar{\psi})ds\right|^2, \tag{4.4}$$

thus

$$\left[1-\frac{a^2}{4\varepsilon_1}-\frac{f_0^2\chi_1}{4\varepsilon_1 c_s^2}a^2\left(1+\frac{1}{\varepsilon_3}\right)\right]\frac{1}{2}\|\nabla(\psi-\bar{\psi})\|_M^2$$

$$+\left[1-\frac{f_0^2\chi_1}{2\varepsilon_1 c_s^2}\varepsilon_3\left(\max\frac{c^2}{f_0^2}\right)\right]\frac{1}{2}\left\|\frac{f_0}{c}\zeta\frac{\partial}{\partial\zeta}(\psi-\bar{\psi})\right\|_M^2$$

$$+\left(1-\frac{\chi_2}{2\varepsilon_1\alpha_s}\right)\frac{1}{2}\frac{f_0^2}{c_s^2}\alpha_s\|\psi_s-\overline{\psi}_s\|^2$$

$$\leqslant\frac{1}{4}\frac{f_0^2\varepsilon_1}{c_s^2}\|b-\overline{b}\|_{s^2}^2+\frac{\varepsilon_2}{4}\|q-\overline{q}\|_M^2$$

$$+\left[\frac{f_0^2\chi_1}{4\varepsilon_1 c_s^2}\left(1+\frac{1}{\varepsilon_3}\right)+\frac{1}{4\varepsilon_2}\right]\frac{1}{4\pi a^2}\int_0^1 d\zeta\left|\int_{s^2}(\psi_0-\overline{\psi})ds\right|^2. \tag{4.5}$$

Denote

$$M_1=\min\{1-\frac{a^2}{4\varepsilon_2}-a^2\left(1+\frac{1}{\varepsilon_3}\right)\frac{f_0^2\chi_1}{4\varepsilon_1 c_s^2},$$

$$1-2\varepsilon_3\max\left(\frac{c^2}{f_0^2}\right)\frac{f_0^2 x_1}{4\varepsilon_1 c_s^2},1-\frac{\chi_2}{2\varepsilon_1\alpha_s}\}, \tag{4.6}$$

let $M_1>0$ then

$$H_3(t)\leqslant\frac{1}{4M_1}\frac{f_0^2\varepsilon_1}{c_s^2}\|b-\overline{b}\|_{s^2}^2+\frac{\varepsilon_2}{4M_1}\|q-\overline{q}\|_M^2$$

$$+\left[\frac{f_0^2\chi_1}{4M_1\varepsilon_1 c_s^2}\left(1+\frac{1}{\varepsilon_3}\right)+\frac{1}{4M_1\varepsilon_2}\right]\frac{1}{4\pi a^2}\int_0^1 d\zeta\left|\int_{s^2}(\psi_0-\overline{\psi})ds\right|^2. \tag{4.7}$$

On the other hand, from (2.27) of Mu and Zeng (1991) we have

$$\frac{C_1}{2}\|q-\overline{q}\|_M^2+\frac{C_3}{2}\frac{f_0^2}{c_s^2}\|b-\overline{b}\|_{s^2}^2\leqslant H_3(t)-H_3(0)-H_4(0), \tag{4.8}$$

thus

$$\left(\frac{C_3}{2}\frac{f_0^2}{c_s^2}-\frac{1}{4M_1}\frac{f_0^2\varepsilon_1}{c_s^2}\right)\|b-\overline{b}\|_{s^2}^2+\left(\frac{C_1}{2}-\frac{\varepsilon_2}{4M_1}\right)\|q-\overline{q}\|_M^2$$

$$\leqslant\left[\frac{f_0^2\chi_1}{4M_1\varepsilon_1 c_s^2}\left(1+\frac{1}{\varepsilon_3}\right)+\frac{1}{4M_1\varepsilon_2}\right]\frac{1}{4\pi a^2}\int_0^1 d\zeta\left|\int_{s^2}(\psi_0-\overline{\psi})ds\right|^2$$

