# Some Possible Solutions of Nonlinear Internal Inertial Gravity Wave Equations in the Atmosphere

*Li Guoping* (李国平) and *Lu Jinghua* (卢敬华)
Chengdu Institute of Meteorology, Chengdu, 610041
Received January 19, 1995; revised September 11, 1995

ABSTRACT

In this paper, the nonlinear internal inerntial gravity wave equation is derived by the analysis method of phase plane and is solved by integration method. The results showed that this nonlinear equation not only has ordinary solitary wave solution but also has another extra–ordinary solutions, and the form of solution is related to stratification stability, wave velocity and direction of wave motion.

**Key words:** Internal inertial gravity wave, Nonlinear wave solution, Solitary wave

## I. INTRODUCTION

In the recent 30 years, the investigation on the solitary and cnoidal wave of the nonlinear motion has attracted great interests in the atmosphere and ocean sciences. For the barotropic fluid, Long (1964) did creative work. Under the condition of stationary and quasi–geostrophic fluid on $\beta$–plane, supposing that the basic flow varies slowly with latitudes he discussed the barotropic fluid problem and obtained the solitary solution. Redekopp (1977) investigated the soliton of stratified fluid in the quasi–geostrophic potential vorticity model and discovered that the Rossby wave satisfies the deformation KdV equation (i. e. mKdV equation). Malguzzi and Malanotte–Rizzoli (1984) studied the nonlinear and stationary Rossby solitary wave under the basic flow with horizontal and vertical shear. McWilliams et al. (1981) studied strong nonlinear effect, and got solitary vertex solution. Chao and Huang (1980) studied the cnoidal wave of barotropic atmosphere. Liu and Liu (1982) presented some typical solutions of cnoidal and solitary wave, through transforming the nonlinear partial differential equation to the nonlinear ordinary differential equation by means of a phase–plane method. Cheng (1993) reanalyzed the KdV equation of liu et al. and derived the general solution of solitary wave by Bargmann potential method. He (1985) studied the boundary problem of second order ordinary differential equation and his results showed that the soliton exists in the atmosphere widely. In this paper, for a mesoscale synoptic system (e. g. China South West Vortex), the KdV equation describing nonlinear internal inertial gravity wave is given. Applying directive integration method, some typical solutions of KdV equation are derived and discussed in detail.

## II. THE DERIVATION OF KDV EQUATION

In the $p$ coordinates, the equations decribing nonlinear internal inertial gravity wave related to the mesoscale synoptic system are in the form:

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+\omega\frac{\partial u}{\partial p}=-\frac{\partial \varphi}{\partial x}+fv\ , \tag{1}$$

$$\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+\omega\frac{\partial v}{\partial p}=-\frac{\partial \varphi}{\partial y}-fu\ , \tag{2}$$

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial \omega}{\partial p}=0\ , \tag{3}$$

$$\frac{\partial T}{\partial t}+u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}-\sigma\omega=0\ , \tag{4}$$

$$\frac{\partial \varphi}{\partial p}=-\frac{RT}{p}\ , \tag{5}$$

where $\sigma=-(\gamma_d/\rho g+\partial T/\partial p)$ represents the parameter of atmospheric stratification stability, other symbols are as usual. For simplicity, assuming $\partial(\ \ )/\partial y=0$ (i. e. the disturbances are not relative to $y$), combining Eqs.(4) and (5) we yield

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+\omega\frac{\partial u}{\partial p}=-\frac{\partial \Phi}{\partial x}+fv\ , \tag{6}$$

$$\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+\omega\frac{\partial v}{\partial p}=-fu\ , \tag{7}$$

$$\frac{\partial u}{\partial x}+\frac{\partial \omega}{\partial p}=0\ , \tag{8}$$

$$\frac{\partial}{\partial t}\left(\frac{\partial \varphi}{\partial p}\right)+u\frac{\partial}{\partial x}\left(\frac{\partial \varphi}{\partial p}\right)+\sigma_p\omega=0\ , \tag{9}$$

where $\sigma_p=R\sigma/p$. For solving Eqs.(6)–(9), let

$$u=U(\theta)\ ,\quad v=V(\theta)\ ,\quad \omega=\Omega(\theta)\ ,\quad \varphi=\Phi(\theta)\ , \tag{10}$$

where $\theta=kx+np-\nu t$. Substituting Eq.(10) into Eqs.(6)–(9), we have

$$(-\nu+kU+n\Omega)U'=-k\Phi'+fV\ , \tag{11}$$

$$(-\nu+kU+n\Omega)V'=-fU\ , \tag{12}$$

$$kU'+n\Omega'=0\ , \tag{13}$$

$$n\Phi''(-\nu+kU)+\sigma_p\Omega=0\ , \tag{14}$$

where $\nu=kc$ represents wave frequency, $c$ is wave speed. Assume the conditions of determinant solution as follows

when $\theta\to\infty$

$$U_0\to U_0\ \text{(constant)}, U'\to 0,\ U''\to 0,\ \Phi'\to 0\ . \tag{15}$$

With these conditions, we integrate Eq.(13) and take an integration constant equal to zero after considering expression (15), and yield

$$n\Omega=-kU\ . \tag{16}$$

Substituting Eq.(15) into Eqs.(11), (12) and (14), after eliminating $V$ and $\Phi$, the differential equation with a single variable $U$ is got:

$$U''=-\frac{f^2}{\nu^2}U+\frac{\sigma_p k^2}{n^2\nu}\cdot\frac{U}{-\nu+kU}\ . \tag{17}$$

Eq.(17) is a nonlinear differential equation. evidently, $U=0$ is one of its balance points. By expansion method of nonlinear term, $U/(-v+kU)$ is expanded into Taylor's series at $U=0$ , i.e.

$$\frac{U}{-v+kU}=-\frac{U}{v}\cdot\frac{1}{1-kU/v}=-\frac{U}{v}-\frac{k}{v^2}U^2-\cdots \tag{18}$$

Taking the first order approximation of Eq.(18), we yield

$$U''=-\frac{f^2n^2+\sigma_p k^2}{n^2v^2}U \ . \tag{19}$$

Through the translations above, Eq(17) can be expressed as a linear differential equation (i. e. Eq.(19)). This equation can be solved as follows.

2.1 For $f^2n^2+\sigma_p k^2>0$ (i. e. the stratification is stable or weakly unstable), we have

$$U=c_1^* \cos\left(\frac{f^2n^2+\sigma_p k^2}{n^2v^2}\right)^{1/2}\theta+c_2^* \sin\left(\frac{f^2n^2+\sigma_p k^2}{n^2v^2}\right)^{1/2}\theta \ , \tag{20}$$

which represents the linear internal inertial gravity wave.

2.2 For $f^2n^2+\sigma_p k^2<0$ (i. e. the stratification is strong unstable), we have

$$U=c_3^* \exp\left[\left(\frac{-f^2n^2-\sigma_p k^2}{n^2v^2}\right)^{1/2}\theta\right]+c_4^* \exp\left[-\left(\frac{-f^2n^2-\sigma_p k^2}{n^2v^2}\right)^{1/2}\theta\right] \ , \tag{21}$$

where $c_2^*$ , $c_2^*$ , $c_3^*$ and $c_4^*$ are all integration constants, they could be determined by the conditions of determinant solution.

The purpose of this paper is to discuss nonlinear characteristics of Eq.(17), therefore we take the second order approximation of Eq.(18):

$$\frac{U}{-v+kU}=-\frac{U}{v}-\frac{k}{v^2}U^2 \ . \tag{22}$$

Substituting Eq.(22) into Eq.(17) yields

$$U''=-\frac{f^2n^2+\sigma_p k^2}{n^2v^2}U-\frac{\sigma_p k^3}{n^2v^3}U^2 \ . \tag{23}$$

Differentiating Eq.(23) with respect to $\theta$, we get

$$U'''+\frac{2\sigma_p k^3}{n^2v^3}UU'+\frac{f^2n^2+\sigma_p k^2}{n^2v^3}U'=0 \ . \tag{24}$$

Letting $a=\sigma_p k^3/n^2v^3$ , $b=(f^2n^2+\sigma_p k^2)/n^2v^3$ , then Eq.(24) can be rewritten as

$$U'''+2aUU'+bU'=0 \ . \tag{25}$$

Eq.(25) is an ordinary differential equation corresponding to the famous KdV equation. Neglecting stratification effect, i.e. $a=0$, or $\sigma_p=0$ (neutral stratification), the nonlinear wave disappears, i. e Eq.(25) would degenerate to a linear equation (similar to Eq.(19). Its solution was discussed above.