$$-H_3(0)-H_4(0). \tag{4.9}$$

Obviously if there exist $\varepsilon_1>0$, $\varepsilon_2>0$, $\varepsilon_3>0$, $\chi_1>0$ such that

$$\begin{cases} C_1-\dfrac{\varepsilon_2}{2M_1}>0 \\ C_3-\dfrac{\varepsilon_1}{2M_1}>0 \\ M_1>0, \end{cases} \tag{4.10}$$

then, by (4.7) and (4.9), the disturbance energy, distrubance potential enstrophy and disturbance boundary energy could be upper bounded in terms of the initial disturbance field. In Appendix B, we have proved

**Lemma 2.** There exists a positive solution ($\chi_1$, $\varepsilon_1$, $\varepsilon_2$, $\varepsilon_3$) to the system of (4.10) if and only if

$$\begin{cases} C_1 \dfrac{2}{a^2} > 1 \\ C_3 \alpha_s > 1 - \dfrac{C_3\left(\sqrt{1+4\max\left(\dfrac{c^2}{f_0^2}\right)\left(\dfrac{2}{a^2}-\dfrac{1}{C_1}\right)}-1\right)}{2\max\left(\dfrac{c^2}{f_0^2}\right)\dfrac{f_0^2}{c_s^2}} \end{cases} \quad (4.11)$$

Similar to the above section, by (4.7), (4.9) and Lemma 2, we have

**Theorem 4.1** Suppose the basic flow satisfies (2.11)–(2.14) and (4.11), then it is nonlinearly stable, i.e. for any $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\|\nabla(\psi - \bar{\psi})\|_M^2 < \varepsilon, \quad \|\frac{f_0}{c}\zeta\frac{\partial}{\partial\zeta}(\psi - \bar{\psi})\|_M^2 < \varepsilon, \quad \|q - \bar{q}\|_M^2 < \varepsilon,$$

$$\|b - \bar{b}\|_S^2 < \varepsilon, \quad \|\frac{f_0}{c}\sqrt{\alpha_s}(\psi_s - \bar{\psi}_s)\|_S^2 < \varepsilon, \ \forall t \geq 0,$$

provided that

$$\|\nabla^2(\psi_0 - \bar{\psi})\|_M^2 < \delta,$$

$$\|\frac{f_0}{c}\frac{\partial^2}{\partial\zeta^2}(\psi_0 - \bar{\psi})\| < \delta,$$

$$\|(\psi_0 - \bar{\psi})\|^2 < \delta.$$

Obviously, Theorem 4.2 is better than Theorem 2 of Mu and Zeng (1991), where for the steady state to be nonlinearly stable, we require

$$C_1 \cdot \frac{2}{a^2} > 1, \qquad C_3 \cdot \alpha_s > 1.$$

Now we would like to show that this improvement is of importance by the following argument.

Consider the case $c^2 = \text{constant} = c_s^2$. The typical value for $f_0$ is $10^{-4}$ $s^{-1}$, $c_s$ is $10^2$ $ms^{-1}$ and $\alpha_s$ is 0.1, $\lambda = \frac{2}{a^2} \approx 0.5 \times 10^{-13}$ (c.f. Zeng, 1983). Then (4.11) becomes

$$C_3(\alpha_s + 0.024) > 1.$$

Since $\frac{0.024}{\alpha_s} = 0.24$, omitting it certainly would influence the criterion significantly.

Finally we point out that for the sake of simplicity, we consider only the perturbation of initial data in this paper, but the results of Section 3 and Section 4 can also be applied to perturbations of initial data and the parameters in the model (c.f. Mu and Zeng (1991)).