III. SOLVING OF THE NONLINEAR INTERNAL INERTIAL GRAVITY EQUATION

There are some methods solving the KdV equation, e. g. the expansion of nonlinear term method, Backlund translation method, inverse scattering method, Bargmann potential method, singular perturbation method, reductive perturbation method and so on. In this paper, we are going to solve Eq.(25) by a simple method, different from the methods mentioned above, i. e. not directly solving the KdV equation as usual method but solving its lower order equation (i.e Eq.(23)). For general KdV equation, the expression same as Eq.(22) can be obtained only after integrating it. Multiplication of (23) by $U'$ yields

$$U'U'' + aU'U^2 + bU'U = 0 \ , \tag{26}$$

or

$$\frac{dU}{d\theta}\frac{d}{d\theta}\left(\frac{dU}{d\theta}\right) + a\frac{dU}{d\theta}U^2 + b\frac{dU}{d\theta}U = 0 \ . \tag{27}$$

Integrating Eq.(27) with respect to $\theta$, we have

$$\frac{1}{2}\frac{d}{d\theta}\left[\left(\frac{dU}{d\theta}\right)^2\right] + \frac{d}{d\theta}\left(\frac{a}{3}U^3 + \frac{b}{2}U^2\right) \ . \tag{28}$$

· or

$$\frac{1}{2}\left(\frac{dU}{d\theta}\right)^2 + \frac{a}{3}U^3 + \frac{b}{2}U^2 = c_5^* \tag{29}$$

The integration constant can be determined by expression (15) as follows:

$$c = \frac{2a}{3}U_0^3 + bU_0^2 (c = 2c_5^*) \ . \tag{30}$$

Under $U_0 = -b/a$, i. e. $U_0 = -v(f^2n^2 + \sigma_p k^2)/\sigma_s k^3$, Eq. (29) can be written as

$$\left(\frac{dU}{d\theta}\right)^2 = \frac{2a}{3}(U_0^3 - U^3) + b(U_0^2 - U^2) \ , \tag{31}$$

and the right hand of Eq.(30) can be rewritten as

$$\frac{2a}{3}(U_0^3 - U^3) + b(U_0^2 - U^2) = (U + A)^2(BU + D) \ , \tag{32}$$

where $A = b/a = -U_0$, $B = -2a/3$, $D = a^2c/b = b/3$.

Therefore, we take square root of Eq.(31) after considering expression (32), and integrate it, thus yield

$$\int \frac{dU}{(U + A)\sqrt{BU + D}} = c^* \pm \theta \ , \tag{33}$$

where $c^*$ is integration constant. Assuming $F = U + A$, Eq.(33) can be rewritten as

$$\int \frac{dF}{F\sqrt{BF + (D - AB)}} = c^* \pm \theta \ , \tag{34}$$

3.1 If $D - AB < 0$, the intergration of Eq.(34) is

$$\frac{2}{\sqrt{AB-D}}\arctan\left[\frac{BF+(D-AB)}{AB-D}\right]^{1/2}=c^{*}\pm\theta \quad . \tag{35}$$

Regularizing Eq.(35), we have one of solutions of KdV equation as follows:

$$U=-\frac{D}{B}+\frac{(AB-D)}{B}\tan^{2}\left[\frac{\sqrt{AB-D}}{2}(c^{*}\pm\theta)\right] , \tag{36}$$

i. e.

$$U=\frac{b}{2a}+\frac{3b}{2a}\tan^{2}\left[\frac{\sqrt{-b}}{2}(c^{*}\pm\theta)\right] . \tag{37}$$

Because the origin of coordinates can be moved along $\theta$ axis, taking $c^{*}=0$, yield

$$U=-\frac{U_0}{2}-\frac{3}{2}U_0\tan^{2}\left[\frac{\sqrt{-(f^2n^2+\sigma_p k^2)}}{2n\nu}\theta\right] . \tag{38}$$

Letting $N^2=f^2n^2+\sigma_p k^2$, this solution can be rewritten as

$$U=\frac{\nu N^2}{2\sigma_p k^3}+\frac{3\nu N^2}{2\sigma_p k^3}\tan^{2}\left(\frac{\sqrt{-N^2}}{2n\nu}\theta\right) . \tag{39}$$

Evidently, this solution does not accord with the conditions of determinant solution (i.e. expression (15)), therefore it is an extra solution (i. e. nonsensical solution) which will not be discussed in this paper.

3.2 If $D-AB>0$, i. e. $3b>0$, or $3(f^2n+\sigma_p k^2)>0$, i.e.

$$\sigma_p>-\frac{f^2n^2}{k^2} \qquad (\text{or} \quad N^2>0) , \tag{40}$$

which shows that the stratification is weakly unstable or stable, then integrating the left hand of Eq.(34) yields

$$\frac{1}{\sqrt{D-AB}}\ln\left|\frac{\sqrt{BF+(D-AB)}-\sqrt{D-AB}}{\sqrt{BF+(D-AB)}+\sqrt{D-AB}}\right|=c^{**}\pm\theta . \tag{41}$$

3.2.1 If $\sqrt{BF+(D-AB)}-\sqrt{D-AB}>0$ in Eq.(41), we may derive $F>0$ or $U>-b/a=U_0$, where $U_0$ is the minimum of $U$. Then, Eq.(41) can be rewritten as

$$\ln\frac{\sqrt{BF+(D-AB)}-\sqrt{D-AB}}{\sqrt{BF+(D-AB)}+\sqrt{D-AB}}=\sqrt{D-AB}\quad(c^{**}\pm\theta) , \tag{42}$$

or

$$\zeta=\ln\frac{\sqrt{BU+D}-\sqrt{D-AB}}{\sqrt{BU+D}+\sqrt{D-AB}}=\ln\left[\frac{\sqrt{BU+D}-\sqrt{D-AB}}{\sqrt{BU+AB}}\right]^{2} , \tag{43}$$

where $\zeta=\sqrt{D-AB}(c^{**}\pm\theta)$. Especially, we want to point out that the sign of $\sqrt{BU+AB}$ in expression (43) does not affect the value of the right hand of expression (43).

Because of $U+A=F>0$, we have $B>0$, i.e. $-2a/3>0$ when $(BU+AB)>0$. According to a definite value, it would have the following conclusions: For $\sigma_p>0$, $\nu<0$;

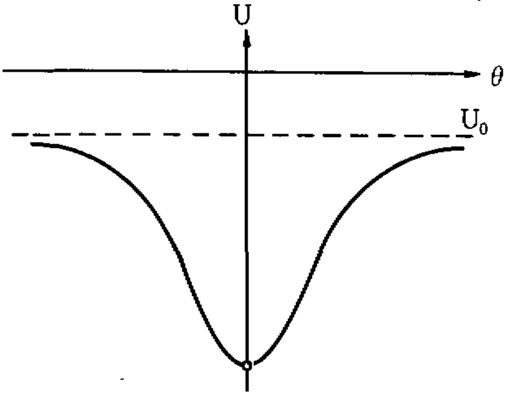


Fig. 1. The first type of wave solutions

for $\sigma_p < 0$, $v > 0$. i.e. the disturbance propagates westward in the stable stratification; on the other hand, the disturbance propagtes eastward in the unstable stratification.

For expression (44), we have

$$\frac{\zeta}{2} = \ln\frac{\sqrt{BU+D} - \sqrt{D-AB}}{\sqrt{BU+AB}} , \qquad -\frac{\zeta}{2} = \ln\frac{\sqrt{BU+AB}}{\sqrt{BU+D} - \sqrt{D-AB}} \tag{44}$$

$$\frac{1}{2}(e^{\zeta/2} - e^{-\zeta/2}) = sinh\frac{\zeta}{2} = -\frac{\sqrt{D-AB}}{\sqrt{BU+AB}} . \tag{45}$$

Through appropriate mathematical translation, one of the solutions of KdV equation can be obtained as follows:

$$U = \frac{(D-AB)}{B}\text{csch}^2\frac{\zeta}{2} - A , \tag{46}$$

or

$$U = -\frac{3b}{2a}\text{csch}^2\frac{\zeta}{2} - A , \tag{47}$$

i. e.