**Appendix A**. The proof of Lemma 1.

Denote $F_1 = \frac{f_0^2}{c_s^2}\left(1 + \frac{1}{\varepsilon_3}\right)$, $F_2 = \frac{f_0^2}{c_s^2} max\left(\frac{c^2}{f_0^2}\right)\varepsilon_3$ and $\lambda = \frac{2}{a^2}$, then

$$M = \min\left\{1 - \frac{1}{2\lambda\varepsilon_2} - \frac{F_1}{2\lambda\varepsilon_1}, 1 - \frac{F_2}{2\varepsilon_1}\right\}.$$

**Proposition A1.** Suppose $C_1$, $C_3$, $\lambda$, $F_1$ and $F_2$ are given positive constants, then there exists $(\varepsilon_1, \varepsilon_2)$ satisfying

$$\begin{cases} 2MC_1 > \varepsilon_2 > 0 \\ 2MC_3 > \varepsilon_1 > 0, \end{cases} \quad (A1)$$

if and only if

$$\begin{cases} \lambda > \dfrac{1}{C_1} + \dfrac{F_1}{C_3} \\ C_3 > F_2. \end{cases} \tag{A2}$$

**Proof:** First we define

$$\delta_1 = \frac{1}{2\lambda\varepsilon_2} + \frac{F_1}{2\lambda\varepsilon_1}, \qquad \delta_2 = \frac{F_2}{2\varepsilon_1}.$$

(I) Necessity. (i). When $\delta_1 \geqslant \delta_2$, then $1-\delta_1 \leqslant 1-\delta_2$, thus

$$M = 1-\delta_1$$

and

$$\varepsilon_1 = \frac{F_2}{2\delta_1}, \qquad \varepsilon_2 = \frac{F_2}{2(\lambda F_1\delta_1 - F_1\delta_2)}.$$

Meanwhile (A1) becomes

$$2C_1(1-\delta_1) > \frac{F_2}{2(\lambda F_1\delta_1 - F_1\delta_2)} > 0, \tag{A3}$$

$$2C_3(1-\delta_1) > \frac{F_2}{2\delta_1} > 0. \tag{A4}$$

$$0 < \delta_2 \leqslant \delta_1 \quad . \tag{A5}$$

From (A3) and (A4), we have

$$\delta_2 < \frac{\lambda F_2\delta_1}{F_1} - \frac{F_2}{4C_1(1-\delta_1)F_1}, \tag{A6}$$

$$\delta_2 > \frac{F_2}{4C_3(1-\delta_1)}. \tag{A7}$$

Hence

$$\frac{F_2}{4C_3(1-\delta_1)} < \frac{\lambda F_2\delta_1}{F_1} - \frac{F_2}{4C_1(1-\delta)F_1}. \tag{A8}$$

From (A5) and (A7), we have

$$\frac{F_2}{4C_3(1-\delta_1)} < \delta_1. \tag{A9}$$

Since $1-\delta_1 > 0$, (A8) and (A9) become

$$\delta_1^2 - \delta_1 + \frac{1}{4\lambda}\left(\frac{F_1}{C_3} + \frac{1}{C_1}\right) < 0. \tag{A10}$$

$$\delta_1^2 - \delta_1 + \frac{F_2}{4C_3} < 0. \tag{A11}$$

Obviously, $\delta_1$ satisfies (A10) and (A11) if and only if their discriminants are positive, hence we obtain the inqualities (A2).

(ii). when $\delta_1 < \delta_2$, then $1-\delta_1 > 1-\delta_2$, and

$$M = 1-\delta_2$$

From (A1), we have

$$2C_1(1-\delta_2) > \frac{F_2}{2(\lambda F_1\delta_1 - F_1\delta_2)} > 0 \tag{A12}$$

$$2C_3(1-\delta_2) > \frac{F_2}{2\delta_1} > 0 \tag{A13}$$

$$0 < \delta_1 < \delta_2 \tag{A14}$$

By (A12) and (A14), we obtain

$$\delta_2^2-\delta_2+\frac{F_2}{4C_1(\lambda F_2-F_1)}>0. \tag{A15}$$

And (A13) can be written as

$$\delta_2^2-\delta_2+\frac{F_2}{4C_3}<0. \tag{A16}$$

Since $\delta_2$ is a solution to (A15) and (A16), we have

$$\lambda>\frac{F_1}{F_2}+\frac{1}{C_1},\qquad C_3>F_2, \tag{A17}$$

which also implies (A2).