$$U = -\frac{3b}{2a}\text{csch}^2\left[\left(\frac{\sqrt{b}}{2}\right)(c^{**} \pm \theta)\right] - \frac{b}{a} . \tag{48}$$

By the method as 3.1, the orgin of coordinates can be moved along $\theta$, so that $c^{**} = 0$, and considering that $U_0 = -b/a$ and $\text{csch}^2\theta$ is not negative, we have

$$U = U_0 + \frac{3}{2}U_0\text{csch}^2\left(\frac{\sqrt{f^2n^2 + \sigma_p k^2}}{2nv}\theta\right) , \tag{49}$$

or

$$U = -\frac{vN^2}{\sigma_p k^3} - \frac{3vN^2}{2\sigma_p k^3}\operatorname{csch}^2\frac{\sqrt{N^2}}{2nv}\theta \ . \tag{50}$$

The solution as expression (50) is discontinuous at $\theta = 0$. Its shape is shown in Fig. 1.

At $\theta = 0$ (i.e. the orgin point), function $\operatorname{csch}^2\theta$ is discontinuous, it would represent the weather of vortex eye. It is known that there is an eye structure in the center of tropical cyclone. Recently, the meteorologists in U. S. A. have observed the similar structure in the model expeiments for the genesis and development of midlatitude cyclone with computer. Moreover, based on the first type of solution and the weather facts above, we could infer that there may be an eye structure in the center of anticyclone (i.e. high pressure, when $U_0 < 0$). Of course, this inference should be testified by the new discovery from the weather observation and analysis, for example, the satellite cloud picture and mesoscale weather map.

3.2.2 If $\sqrt{BF+(D-AB)} - \sqrt{D-AB} < 0$, then $U + A < 0$, i.e. $U < -B/A = U_0$, therefore $-b/a$ is the maximum of $U$.

Similary, letting $\zeta = \sqrt{D-AB}\,(c^{**} \pm \theta)$ yields

$$\zeta = \ln\frac{\sqrt{D-AB} - \sqrt{BU+D}}{\sqrt{D-AB} + \sqrt{BU+D}} = \ln\left[\frac{\sqrt{D-AB} - \sqrt{BU+D}}{\sqrt{-BU-AB}}\right]^2 \ . \tag{51}$$

Through simple mathematical translation, we yield

$$\frac{\zeta}{2} = \ln\frac{\sqrt{D-AB} - \sqrt{BU+D}}{\sqrt{-BU-AB}} \ , \qquad -\frac{\zeta}{2} = \ln\frac{\sqrt{-BU-AB}}{\sqrt{D-AB} - \sqrt{BU+D}} \ , \tag{52}$$

$$\frac{1}{2}(e^{\zeta/2} + e^{-\zeta/2}) = \operatorname{ch}\frac{\zeta}{2} = \frac{\sqrt{D-AB}}{\sqrt{-BU-AB}} \ . \tag{53}$$

Finally, we get

$$U = -\frac{b}{a} + \frac{3b}{2a}\operatorname{sech}^2\frac{\zeta}{2} \ . \tag{54}$$

By the method similar to above, we get another type of solution

$$U = U_0 - \frac{3}{2}U_0\operatorname{sech}^2\left(\frac{\sqrt{f^2n^2 + \sigma_p k^2}}{2nv}\theta\right) \ , \tag{55}$$

or

$$U = -\frac{vN^2}{\sigma_p k^3} + \frac{3vN^2}{2\sigma_p k^3}\operatorname{sech}^2\frac{\sqrt{N^2}}{2nv}\theta \ . \tag{56}$$

It is the typical solution of the KdV equation, i. e. solitary wave solution.