(II). Sufficiency. If (A2) holds, there exists $0<\delta_1<1$ such that (A10) holds. Hence (A8) (A9) are true. By (A8) and (A9), we can choose $0<\delta_2<\delta_1$ such that (A6) and (A7) hold. By (A6) we have

$$\lambda F_2\delta_1-F_1\delta_2>0. \tag{A18}$$

By $1-\delta_1>0$, (A6) and (A18), we have (A3). By $1-\delta_1>0$ and (A7), (A4) holds. Now let

$$\varepsilon_1=\frac{F_2}{2\delta_1},\qquad \varepsilon_2=\frac{F_2}{2(\lambda F_1\delta_1-F_1\delta_2)}, \tag{A19}$$

we can obtain (A1).

**Proposition A2**. There exists $\varepsilon_3>0$ such that (A2) holds if and only if (3.26) holds.
**Proof:** (I). Necessity. If (A2) holds with $\varepsilon_3>0$, then

$$\begin{cases}\lambda>\dfrac{1}{C_1}+\dfrac{1}{C_3}\dfrac{f_0^2}{c_s^2}\left(1+\dfrac{1}{\varepsilon_3}\right)\\ C_3>\dfrac{f_0^2}{c_s^2}\max\left(\dfrac{c^2}{f_0^2}\right)\varepsilon_3\end{cases}. \tag{A20}$$

Hence, we have

$$\begin{cases}\varepsilon_3>\dfrac{\dfrac{1}{C_3}\dfrac{f_0^2}{c_s^2}}{\left(\lambda-\dfrac{1}{C_1}-\dfrac{1}{C_3}\dfrac{f_0^2}{c_s^2}\right)}\\ \varepsilon_3<\dfrac{C_3}{\left[\dfrac{f_0^2}{c_s^2}\max\left(\dfrac{c^2}{f_0^2}\right)\right]}\end{cases}. \tag{A21}$$

Thus,

$$\frac{\dfrac{1}{C_3}\dfrac{f_0^2}{c_s^2}}{\left(\lambda-\dfrac{1}{C_1}-\dfrac{1}{C_3}\dfrac{f_0^2}{c_s^2}\right)}<\frac{C_3}{\left[\dfrac{f_0^2}{c_s^2}\max\left(\dfrac{c^2}{f_0^2}\right)\right]}. \tag{A22}$$

Then (3.26) holds.

(II). Sufficiency. If (3.26) holds, we have

$$\lambda-\frac{1}{C_1}-\frac{1}{C_3}\frac{f_0^2}{c_s^2}>\frac{1}{C_3^2}\frac{f_0^4}{c_S^4}\max\left(\frac{c^2}{f_0^2}\right),$$

thus (A22) holds, we can choose $\varepsilon_3>0$ such that (A21) holds, and (A2) is true.

By Proposition A1 and Proposition A2, we can easily prove Lemma 1.

**Appendix B** The proof of Lemma 2.

Let $\tilde{F}_1 = \frac{f_0^2}{c_s^2}\chi_1\left(1+\frac{1}{\varepsilon_3}\right)$, $\tilde{F}_2 = \frac{f_0^2}{c_s^2}\chi_1 \max\left(\frac{c^2}{f_0^2}\right)\varepsilon_3$ and $\tilde{F}_3 = \frac{(1-\chi_l)}{\alpha_3}$, then

$$M_1 = \min\left\{1-\frac{1}{2\lambda\varepsilon_2}-\frac{\tilde{F}}{2\lambda\varepsilon_1}, 1-\frac{\tilde{F}_2}{2\varepsilon_1}, 1-\frac{\tilde{F}_3}{2\varepsilon_1}\right\} .$$