Assume $U$ and $d$ to be the amplitude and width of solitary wave, respectively. For the internal inertial gravity solitary wave, we have

$$U^* = \frac{3c}{2k^2}\left|\frac{N^2}{\sigma_p}\right| \tag{57}$$

and

$$d = \frac{2nkc}{\sqrt{N^2}} \quad , \tag{58}$$

which indicate that $U^*$ and $d$ are related to both wave speed $c$ and stratification stabitity $\sigma_p$. The dispersion relationship of internal inertial gravity solitary wave is greatly different from that of linear internal inertial gravity wave. In the stable stratification, $c > 0$ i.e the wave moves eastward; in the unstable stratification, $c < 0$, i.e. the wave moves westward. Obviously, the wave speed is related to the wave number, the stratification stabitity and the wave ampliutude (as shown in Fig. 2). From Eq.(58), the amplitude of internal gravity solitary wave is proportional to its wave speed. Its wave width is proportional to its wave speed and inversely proportional to stratification ($\sqrt{N^2}$). The results show that the larger the amplitude of wave is, the wider the wave is and the quicker the wave moves. In addition, the more stable the stratification, the narrower the wave.

The analysis of solitary wave associated with weather facts may be as follows. It is generally considered that thundery weather in summer is caused by mesoscale systems, such as cumulonimbus. In the view point of wave, the internal inertial gravity wave is one of the main factors in this weather processes. If solitary wave enhances (it is shown that wind speed increases, vertical motion strengthens and so on), the wave motion speeds up. In addition, the more stable the stratification, the weaker the system. These features all accord with the actual situations. If $f = 0$, i.e. the inertial effect is neglected, the results were already shown by Liu and Liu (1982).

## IV. CONCLUDING REMARKS

In this paper, we investigete the nonlinear equation describing the internal inertial gravity wave. By using a direct integration method, two types of analysis solutions are obtained. The results show that for the ordinary differential equation corresponding to the KdV equation, there is another solution besides the solitary solution. We not only attend to the mathematical

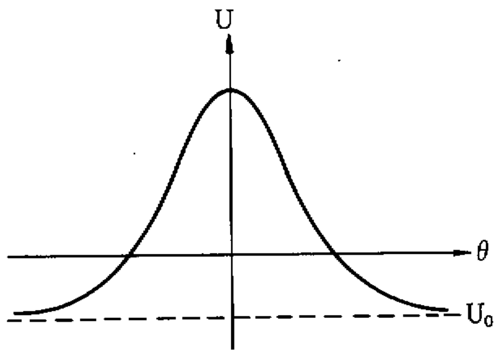


Fig. 2. The second type of solutions (i.e. soliton solution).

form of various solutions in this paper, but also discuss the physical features of them, Furthermore, these results would make us understand characteristics of the KdV equation and the solitary wave and mesoscale weather system more deeply.

The famous characteristics of the KdV equation are that there is a solitary solution, however, it only represents the special nature of KdV equation (Xue and Guo 1993). Specially, we must indicate out that the solving method and results are generally significant not only for the nonlinear internal inertial gravity wave, but also for the KdV equation describing other nonlinear waves. Besides the solitary wave, other wave solutions would represent some synoptic systems and weather phyenamena which are not known well up to now. It will be discussed further in the next paper.

We would like to thank Professor Zhang Qingde for his helpful discussion on mathematics.

REFERENCES

Chao Jiping and Huang Ruixin (1980), The cnoidal in the rotating barotropic atmosphere, *Scientia Sinica (Ser. B)*, **7**: 697–705.

Chen Zhongming (1992), General solutions to some types of solitons in the atmosphere, *Scientia Sinica (Ser. B)*, **3**: 328–336.

He Mengxing (1985), The soliton in the atmosphere, *Scientia Sinica (Ser. B)*, **1**: 79–86.

Liu Shida and Liu Shikuo (1982), The nonlinear cnoidal and soliton in the atmosphere, *Scientia Sinica (Ser. B)*, **4**: 372–385.

Long, R. R. (1964), Solitary wave in the westerlies. *J. Atmos. Sci.*, **21**: 197–200.

Malguzzi, P. and Malanotte R. (1984), Nonlinear stationary Rossby waves on nonuniform zonal winds and atmospheric blocking. Part II: The analytical theory, *J. Atmos. Sci.*, **41**: 2620–2628.

McWilliams, J. C., et al. (1981), Numerical studies of barotropic models. *Dyn. Atmos. Ocean S.*, **5**: 219–238.

Redekopp, L. G. (1977), On the theory of solitary Rossby waves. *J. Fluid Mech.*, **82**: 725–745.

Xue Jishan and Guo Bingrong (1993), Nonlinearity of weakly dispersive atmospheric motions, *Acta Meteorologica Sinica.*, **51**: 248–252.