**Proposition B1.** Suppose $C_1>0, C_3>0$ and $\tilde{F}_i>0$ $(i=1,\cdots,3)$ are given positive constants, then there exists $(\varepsilon_1, \varepsilon_2)$ satisfying (4.10) if and only if

$$\begin{cases} \lambda > \frac{1}{C_1}+\frac{\tilde{F}_1}{C_3} \\ C_3 > \tilde{F}_2 \\ C_3 > \tilde{F}_3 \ . \end{cases} \tag{B1a,b,c}$$

**Proof:** (I). Necessity. (i). If $\tilde{F}_2 \leqslant \tilde{F}_3$, then

$$M_l = \min\left[1-\frac{1}{2\lambda\varepsilon_2}-\frac{\tilde{F}_2}{2\lambda\varepsilon_1}, 1-\frac{\tilde{F}_2}{2\varepsilon_1}\right] .$$

By Proposition A1 and (4.10), we can obtain (B1a,b), and (B1c) is also true.

(ii) If $\tilde{F}_2 > \tilde{F}_3$, then we can prove (B1) by (4.10) similarly.

(II). Sufficiency. If (B1) holds, then by Proposition A1 we can prove that there exists $(\varepsilon_1, \varepsilon_2)$ satisfying (4.10).

**Proposition B2.** Suppose $\varepsilon_3>0, 0<\chi_1<1$ and $\tilde{F}_i>0$ $(i=1,\cdots,3)$, then $(\varepsilon_3, \chi_1)$ such that (B1) holds if and only if (4.11) holds.

**Proof:** (I). Necessity. If (B1) holds, then by Proposition A2 and (B1a,b), we have

$$\frac{\chi_1 f_0^2}{C_3 c_s^2}\max\left(\frac{c^2}{f_0^2}\right)<\frac{1}{\varepsilon_3}<\left(\lambda-\frac{1}{C_1}-\frac{1}{C_3}\frac{f_0^2\chi_1}{c_s^2}\right)\frac{C_3 c_s^2}{\chi_1 f_0^2} \ , \tag{B2}$$

thus

$$\left(\frac{f_0^2}{C_3 c_s^2}\max\left(\frac{c^2}{f_0^2}\right)\right)\chi_1^2+\chi_1+\left(\frac{1}{C_1}-\lambda\right)\frac{C_3 c_s^2}{f_0^2}<0 \ . \tag{B3}$$

There exists a solution $0<\chi_1<1$ to (B2) if and only if (4.11a) holds, and

$$0<\chi_1<\frac{C_3\left(\sqrt{1+4\max\left(\frac{c^2}{f_0^2}\right)\left(\frac{2}{a^2}-\frac{1}{C_1}\right)}-1\right)}{2\max\left(\frac{c^2}{f_0^2}\right)\frac{f_0^2}{c_s^2}} \ . \tag{B4}$$

Since (B1c) can be written as

$$1-C_3\alpha_s<\chi_1, \tag{B5}$$

hence

$$1-C_3\alpha_s<\chi_1<\frac{C_3\left(\sqrt{1+4\max\left(\frac{c^2}{f_0^2}\right)\left(\frac{2}{a^2}-\frac{1}{C_1}\right)}-1\right)}{2\max\left(\frac{c^2}{f_0^2}\right)\frac{f_0^2}{c_s^2}} \ . \tag{B6}$$

Omitting $\chi_1$, we can obtain (4.11b).

(II). Sufficiency. If (4.11) holds, we can choose $0<\chi_1<1$ such that (B6) holds, then (B4) and (B5) hold, By (B5), we can obtain (B1c), and by (B4) we have (B3). Hence we can choose $\varepsilon_1>0$ such that (B2) holds, then (B2a) and (B1b) hold.

Finally, Lemma 2 is proved by using Proposition B1 and Proposition B2.

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